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Difference Not Deficit: Reconceptualizing Mathematical Learning Disabilities

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Mathematical learning disability (MLD) research often conflates low achievement with disabilities and focuses exclusively on deficits of students with MLDs. In this study I adopt an alternative approach using a response-to-intervention MLD classification model and identify the resources students draw upon rather than the skills they lack. The intervention model involved videotaped one-on-one fraction tutoring sessions implemented with students with low mathematics achievement. This article presents case studies of two students who did not benefit from the tutoring sessions. Detailed diagnostic analyses of the sessions revealed that the students understood mathematical representations in atypical ways and that this directly contributed to the persistent difficulties they experienced. Implications for screening and remediation approaches are discussed.

Key words: Dyscalculia; Fractions; Learning disability; Mathematical learning disability

Many students struggle with mathematics, but not all students struggle for the same reasons. For the estimated 6% of students with a mathematical learning disability (MLD) (Shalev, 2007), their difficulties stem from a cognitive origin, leading to qualitatively different error patterns than those experienced by their low-achieving peers (Mazzocco, Devlin, & McKenney, 2008; Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013). Teachers are faced with the challenge of addressing the unique learning difficulties encountered by students with MLDs without accurate methods to identify students with MLDs or effective instructional supports. Standard instruction is unlikely to benefit these students, as longitudinal studies have demonstrated that the difficulties experienced by students with MLDs persist

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over years (Andersson, 2010; Geary, Hoard, Nugent, & Bailey, 2012; Mazzocco et al., 2013). Understanding the nature of the difficulties faced by these students is an essential first step toward the design of identification tools and alternative instructional approaches. In this article, I present detailed diagnostic case studies of two adult students with MLDs. I focus on how and what each student understood about the mathematics and refer to these analytically as *understandings*. In these case studies, I identify the understandings the student relied upon and the ways in which these understandings persisted and were incompatible with standard instructional approaches. This study contributes an analysis of MLDs that addresses several of the methodological challenges facing the field of MLD research.

Prior Research on Mathematical Learning Disabilities

Research on MLDs involves methodological challenges related to both the identification of students with MLDs and the complexities of characterizing learning disabilities in a hierarchical topic domain. Although there is agreement among researchers that MLDs have a biological (i.e., cognitive) origin (Mazzocco, 2007),[1] the field is still struggling with foundational issues around accurate identification of students. Currently there is no consensus on the operational definition of MLDs (Mazzocco, 2007), and researchers often rely on achievement test score thresholds (commonly the 25th percentile) to identify students with MLDs (Geary & Hoard, 2005). This method alone cannot determine if a student's low mathematics test score is due to cognitive or environmental factors, and thus its use results in overclassification of minority, low socioeconomic status, and nonnative-English-speaking students in the MLDs group (Hanich, Jordan, Kaplan, & Dick, 2001). This conflation of low achievement and MLDs continues to be a central challenge in the field.

In addition to these identification issues, prior research on MLDs has predominantly focused on elementary-aged students' speed and accuracy on written assessments of basic arithmetic calculation. Because of this, the defining characteristic of MLDs is often considered to be insufficient automaticity of arithmetic number facts, such as "4 + 5 = 9" (e.g., Gersten, Jordan, & Flojo, 2005; Swanson, 2007; Swanson & Jerman, 2006). The predominant focus on performance deficits in arithmetic calculation reduces mathematical cognition to the accurate and efficient production of an answer and leaves unexplored many of the conceptual, procedural, and representational issues core to mathematics. The few studies that have begun examining MLDs in more complex mathematical domains have found conceptual and representational issues—not difficulties with number facts—to be central to the errors made by students with MLDs (e.g., Hecht & Vagi, 2010; Mazzocco & Devlin, 2008). These studies suggest that MLDs should be

[1]Other researchers have argued for the social construction of disability (Gallagher, 2004; McDermott, 1993; McDermott & Varenne 1995). Although this perspective—which distinguishes impairment and disability (Baglieri, Valle, Connor, & Gallagher, 2011; Sherry, 2006)—is consistent with my theoretical framing, the point made here is that researchers studying MLDs conceptualize MLDs as having a cognitive origin.

considered in the context of more complex mathematical topics, which raises methodological questions around how to investigate mathematics that cannot be so easily reduced to measures of speed and accuracy.

Overview of the Present Study

In this study, I adopt an alternative approach that attempts to address the challenges facing the field and provides a new vantage point on MLDs. I avoid conflation of low achievement and MLDs by selectively recruiting students and employing a hybrid model of learning disability identification (Fletcher, Lyon, Fuchs, & Barnes, 2007), which attempts to empirically evaluate the likelihood that a student's low achievement is due to something other than a disability (i.e., confounding factors, including: poor teaching, environmental causes, or affective factors). This model requires that in addition to a student exhibiting low mathematics achievement, which cannot be attributed to another factor, the student must not benefit from a validated intervention—in this case, a series of tutoring sessions focused on fractions that were effective for typically achieving students (Lewis, 2011). To move beyond an understanding of MLDs as deficits in basic calculation, I conceptualize MLDs in terms of cognitive differences and identify those differences through a detailed analysis of the understandings a student relied upon when attempting to learn fractions—a conceptually and representationally complex topic. My intent was to identify what understandings are consequential (and sometimes detrimental) to the student's learning of mathematics.

To this end, in the context of weekly tutoring sessions I explored the understandings of fractions that emerged and persisted for students with MLDs. The purpose of this study was to provide an in-depth analysis of the nature of the difficulties experienced by students with MLDs. The tutoring data provided a rich context to explore how the students with MLDs understood fractions, often in problematic ways. In particular I was interested in investigating (a) what persistent understandings were underlying each student's difficulties and (b) what similarities, if any, existed between the identified persistent understandings. This in-depth look at two students with MLDs allowed for a view of MLDs beyond errors on outcome measures and instead captured the MLDs in a learning context as it emerged.

Theoretical Framework

In this section I propose an alternative perspective of MLDs, which is grounded in a Vygotskian theoretical perspective and informed by mathematics education research on the teaching and learning of fractions.

Reconceptualizing Disability as Difference

Unlike prior research that has predominantly conceptualized MLDs in terms of *cognitive deficits* (e.g., Geary, 2010), I conceptualize MLDs in terms of *cognitive differences*. My perspective is derived from a Vygotskian perspective of disability in which disabilities are understood to result in different paths of development rather than deficient development (Vygotsky, 1929/1993). Students with MLDs are understood to have biological (i.e., cognitive) differences that may result in mathematical development unlike that of their peers. This theoretical perspective highlights the importance of mediational tools in human development and has implications for the methodologies used to study disabilities.

Mediational tools, which have developed over the course of human history, are central to a sociocultural understanding of learning and development and must be considered in the study of disabilities (Vygotsky, 1978). For students with disabilities the mediational tools may be incompatible with the student's biological development (Vygotsky, 1929/1993). As illustrated in an extreme case, spoken language is not accessible to a deaf child and therefore does not serve the same mediational role to support the child's development of language as it would for a hearing child. In the case of students with MLDs, it is possible that standard mathematical mediational tools (e.g., Arabic numerals, drawings, manipulatives), which support the development of typically achieving students, may not be compatible with how students with MLDs cognitively process numerical information. These kinds of incompatibilities do not simply result in deficient development; instead, the incompatibility results in the recruitment of alternative resources and different developmental paths (Cole, Levitin, & Luria, 2006). Therefore, students with MLDs may develop alternative understandings, particularly around representations, which are different from (and inconsistent with) canonical mathematical understandings.

Methodologies used to study individuals with MLDs should capture these alternative understandings. Quantitative measures of an individual in terms of less and more are inappropriate. In the same way that measuring a deaf child's lack of auditory receptiveness will not provide insight into his or her language development, measuring only what a student with an MLD cannot do (i.e., errors) will not provide insight into his or her mathematical development. Instead, measures sensitive to qualitative differences must be employed. Diagnostic methodologies involve starting with careful observations of the individual in which no a priori analytic schemes are defined, as the researcher cannot know the shape that the divergent developmental paths might take (Cole et al., 2006). Only through repeated observations and iterative analysis can these differences be identified and characterized (Cole et al., 2006).

In the context of MLDs, careful observations of the student engaged in attempts to learn mathematics provides a venue to study how the student makes sense of mathematics. Through repeated observations of the student it is possible to begin to identify and characterize the student's understandings (Schoenfeld, Smith, & Arcavi, 1993). Given that persistent difficulties have been documented in students with MLDs (Geary et al., 2012; Mazzocco et al., 2013), in this study it was important to document the understandings that persisted. Therefore a central analytic construct is *persistent understanding*. The attribution of understandings to an individual is an inherently complex issue, and I make no claims about the cognitive reality of attributed understandings. Instead, I adopt Schoenfeld's (1998) model of analytic attribution: I attribute an understanding to an individual if in that instance the student behaves in a way that is consistent with that operationally defined persistent understanding. The persistent understandings become analytic tools with which to view the complexity of the moment-by-moment interaction and look for patterns within the data.

In this study, MLDs are framed as cognitive differences, and the analysis focuses on identifying the persistent understandings that a student relies upon in the context of learning. It is expected that students with MLDs will have difficulties that may be qualitatively different from those typically experienced by students learning the topic and that these difficulties may revolve around their understanding or use of mathematical representations.

Importance of Mathematical Representations

Representations are not only central to a Vygotskian perspective of MLDs but are fundamental to the very practice of mathematics (Ball, 1993; Kaput, 1987). Engaging with mathematics involves mastering a host of representations (e.g., symbols, diagrams, pictures) to navigate a mathematical context. Yet, a representation itself does not contain meaning; the meaning of the representation is a result of the interpretation of the user (von Glasersfeld, 1987). Because of this inherent subjectivity involved in the use of representations, how a student with an MLD perceives and operates upon a representation might be quite different from how students typically perceive and operate on that representation. For example, prior research on students' understanding of basic numerical symbols has suggested that students with MLDs may have difficulty processing both symbolic forms, such as "7" (De Smedt & Gilmore, 2011; Landerl & Kölle, 2009), and pictorial forms, such as "●●●●●●●" (Piazza et al., 2010), of numerical magnitude. Therefore, studies of MLDs should carefully attend to the ways in which students make sense of representations.

Mathematical Domain of Fractions

Although a student's use of representations could be explored in any mathematical domain, I chose to focus on the extensively researched and mathematically rich topic of fractions. Not only is an understanding of fractions essential for later mathematical development (Bailey, Hoard, Nugent, & Geary, 2012; National Mathematics Advisory Panel, 2008; Siegler et al., 2012), but also research on MLDs suggests that students with MLDs appear to have different learning trajectories than low-achieving students, specifically in the context of fractions (Mazzocco et al., 2013). A fraction is a number that can be expressed as a ratio a/b when a and b are integer values and b is not equal to zero. I use the term *fraction* rather than *rational number* because it represents the predominant notational form used during the tutoring sessions.

For a number of reasons, fractions provide a mathematically rich terrain to explore how students with MLDs make sense of mathematical concepts beyond whole number competencies. Students must learn that the value of the fraction is determined by the coordination of the numerator and denominator, which means that fractions with larger numbers may actually be smaller (e.g., 17/100 < 1/2), that there are multiple ways to write any fractional value (e.g., 1/2, 2/4, 3/6, etc.), and that there is no next fractional value as there is with integers. In addition, students must master a variety of representations used to visualize, manipulate, and make sense of rational numbers (Lamon, 1996, 2007). Developing competency with fractions involves procedural, conceptual, and representational understanding and consequently is an ideal mathematical context in which to explore MLDs.

This study was limited to a part–whole understanding of fractions. A part–whole understanding of fractions requires that students understand that the fraction represents a single value and that the value depends upon the relationship between the (part) numerator and the (whole) denominator (Mack, 1990, 1995; Post, Wachsmuth, Lesh, & Behr, 1985). Although rational numbers can take on a variety of interpretations (i.e., part–whole, quotient, ratio, operator, and measure), the part–whole understanding is the most commonly used interpretation for introducing fraction concepts. It allows for a multitude of representational forms to be explored; and it is the most thoroughly researched with respect to the teaching and learning of fractions.

Methods

The data collected for this study were used for two purposes: determination of the student's MLD status and diagnostic case study analysis. Students were recruited for this study, and all data were collected before a classification determination was made. Only those students who met the MLD classification criteria were included in the diagnostic analysis as case study participants (see Figure 1 for an overview of the design of the study).

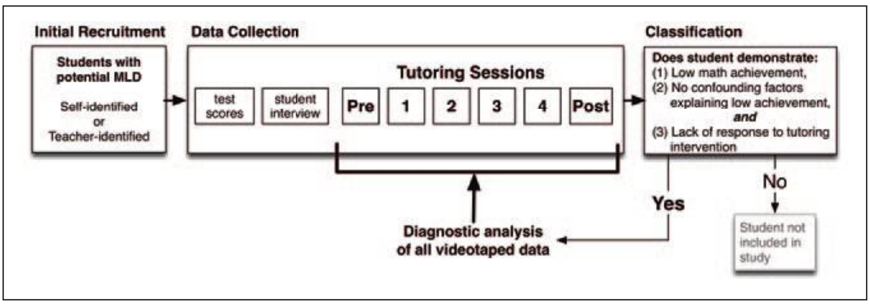


Figure 1. Schematic overview of methods used in this study.

Participants

Eleven students with potential MLDs were selectively recruited from a local middle school, high school, and community college based on self-nomination or teacher nomination. In addition, typically achieving students were recruited to evaluate the effectiveness of the tutoring protocol and empirically establish expected learning gains from pretest to posttest. To ensure that the mathematical content was appropriate for the typically achieving students, fifth-grade students were recruited because, like the students with potential MLDs, these students had prior experience with fractions but had not mastered the topics covered during the tutoring sessions. The parents of all fifth-grade students at a local elementary school received an email inviting their child to participate in a series of mathematics enrichment tutoring sessions. All five students whose parents responded and consented were included as comparison students.

Classification of students. I collected test scores, interview data, and pretest and posttest scores (from the tutoring sessions) to evaluate whether the 11 students met qualifications for having an MLD, defined as (a) low mathematics achievement, (b) no confounding factors that could explain the low achievement, and (c) lack of response-to-intervention (Fletcher et al., 2007). Standardized test scores were collected to establish that the student demonstrated low achievement (below the 25th percentile) comparable to criteria used in other studies of MLDs (see Geary & Hoard, 2005, for a discussion). To evaluate if the student's low achievement could be due to a confounding factor, students were interviewed and asked to reflect upon the origin and cause of their difficulties in mathematics. Confounding factors, correlated with low mathematics achievement, were considered exclusionary criteria for this study; these included lack of English fluency (Halle, Hair, Wandner, McNamara, & Chien, 2012), low socioeconomic status (Chatterji, 2005; Diversity in Mathematics Education Center for Learning and Teaching [DiME], 2007; Jordan, Kaplan, Nabors Oláh, & Locuniak, 2006), anxiety (Ashcraft, Krause, & Hopko, 2007), and behavior or attention issues (Zentall, 2007).[2]

To evaluate the 11 students' response-to-intervention, I compared their scores from the pretest and the posttest given in the fraction tutoring intervention (described below) to those of the fifth-grade students. The fifth-grade students had an average gain of 15% from pretest to posttest and an average posttest score of 84% (pretest mean = 68.7%, SD = 19.4%; posttest mean = 83.6%, SD = 12.6%). A lack of response-to-intervention was defined as less than a 10% gain from pretest to posttest and a posttest score at or below 60% (one standard deviation lower than the average fifth-graders' posttest score).

Out of the 11 students, nine students were excluded from the MLD classification for one of several reasons: performance at ceiling on the pretest (n = 2), observed or self-reported attention or behavior problems (n = 3), failure to complete all data collection sessions (n = 1), or response-to-intervention (n = 2; e.g., substantial gains from pretest to posttest suggesting that poor prior instruction was a possible cause of their low mathematics achievement). Only two students, Lisa (a White,

[2] These exclusion criteria are intended to ensure that the student's low achievement is not *primarily* due to a social or environmental factor. This does not mean that MLDs cannot cause anxiety or behavior issues, nor that nonnative English speakers or students from low SES backgrounds cannot have MLDs.

19-year-old community college student) and Emily (a White, 18-year-old recent high school graduate),[3] met all the qualifications for having an MLD (see Table 1). These two students, although considerably older than the fifth-grade students, scored within one standard deviation on the pretest but not on the post-test. This suggests that they had similar prior understanding to that of the fifth-grade students but did not similarly benefit from the tutoring protocol.

Table 1
Classification and Demographic Information for the Two Students with MLDs

Student	Math Achievement	Confounding factors	Response-to-intervention
Lisa	**Low** College placement test placed her in a remedial arithmetic class, which she failed. (Standardized achievement test scores not available.)	**None identified** Native English speaker Not low SES No attention or behavior issues	**None** (below threshold) Pretest = 59% Posttest = 44% Change = –15%
Emily	**Low** California state mandated STAR test mathematics score <25th percentile.	**None identified** Native English speaker Not low SES No attention or behavior issues	**None** (below threshold) Pretest = 49% Posttest = 54% Change = +5%

Note. Lisa was administered a truncated version of the pretest and the complete posttest. For the comparison of pretest to posttest score, only items that had a corresponding pretest item were included. On the complete posttest Lisa scored 34%, suggesting that the classification of "no response-to-intervention" is warranted.

Data Collection

Data were collected during six weekly videotaped sessions with each student. The student interview and pretest were administered in the first session, and the posttest was administered in the last session.

Pretest and posttest. The videotaped pretest and posttest were administered to all participants using a semi-structured clinical interview protocol (Ginsburg, 1997). The test was designed to cover the fraction concepts targeted in the tutoring sequence (see Appendix A for pretest and posttest questions and scoring).

Tutoring sessions. I conducted four hour-long videotaped tutoring sessions with each student focused on part–whole fraction concepts, building on research on dynamic assessment (Campione & Brown, 1987) and teaching experiments

[3]All participant names are psuedonyms.

(Saxe et al., 2010; Steffe & Thompson, 2000). Given the short duration of the tutoring intervention, the instructional goals were modest. Drawing upon Mack's (1990, 1995) tutoring studies, the instructional goals of the tutoring session were (a) to build an understanding of fractional magnitude through supporting an understanding of a fraction as a single value, which is determined by the relationship between the numerator and denominator, and (b) to use manipulatives and representations to explore the concepts of fraction equivalence and fraction operations. An area model representation was selected as the primary representational tool for these sessions, given that students with MLDs have been shown to have difficulties with number line representations of numerical magnitude (Geary, Hoard, Nugent, & Byrd-Craven, 2008). These sessions were designed based on prior research on the teaching and learning of fractions.

- Tutoring Session 1 involved the use of foam fraction pieces and focused on establishing the meaning of the numerator and the denominator (Armstrong & Larson, 1995; Hunting & Davis, 1991; Ni, 2001; Saxe, Taylor, McIntosh, & Gearhart, 2005).
- Tutoring Session 2 continued to build upon these concepts, explored the conventions of representing fractions with drawn area models, and focused on the use of area models to compare fractional amounts (Armstrong & Larson, 1995; Ball, 1993; Mack, 1993; Post et al., 1985; Saxe et al., 2010).
- Tutoring Session 3 addressed fair sharing in conjunction with area models to explore equivalent fractions (Ball, 1993; Empson, 2001; Hunting & Davis 1991; Lamon, 1996; Mack, 1993; Post et al., 1985; Saxe et al., 2005).
- Tutoring Session 4 focused on the use of manipulatives and area models to explore fraction operation problems (Mack 1995; Steffe, 2003).

Throughout all the tutoring sessions, meaningful engagement with representations was a central objective. Explicit attention was paid to connecting the symbols to their underlying referents (Hiebert, 1988), focusing upon explicitly establishing the conventions of standard pedagogical representations of fractions (e.g., Saxe et al., 2010). Following Lesh, Post, and Behr (1987), it was an instructional focus for the student to consider both translations between representational systems (e.g., representing 2/3 in fractional notation and with an area model) and transformations within representational systems (e.g., using an area model to create an equivalent fraction for 2/3).

To support these objectives, a tutoring protocol was developed in which the problems were carefully sequenced to ensure that each question built upon previously established mathematical content (see Appendix B; for complete protocol see Lewis, 2011). Similar to prior tutoring work, each question was conceptualized as an opportunity for the student to learn and as a means of assessing the student's understanding (Gutstein & Mack, 1998; Mack, 1995). Because building upon the student's prior knowledge was central to these tutoring sessions, students were asked to write a journal entry at the end of each session to create a record of what he or she had learned.

It is worth noting that although the tutoring intervention was essential for the MLD classification criteria employed in this study, the focus of this research was not on the tutoring intervention itself. Instead, the videotaped tutoring sessions, along with the pretest and posttest, provided a context in which the difficulties that arose for the students with MLDs could be analyzed. That several fifth-grade students benefited from this tutoring protocol suggests that this intervention should be considered a reasonable learning environment.

Analytic Approach

For the two students classified as having MLDs, a detailed analysis of the pretest, tutoring sessions, and posttest was conducted in an attempt to explore the difficulties they experienced. All videotapes of the sessions were transcribed and all artifacts were scanned. Each session was parsed into individual problems, each of which began with the posing of a question and ended with the student's answer. I conducted a grounded (Glaser & Strauss, 1967) diagnostic analysis for each student to generate analytic categories that capture the nature of the student's understanding (Schoenfeld et al., 1993). The goal was to identify the understandings that contributed to the student's difficulties. Therefore, in order to qualify as such, a persistent understanding needed to occur across multiple sessions and at least sometimes lead to an incorrect answer. Operational definitions for persistent understandings were developed and refined through iterative passes through the data, which is common in video analysis (Barron, Pea, & Engle, 2013). In this case, the iterative analysis involved identifying candidate persistent understandings, specifying inclusion and exclusion criteria, considering alternative explanations, and attempting to identify counterexamples that would contradict the proposed persistent understandings. This process involved considering alternative hypotheses to explain the data and then reviewing the data to determine if each hypothesis was supported or refuted.

My multiple roles in this study as both the tutor and the researcher warrants discussion of the researcher-as-instrument (Jaworski, 1998) in which I, as the researcher, was an active participant in the research process. Here, I describe how I viewed each of these roles and attempt to be explicit about the decisions made in the study design and analysis to mitigate bias that might arise from my multiple roles. In my role as the tutor, my goal was to maintain fidelity to the tutoring protocol and engage meaningfully with the student. Because the coding scheme emerged from the analysis, in my role as the tutor I was not hampered with preconceived notions of what form my analytic scheme might take. In my role as the researcher, my goal was to capture in a comprehensive manner what contributed to the difficulties that the student experienced. Because the tutoring sessions were conducted before analysis began, it was with some distance that I was able to analyze the video data. The video recording allowed for retrospective analysis of whether or not specific features of the tutoring sessions or tutoring itself might have contributed to any of the persistent understandings identified.

To ensure that my role as the tutor did not unduly shape my subsequent analysis of the data, after the operational definitions of persistent understandings were refined, a team of four coders, including myself, used these operational definitions to code the data. Each tutoring session was coded by at least two coders with each problem instance coded for correctness and evidence of any persistent understanding (see Figure 2 for an illustration of the parsing and coding of problems). Reliability for this coding was 95% (94.6% for Lisa and 95.4% for Emily). Any discrepancies

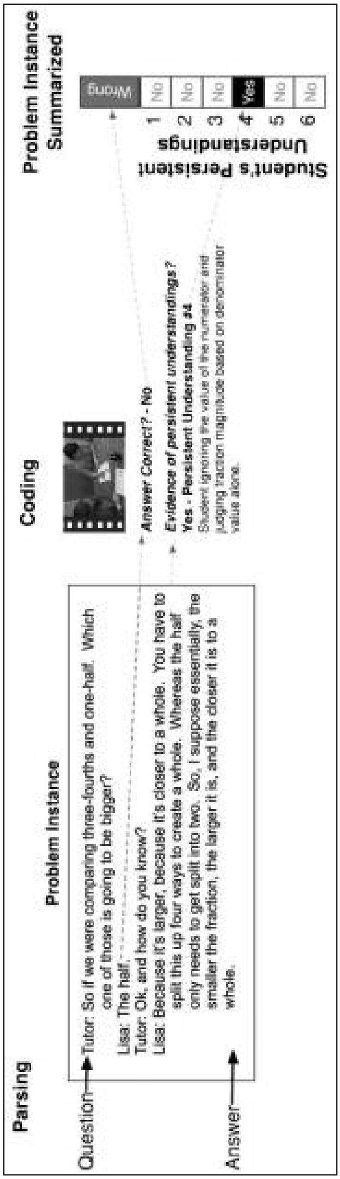


Figure 2. Illustration of the analytic process of problem instance parsing and coding for correctness and evidence of persistent understanding.

were discussed in a research meeting with all four coders and were resolved by watching the video from that problem instance and discussing whether there was sufficient evidence in the video to warrant the attribution of the operational definition (for a similar approach, see Schoenfeld et al., 1993). All discrepancies were resolved using stringent criteria for coding—if one of the four coders was not convinced that the episode matched the operational definition, it was not coded as such.

After the completion of the individual case studies, a comparison across cases was conducted to determine if similarities existed between students. Persistent understandings were considered to be similar if they led to similar errors. In addition, the video data collected with the typically achieving fifth-grade students were content logged (e.g., each question answered was briefly summarized; see Sawyer, 2013) and selectively transcribed to evaluate whether any of the persistent understandings identified for either student with MLDs were evident in any of the fifth-grade students and therefore could be attributed to the tutoring protocol itself. Data from the fifth-grade students will be discussed with respect to the comparison of case studies.

Results

Analysis of Cases

For each case study student, the analysis of the videotaped sessions revealed a unique collection of persistent understandings. Six persistent understandings were identified for Lisa (labeled L1 through L6), and six persistent understandings were identified for Emily (labeled E1 through E6; see Appendices C and D for the operational definition of each persistent understanding). First, I present an overview of analyses for both cases, including a graphic representation of their performance across the sessions and a narrative description of the persistent understandings identified for each student. The purpose of this section is to provide a high-level view of each case and establish that the persistent understandings provide a relatively comprehensive account of the difficulties experienced during the sessions. Second, I provide a detailed view of one persistent understanding for each student. Through transcript excerpts, I illustrate the persistent understanding as it occurred in the tutoring context, I demonstrate how the persistent understanding was detrimental in the student's attempts to reason about more complex fraction concepts, and I illustrate the ways in which the understanding was robust. Finally, I present a cross case comparison of Lisa and Emily, focusing on areas of similarity in their persistent understandings and contrasting them with the typically achieving fifth-grade students. In particular, I identify how both students' understandings led to common kinds of errors, which involved (a) focusing on the fractional complement and (b) representing 1/2 by halving.

Overview of Case Studies

To provide a high-level overview of the case studies, the correctness of the student's answer and evidence of persistent understandings is illustrated in

Figure 3. Each vertical segment represents an individual problem instance, and each problem instance is represented in chronological order. The number of problems in each tutoring session varied somewhat because the parsing of the data was based largely on the student's responses. This high-level view of the data provides a sense of each student's difficulties throughout the tutoring session (as illustrated by the prevalence of the incorrect answers) and reveals that the persistent understandings occurred across all sessions and sometimes in combination.

Description of the Persistent Understandings

Below I provide narrative descriptions of the persistent understandings for each student. These are intended to provide the reader with a sense of the variety of persistent understanding demonstrated by the students during the sessions. Each persistent understanding will be referenced by label (e.g., L1, which corresponds to Lisa's first persistent understanding) and illustrated with a graphic example. Although some of these persistent understandings may appear similar to difficulties that all students may experience when learning fractions, the goal here is not to argue for the commonality of the persistent understandings but to provide a comprehensive account of the persistent understandings that emerged in each of the cases. Note that the description presented here is a gloss in which an area model representation and example problem type is often used for illustrative purposes; however, these persistent understandings occurred in conjunction with multiple representational forms and across multiple problem types. Additional information, including the operational definition and number of instances, is available in Appendices C and D.

Lisa's six persistent understandings. Six persistent understandings were identified in Lisa's case, which involved the ways in which she understood, represented, interpreted, and manipulated representations of fractional quantity (see Figure 4).

Emily's six persistent understandings. Emily also experienced difficulties in representing, interpreting, and manipulating fractional quantities (see Figure 5). Although there was some similarity (that will be discussed later), the six persistent understandings identified were not the same as Lisa's.

Comprehensiveness of persistent understandings. For both Lisa and Emily, the persistent understandings were often associated with problem instances in which the student's answer or explanation was incorrect. An analysis of all incorrect answers was conducted to determine if the persistent understandings identified for each student provided a comprehensive view of the student's difficulties. Indeed, most incorrect answers given during the tutoring sessions were associated with at least one of the identified persistent understandings (82% for Lisa and 75% for Emily). The remaining incorrect answers were further analyzed. These incorrect answers generally involved errors in counting or calculation, were

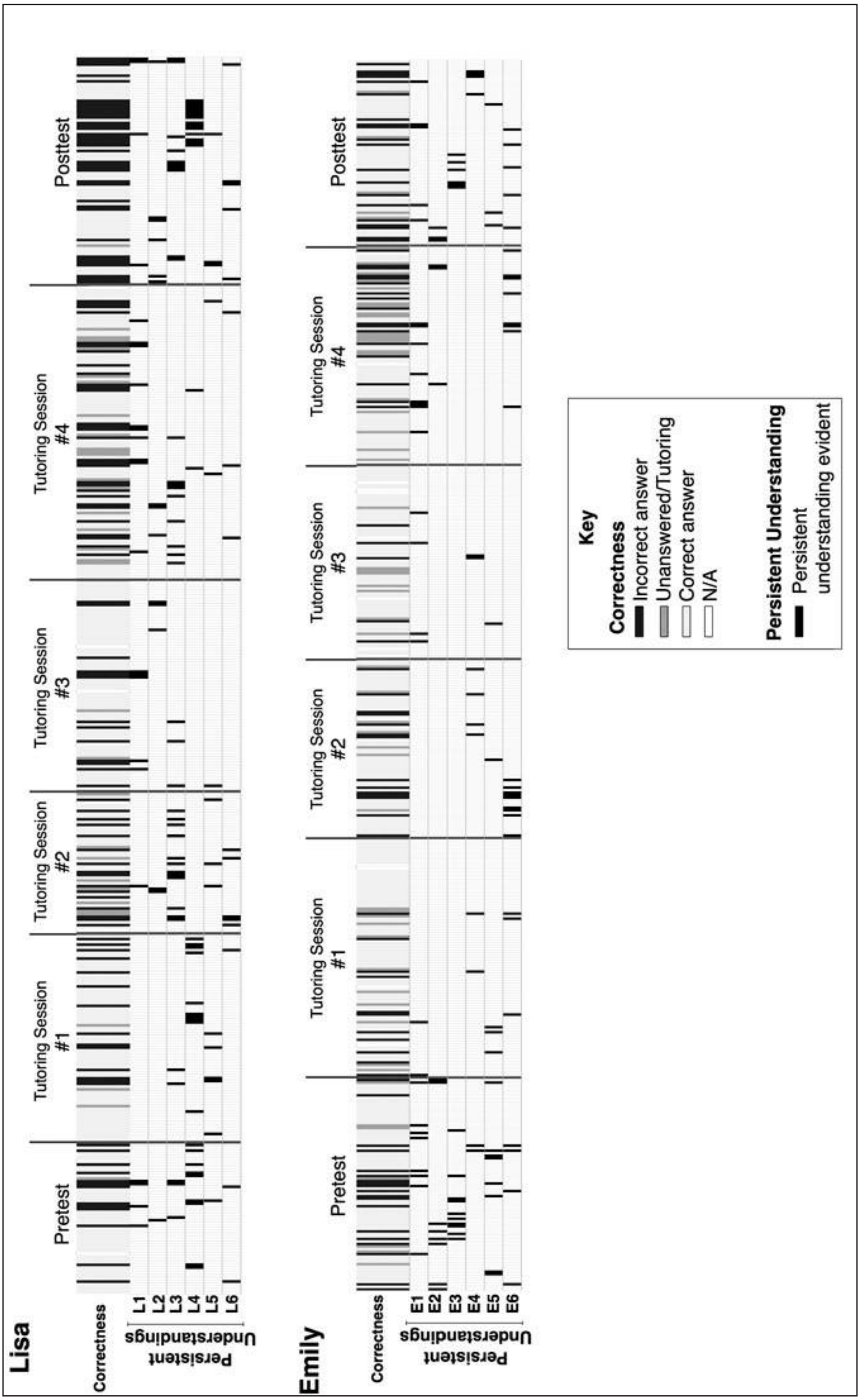
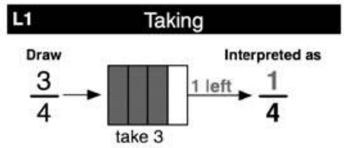
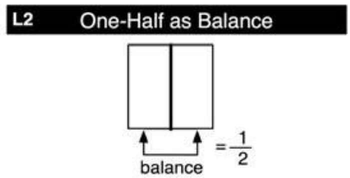


Figure 3. Problem-by-problem coding across all the data from the sessions with Lisa and Emily. Each vertical segment represents an individual problem posed and answered during the sessions.

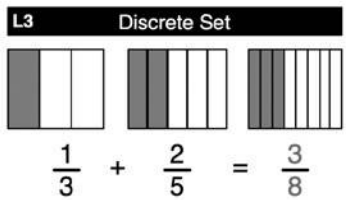
L1: When drawing and then interpreting fractions, she understood the shaded region to be the "amount taken," which resulted in her sometimes incorrectly attending to the fractional complement—the amount "left."



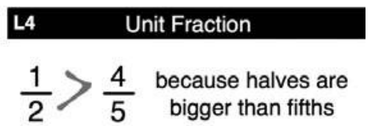
L2: When interpreting the fraction 1/2, she focused on the balance between parts, and when drawing 1/2, she focused on the partition line itself as the representation of 1/2.



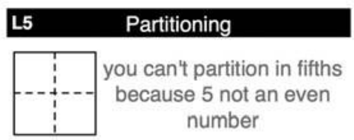
L3: When interpreting or manipulating fractional representations, she often over-applied a discrete set model to continuous models in which she ignored the size of the parts and treated all parts as if they were interchangeable.



L4: When comparing fractional amounts, she would judge the magnitude of the fraction based on the denominator or numerator alone. For example, she would incorrectly argue that 1/2 was larger than 4/5 because halves were larger than fifths.



L5: When drawing representations of fractions, she experienced difficulty with the mechanics involved in partitioning the shape, which resulted in an incorrect number of pieces or led her to incorrectly conclude that it was impossible to divide a shape into an odd number of pieces.



L6: When operating on representations of fractions, Lisa would modify the representations without attending to whether or not the modification would alter the fractional quantity involved (e.g., she might rewrite 1/2 + 1/4 as 1/2 + 1/2).

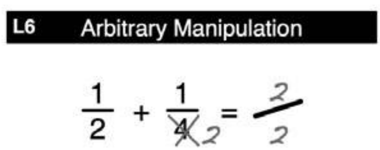
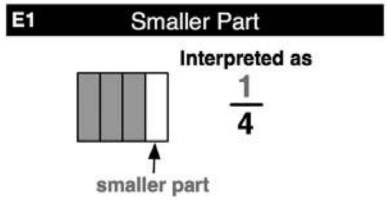
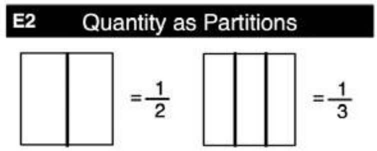


Figure 4. Description and graphic illustration of Lisa's six persistent understandings.

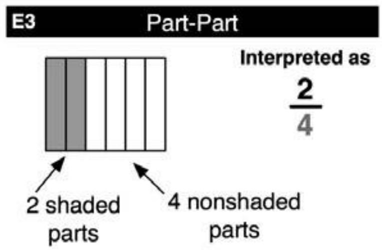
E1: When interpreting representations, she would sometimes determine the numerator value based on the nonshaded pieces, particularly if the nonshaded pieces comprised the smaller part.



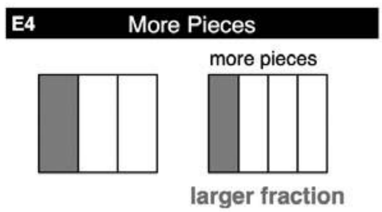
E2: When drawing fractions, she sometimes understood the partitioning of a shape to be the representation of the unit fraction (e.g., 1/2 is represented by partitioning a shape in 2 parts).



E3: When interpreting representations of fractions, she sometimes understood the fraction representation as comprised of two parts—shaded and nonshaded—rather than a part out of a whole.



E4: When comparing fractions, she would attend to the number of pieces rather than the size of the piece. For example, she would argue that 1/4 is larger than 1/3 because 1/4 has more total pieces.



E5: When interpreting fractions, she assumed that parts that perceptually "looked like" quarters (involved an angle close to 90 degrees) were equal to 1/4 and associated with the numeral 25.

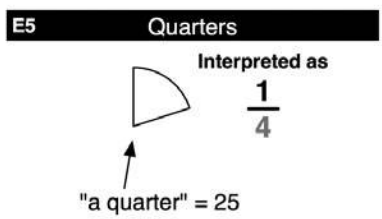


Figure 5. Description and graphic illustration of Emily's six persistent understandings.

E6: When constructing or interpreting representations, Emily treated those representations as if they were answers to specific questions rather than representations of quantity. This often caused her to interact with her own representations in a way that was not connected to their underlying meaning.

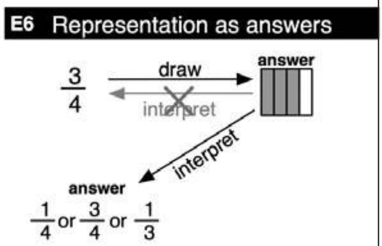


Figure 5. (*Continued*) Description and graphic illustration of Emily's six persistent understandings.

self-corrected by the student, or were lacking sufficient data to warrant classification. No predominant reason surfaced to explain any subset of these errors for either student. Therefore, the persistent understandings identified for each student provided a relatively comprehensive account for the difficulties that the students experienced during the tutoring sessions.

Detailed View of Persistent Understandings

To provide a detailed view of how the persistent understandings occurred in the context of the sessions, excerpts are provided for one persistent understanding for Lisa (L1: "taking") and one for Emily (E1: "smaller part"). These excerpts are presented with the purposes of (a) illustrating a prototypical instance of the persistent understanding and its relationship to the operational definition, (b) providing an example of how the persistent understanding was detrimental to the student's ability to make sense of more complex fraction concepts, and (c) illustrating that this persistent understanding was robust and not easily resolved in the tutoring intervention. This in-depth look at one persistent understanding for each student will serve as background for the comparison of Lisa's case and Emily's case in the subsequent section.

Lisa's "taking" understanding. Lisa's "taking" understanding involved an instability in her understanding of representations of fractional quantity (specifically the numerator value). Recall that Lisa understood representations of fractions in terms of an amount "taken" from the whole, which often caused Lisa to shift her focus to the amount she understood to be "left." Problems were coded as indicative of a "taking" understanding if Lisa (a) used the words *take*, *gone*, or *missing* (or any derivation) to refer to the numerator quantity; (b) used the word *left* to refer to the fractional complement; (c) gestured or referred to the fractional complement (represented by the nonshaded region of an area model or missing fraction pieces) as the focal fractional quantity; (d) used shading to represent the removal of pieces; or (e) interpreted a fractional quantity as the fractional complement (e.g., interpreting a drawing of 3/4 as 1/4). There were 24 instances of "taking" (5.5% of all problems), with most instances (91.7%) associated with an incorrect answer.

A prototypical example. Lisa's "taking" understand was evident in a problem in which she was asked to compare the fractions 7/12 and 1/2. Although drawn correctly (see Figure 6), Lisa interpreted her representation of 7/12 in terms of the amount "left" and incorrectly determined that 7/12 was smaller than 1/2 (i.e., 6/12).

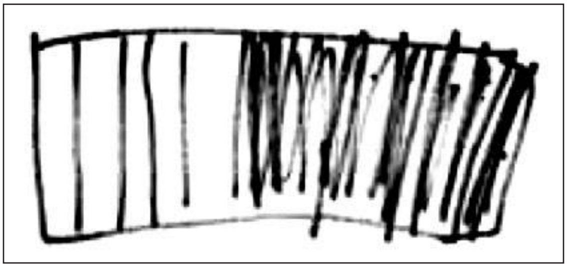


Figure 6. Lisa's drawing of 7/12 in which she partitioned a shape into 12 pieces and then colored seven of the pieces.

Lisa explained that for the fraction 7/12, "we would divide it into 12 pieces and take seven of those." As she attempted to determine if 7/12 was bigger or smaller than 1/2 she referred to her drawing as a cake. "I mean, ok, so let's say that this is the cake [gestures back and forth over entire shape; see Figure 6] and seven pieces are gone [makes sweeping motion over the shaded pieces]." This example was consistent with a "taking" understanding because she described her drawing of 7/12 as "take seven of those" and referred to the shaded region as "gone." As was often the case, her "taking" understanding ultimately led to an incorrect answer. During this particular problem, she had no difficulty determining the equivalent fraction for 1/2 would be 6/12; however, she continued to believe that "the half" was larger. This prototypical example highlights how shading was understood as the amount taken, which was ultimately problematic.

"Taking" as detrimental to learning. Lisa's "taking" understanding was consequential in that it hindered her ability to engage with more complex fraction concepts. In the previous example, her understanding of the shaded region as "gone" was detrimental to her ability to compare fractional quantities. In this next example, Lisa's "taking" understanding became problematic when attempting to interpret an equivalent fraction as she solved a fraction subtraction problem. In this example, she attended to the nonshaded pieces and referred to them as the pieces "left."

Lisa had begun solving the problem "7/8 – 3/4 =" by correctly drawing the area models for both 7/8 and 3/4. She then determined that to subtract the quantities, one would need to further partition the area model for 3/4. In the following excerpt, she correctly partitioned the area model—creating a representation of 6/8—but instead of attending to the six shaded pieces, she began attending to the two nonshaded pieces.

Lisa: If I were to like switch this [gestures with pen down the horizontal middle of the 3/4 area model; see Figure 7] like that, it would be two . . .

Tutor: OK, so let's cut it in half like that.

Lisa: [Draws partition line] There would be two left? Or two out of, two-sixths left. [Writes 2/6] No. I'm not sure [crosses out 2/6].

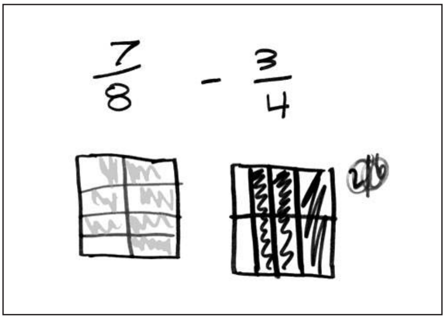


Figure 7. Lisa's drawn representation for the problem 7/8 – 3/4 = .

Lisa began attending to the two nonshaded pieces even before drawing the new partition line. After the partition line was drawn she restated her answer of "two" in fractional form, "two-sixths left." She herself had shaded the 3 pieces to represent 3/4, yet she interpreted the *nonshaded* pieces as indicative of the fractional quantity. Although she crossed out her answer of 2/6 at the end of the excerpt, her attention to the two nonshaded pieces persisted. She went on to interpret this representation as 2/8 (nonshaded/total), explaining that "there's eight . . . and then two [points to two nonshaded pieces of area model 6/8; see Figure 7] of which aren't shaded in." When I asked her what it meant if the pieces were not shaded in, she said, "Ok, then that wouldn't count" and quickly corrected her answer to 6/8 (shaded/total). After Lisa had correctly determined the value of the equivalent fraction, she had no difficulty completing the rest of the problem and determining that 7/8 – 6/8 would equal 1/8. Her main difficulty with this problem was therefore her interpretation of the area model for 6/8.

This example highlights how Lisa's "taking" understanding was invoked as she attempted to solve a problem. Despite her ability to represent both fractions with area models and repartition an area model to produce common-sized pieces, her understanding of the shaded region as "taken" and nonshaded as "left" made her reinterpretation of her area model problematic. As in this case, analysis across all "taking" instances suggested that Lisa's understanding of fractions, in terms of an amount taken, appeared to disrupt her ability to build a more complete understanding of fraction equivalence, fraction comparison, and fraction addition. I now turn to an exploration of why this kind of problematic understanding persisted through the tutoring sessions and did not get resolved.

Robustness of Lisa's "taking" understanding. Lisa's "taking" understanding recurred across sessions, despite the explicit instructional focus on representational conventions and attempts to address the "taking" understanding when it occurred. As previously discussed, developing an understanding of representations of fractions was one of the central goals for these tutoring sessions. The tutoring protocol introduced the primary fraction representations (i.e., area models and fraction pieces) with explicit attention to identifying, discussing, and establishing the conventions used for each representation. One such activity involved Lisa's generation and iterative refinement of directions for interpreting area models of fractions. At the end of this activity, Lisa's directions read: "Count out all portions of the fraction, then count all the shaded in parts, the shaded in part is written on top." The goal was to make explicit and to have the student articulate conventions for interpreting area models. Although in this activity she demonstrated evidence of a conventional understanding of the shaded region, this did not prevent her "taking" understanding from surfacing later.

In addition to questions that were explicitly focused on representational conventions, each time Lisa's "taking" understanding was problematic during the tutoring sessions, it was immediately addressed. For example, during the third tutoring session, when Lisa's tendency to attend to the nonshaded pieces became evident (and began leading to errors), I asked Lisa to imagine the area model was a picture of a cake. I asked her to think of colored frosting to help her conceptualize the shading as a quantity that was there rather than gone. These strategies provided local correction for her interpretation of fractions but did little to displace her tendency to understand the shading as taken. After one such correction, on the next problem she noted, "I keep imagining that this [pointing to the shaded pieces] is being taken away." This suggests that Lisa's understanding of the shaded region as "taken" was not resolved with a simple clarification and that it was robust and central to her understanding of fractions.

Summary. Lisa's "taking" understanding was persistent, detrimental to her attempts to learn more complex fraction concepts, and resistant to instructional attempts to address it. This persistent understanding, which caused her to attend to the fractional complement, was problematic across the tutoring sessions and presents some insight into why tutoring sessions—using standard representations like these—were ineffective for her.

Emily's "smaller part" understanding. Emily's "smaller part" understanding involved an instability in understanding and representation of fractional quantity. Recall that her interpretation of fractional representations often involved assuming that the numerator was represented by the smaller of the two parts (see Figure 5, E1). Problems were coded as indicative of a "smaller part" understanding if Emily interpreted a fraction representation based on the part composed of fewer pieces. In addition, because her attention to the smaller part created ambiguity with respect to how shading should be interpreted, problems were also coded as "smaller part" if she interpreted the fraction as the fractional complement (corresponding to the nonshaded or missing pieces), irrespective of whether it was the smaller of the two quantities. There were 28 instances of "smaller part" (6% of all problems), and most (79%) were associated with an incorrect answer or unanswered question.

A prototypical example of "smaller part." Emily's "smaller part" understanding was evident in the following example, which occurred at the beginning of the third tutoring session. She had just correctly drawn a picture of the fraction 1/5 in which she shaded one piece. However, when I asked her to interpret a drawn representation of the fraction 5/6, she attended to the nonshaded piece and interpreted the fraction as 1/6.

Tutor: So you've just drawn a picture of one-fifth. What if you were to look at something like [draws rectangle partitioned into sixths with 5/6 shaded; see Figure 8] that? What would you say that is a picture of?

Emily: That's um. [Pause: 15 seconds] That's one-sixth.

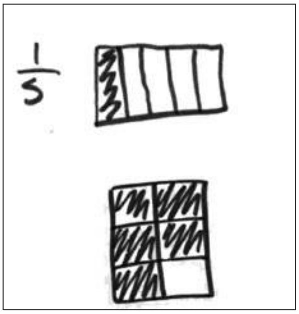


Figure 8. Emily's drawn representation of 1/5 and the tutor's drawn representation of 5/6.

Despite having drawn a correct area model for 1/5, she interpreted the area model that I drew with respect to the nonshaded piece, which comprised the smaller part. When I asked Emily to explain her answer, she said, "There's like, all, except one shaded." For Emily, the fractional quantity was not consistently determined by the shaded pieces. Instead, her understanding of the fractional quantity was dependent upon the part with fewer pieces. Her correct drawing of 1/5 and her incorrect interpretation of 5/6 as 1/6 suggests that Emily's understanding allowed for ambiguity in how area models were interpreted. Although flexibility in use of representations is desirable (e.g., one must redefine the unit when using area models to represent fraction multiplication), the ambiguity characteristic of "smaller part" led to the fraction value being obscured. This episode was considered consistent with a "smaller part" understanding because it involved incorrectly interpreting the fractional representation in terms of the part with fewer pieces. I will return to this example when I discuss the robustness of this understanding.

"Smaller part" as detrimental to learning. Emily's "smaller part" understanding hindered her ability to engage with more complex fraction concepts. In the following example, a simple fraction comparison was problematic for Emily because of her tendency to focus on the smaller part of her own drawn representations. In this problem Emily was asked to compare the fractions 2/8 and 5/8—a relatively easy comparison, given that the denominators were the same in both fractions. To solve the problem, she correctly drew a representation of both fractions, using shading to represent the numerators. Once she drew these two representations, she began interpreting 5/8 as 3/8 and attending to the nonshaded pieces as indicated by her pointing to each nonshaded piece in her drawings.

Tutor: What if we had the problem, two-eighths and five-eighths, which one would be bigger there?

Emily: [Writes 2/8 and 5/8. Draws 8 rectangles, shades in two. Draws 8 rectangles, shades in 5, see Figure 9] So, this is . . . [points to each of the 5 shaded pieces in drawing of 5/8. Points to each of the 3 nonshaded pieces of drawing of 5/8] Um. [Writes 3/8. Points to each of the 6 nonshaded pieces in the drawing of 2/8.] I don't know [scribbles out 3/8]. I don't know.

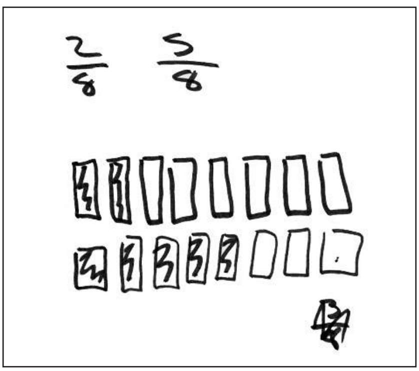


Figure 9. Emily's written work for the comparison of 2/8 and 5/8.

Although Emily correctly represented both fractions, these drawings did not support her comparison of the fractional amounts. Once she had completed the drawings, Emily shifted from attending to the 5 shaded pieces to attending to the 3 nonshaded pieces for 5/8 and interpreted her drawn representation as 3/8—consistent with her "smaller part" understanding. It was only after pointing to each of the nonshaded pieces in the fraction representation for 2/8 that she rejected her interpretation of 5/8 as 3/8. It is possible that counting the nonshaded pieces, which composed the larger part of 2/8, helped her recognize that counting the nonshaded pieces was inappropriate. This example demonstrates how Emily's tendency to focus on the smaller part led to ambiguity with respect to how to interpret her own representations and caused her to be unable to answer this relatively simple comparison problem. As in this example, Emily's "smaller part" understanding of fractions appeared to disrupt her ability to build a more complete understanding of fraction comparison, fraction equivalence, and fraction operations. I now turn to an exploration of why this kind of problematic understanding persisted through the tutoring sessions and did not get resolved.

Robustness of Emily's "smaller part" understanding. Emily's "smaller part" understanding was sufficiently well entrenched that it resisted explicit attempts to address it through standard instruction. Just like Lisa, in the second tutoring session Emily had iteratively refined a list of rules for interpreting area models. Her journal entry read: "rules: put the shaded # of pieces on the top. Put the number of all pieces including shaded pieces on the bottom." When we reviewed her journal at the start of the third tutoring session, she explained her journal entry by reading the rules and then she gave an example of the application of her rule to the area model representation she had drawn for 1/5 (see Figure 8). Despite her apparent focus on the shaded region to determine the fractional quantity, as seen in the previous prototypical example, when I asked her to interpret an area model of 5/6, she switched to attending to the smaller part (the nonshaded pieces). After incorrectly determining that the answer was 1/6, she asked if she should be applying the rules from her journal entry.

Emily: That's um. [Pause: 15 seconds] That's one-sixth.

Tutor: OK, so this one [pointing to area model of 5/6; see Figure 8] is going to be one-sixth?

Emily: Wait, are we doing that? [Points to rules in journal]

Tutor: OK, if we were following your rules, would we end up with one-sixth?

Emily: No.

Tutor: What would we get if we were following your rules?

Emily: Uh, we would get. Uh, five [pause], five-sixths.

Tutor: So, how do we know whether this is a picture of five-sixths or this is a picture of one-sixth?

Emily: I don't know.

Although she was able to determine that the rules would specify the fraction value was 5/6, she was unsure of which was the correct answer. Emily's own method for interpreting fractional representations was clearly at odds with the written rules she had generated in the earlier session. Furthermore, these rules were insufficient to help her correct her answer; instead, she was unsure of which answer was correct.

As the discussion continued, she eventually shifted to attending to the shaded region and correctly interpreted the area model for 5/6. She explained, "Well, I know that this is five-sixths, because it's not one-sixth because, there's like, six-sixths is a whole, and that one would be shaded in [gestures as if shading the last piece]. It's just five-sixths, because only one isn't shaded in. So it's five-sixths." She used her understanding of the whole to make sense of the part of the representation she should attend to. Although she had constructed a relatively articulate explanation, leveraging her understanding of a whole, this did not create a lasting change in her interpretation of fractional representations. Despite reviewing her journal at the start of the fourth tutoring session and immediately before the posttest, Emily's "smaller part" understanding continued to persist and led to errors. This suggests that Emily's understanding of the fraction as defined by the smaller part was robust and resistant to instruction.

Summary. Emily's "smaller part" understanding resulted in instability in the way in which she conceptualized fractional quantity. This understanding was persistent in that it occurred across sessions, detrimental in that it caused difficulty for more complex fraction problems, and robust in that it persisted, despite instructional attempts to correct it. This persistent understanding resulted in ambiguity with standard representational forms and provides some insight into why she did not benefit from the tutoring protocol.

Comparison of Cases

Both Lisa and Emily demonstrated a unique collection of persistent understandings that provide a relatively comprehensive view of the difficulties they experienced in the tutoring sessions when considered together. There were some striking similarities between Lisa's case and Emily's case. In this section, I consider the similarity between cases, first discussing the similarities of Lisa's "taking" and Emily's "smaller part" understandings, both of which involved a focus on the fractional complement. Then I briefly present the second persistent understanding for both Lisa and Emily, which involved understanding the fraction 1/2 as "halving" rather than as the quantity 1/2. For each of these, in addition to providing a comparison, I will draw upon the data from the typically achieving fifth-grade students to highlight how these persistent understandings were not similarly problematic for them.

Similarities of "taking" and "smaller part" understandings. Lisa's "taking" and Emily's "smaller part" were distinct persistent understandings with unique operational definitions. However, in both cases these persistent understandings resulted in similar kinds of errors involving the student inappropriately focusing on the fractional complement (i.e., the nonshaded pieces). This tendency to focus on the fractional complement was not limited to area model representations, suggesting that this was not simply a misunderstanding of the shading convention used with area models. This persistent understanding also occurred when the students used foam fraction piece manipulatives to create fractional quantities. When using the foam fraction pieces, the numerator was represented with physical objects, which one might expect would support an understanding of the number of pieces as the focal fractional quantity. However, when interpreting fraction pieces, both Lisa and Emily sometimes focused on the fractional complement: the missing pieces. For example, when Lisa was asked to interpret seven 1/10 pieces, instead of interpreting the fraction as 7/10, she cupped her hand around the empty space and determined that "a third" would fit in the space.

Tutor: What fraction name would we give this whole amount that is here? [points to seven 1/10 pieces; see Figure 10a]

Lisa: A third?

Tutor: And are you doing that just based on what it looks like, sort of?

Lisa: Yeah. A fourth? No. 'Cause, I mean how much is left? [Gestures with hand around empty space; see Figure 10b]

Tutor: Yeah, we are missing—

Lisa: [Picks up 1/3 fraction piece, knocks pen off table] Whoa! Sorry.

Tutor: —some amount over here.

Lisa: Is a third? [Puts 1/3 piece in empty space; see Figure 10c] Yeah, a third is missing.

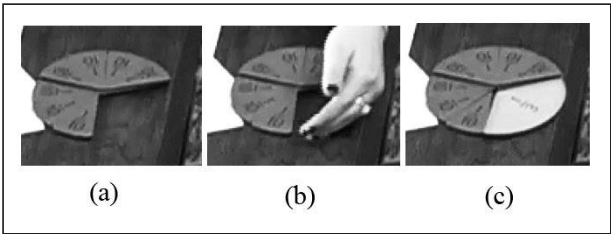


Figure 10. Video screenshots in which Lisa interpreted seven 1/10 pieces in terms of the missing space by filling it with a 1/3 piece.

To interpret the seven 1/10 pieces, Lisa focused on the empty space rather than the fraction pieces. As with the nonshaded pieces of area models, Lisa used the term *left* to refer to the empty space (i.e., fractional complement).

Emily also attended to the empty space when interpreting fraction pieces. In Emily's first interaction with the fraction pieces, I asked her to explain how the fraction pieces were labeled. Although she assembled ten 1/10 pieces and correctly identified the amount as "a whole," she then removed a piece and incorrectly interpreted the remaining 9/10 in terms of the missing piece.

Emily: So there are 10 of the same size pieces, and those all make up a whole [see Figure 11a].

Tutor: OK.

Emily: So. Yeah. So, this [takes piece out and holds it in her fist; see Figure 11b] if you take this out [moves fist away from pieces, continues to look at nine 1/10 pieces on table; see Figure 11c], then that is one-tenth.

Tutor: OK.

Emily: And yeah.

Tutor: OK, so if we take this out then this is one-tenth? [Points to nine 1/10 pieces]

Emily: This [pointing to empty space; see Figure 11d] is one-tenth.

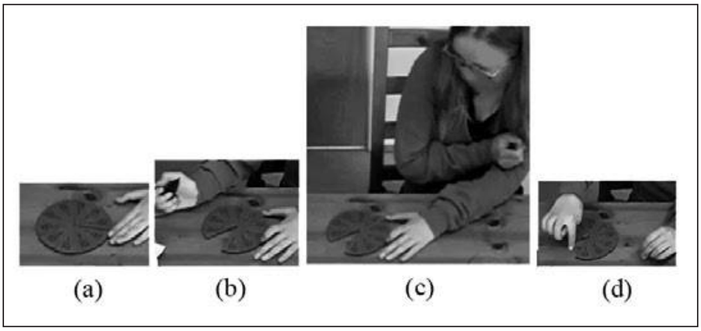


Figure 11. Video screenshots in which Emily interpreted nine 1/10 pieces in terms of the missing piece.

Although she had correctly identified the whole, when she removed one of the 1/10 pieces she named the fractional quantity in terms of the piece that was missing (i.e., the empty space). This understanding persisted beyond the initial work with the fraction pieces; when asked to interpret 7/10 in a later session, Emily attended to and filled in the empty space, just as Lisa had done.

Lisa's "taking" understanding and Emily's "smaller part" understanding led to similar kinds of errors across multiple representational forms. Both students interpreted fraction representations as the fractional complement rather than the fractional quantity. Therefore, these two persistent understandings, although distinct, can be thought of as belonging to a high-order category: focus on the fractional complement.

Comparison to fifth-grade students. The difficulties experienced by Lisa and Emily can be contrasted with the performance of the fifth-grade students. The persistent understanding was either absent in the fifth-grade students or, when it occurred, was quickly and completely resolved. None of the fifth-grade students experienced difficulties attending to the focal fractional quantity with area models, and when asked to interpret seven 1/10 pieces, they all correctly identified the representation as 7/10. Unlike Lisa and Emily, the fifth-grade students attended to the focal fractional quantity and were able to use these representations to engage with more complex fraction concepts.

Halving understanding. The second persistent understanding that was similar for both Lisa and Emily was "halving." Although there were subtle differences between the operational definitions—Lisa's understanding involved seeing "one-half as balance" and Emily's understanding involved seeing "quantity as partitions"—both can be thought of as a "halving" understanding. A "halving" understanding involved representing and understanding the fraction 1/2 not as a quantity but as the act of partitioning or the balance between parts. This persistent understanding can be best illustrated by Lisa's and Emily's answers to the posttest questions in which they were asked to draw or write 1/2. Both drew several different shapes (see Figure 12) that were partitioned in half and omitted the standard shading of one of the two pieces.

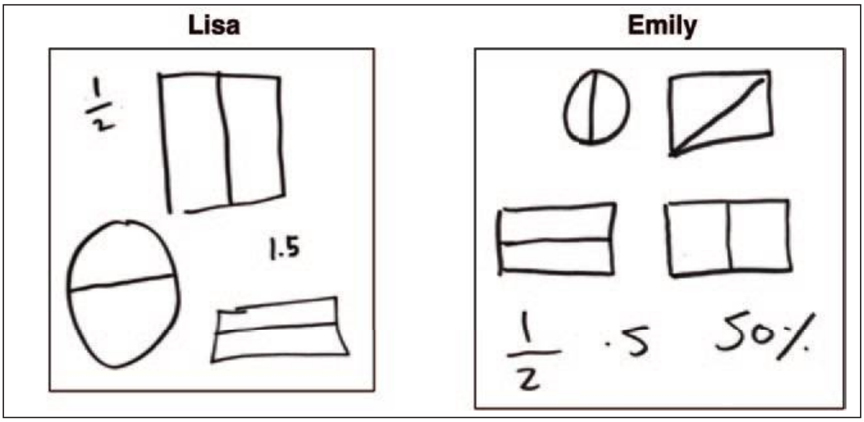


Figure 12. Written work of Lisa's and Emily's representations of 1/2 at the time of the posttest.

When asked to explain why the nonshaded halved shape represented 1/2, Emily explained, "Well, this is one [traces with pen around the circle]. And it's cut in half, and there is two [points with pen back and forth between the two pieces]." This excerpt highlights that her understanding of 1/2 involved the partitioning of the shape and the balance between the pieces rather than the fractional quantity—one out of the two total pieces. Similarly, Lisa used gestures when explaining why a nonshaded shape split in two was equivalent to 1/2; she said, "I just automatically saw something that was [chopping gesture with hand]." Her chopping gesture suggests that she was thinking of the action of "halving" as opposed to the quantity 1/2. Both Lisa and Emily drew 1/2 without shading and understood 1/2 as the process of splitting something in two pieces, embodied by the partition line.

The persistent "halving" understanding was problematic for both students in their representation of 1/2 and in the context of more complex fraction concepts. This halving understanding surfaced 15 times for Lisa (3% of all problems) and 15 times for Emily (3% of all problems) over the course of the tutoring sessions and was often associated with an incorrect answer (67% of cases for Lisa and 87% of cases for Emily).

Comparison of fifth-grade students. Lisa and Emily's persistent "halving" understanding can be contrasted with the understanding exhibited by the fifth-grade students. At the time of the posttest the fifth-grade students' representations of 1/2 all highlighted the fractional quantity (see Figure 13). Almost all the students used shading, and when shading was omitted, the quantity was labeled, clearly signifying a focus on one of the two parts. The explanations given by the fifth-grade students clearly focused on the amount 1/2 rather than the partitioning action. Unlike Lisa and Emily, the fifth-grade students produced representations that indicate the quantity 1/2 is central to their understanding of the fraction 1/2.

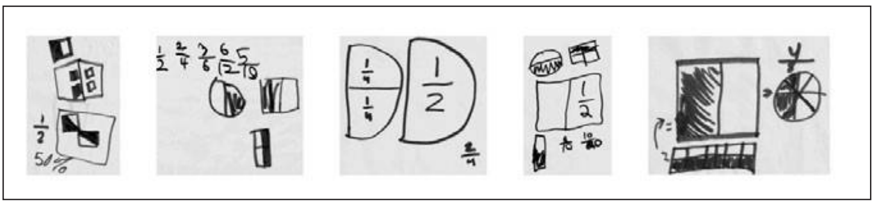


Figure 13. Fifth-grade comparison student's representations of 1/2 at the time of the posttest—all of which included shading or labeling of the focal fractional quantity.

Discussion

Neither Lisa nor Emily benefited from a tutoring protocol that was effective for typically achieving fifth-grade students. Underlying the difficulties the students experienced was a collection of persistent understandings. In this section, I explore two different explanatory frames that can be brought to bear on these findings. First, I discuss the findings with respect to prior research on MLDs. In doing so, I am necessarily engaging with how MLDs are typically framed—from a deficit-model perspective. I address both the consistency of these findings with prior work and the limitations of this explanatory frame. Second, I reframe the presumed cognitive deficit as difference within a Vygotskian explanatory frame. From this perspective, the student's difficulties can be understood as resulting from the inaccessibility of mediational tools. Third, building from this reconceptualization of MLDs, I consider the implications for the identification and remediation of MLDs. Finally, I discuss the limitations of the present study and the need for future research.

Relationship to Prior Research on MLDs

The persistent understandings identified for Lisa and Emily are consistent with, and elaborate upon, the kinds of conceptual and representational issues identified for students with MLDs in the context of fractions (Hecht & Vagi, 2010; Mazzocco & Devlin, 2008). Furthermore, these findings are consistent with the predominant hypothesis that MLDs originate from a domain-specific number processing deficit that involves an inability to mentally represent and manipulate numbers (Butterworth & Reigosa-Crespo, 2007). The two persistent understandings—"fractional complement" and "halving"—can be framed as originating from the student's inability to process or manipulate fractional quantities. "Fractional complement" may reflect an instability in understanding a fraction as a fixed quantity because the fraction itself could transform through the act of representing it (i.e., the student's drawing of 3/4 could subsequently be interpreted as 1/4). Similarly, "halving" involved a focus on the act of splitting a shape rather than on the quantity 1/2. This is particularly notable given that even children as young as four years old have been shown to have an intuitive understanding of the fraction 1/2 (Hunting, Davis, & Bigelow, 1991; Hunting & Davis, 1991). Therefore, the persistent understandings can be thought of as reflecting (and possibly caused by) the student's inability to mentally represent and manipulate numbers.

Although this deficit model provides a reasonable explanatory frame for the data, the derived implications of this frame do not provide sufficient traction on methodological issues facing the field, specifically regarding identification and remediation. Conceptualizing MLDs as a cognitive deficit does not address how identification methods might differentiate the impaired processing of a student with an MLD from a student who has low mathematics achievement due to other causes. Nor does it offer avenues to consider with remediation, because the deficit perspective implies an intractability of these cognitively based deficits. In sum, although these findings can be subsumed within prior research on MLDs, the implications afforded by this explanatory deficit frame are limited.

Reconceptualizing MLDs as Difference

The hypothesized cognitive deficit can be recast from a Vygotskian perspective. The proposed inability to mentally represent and manipulate quantities would create an incompatibility between the student and mediational tools intended to represent quantity. Because of this, a student may have qualitative differences in his or her understanding of these mediational tools and may understand a representation intended to represent quantity as something *other than quantity*. Quantity is most clearly absent in the case of halving. Lisa and Emily represented the fraction 1/2 by drawing a shape and splitting it in two pieces. Rather than representing the quantity 1/2 in the drawing (i.e., one out of two total pieces), they represented by the splitting action. Both students identified the splitting itself as the representation of 1/2. Similarly, "fractional complement" can be thought of as using shading to represent something other than quantity. Lisa understood the shading of an area model as an amount "taken"—reflecting a removal *action* rather than a focal *quantity*. Although Emily did not represent a "taking" action in the same way as Lisa, her representative act produced something that was not the fractional quantity she drew. Therefore, she too appeared to represent something other than fractional quantity. It is not merely that Lisa and Emily could not mentally represent and manipulate quantity, but when given a representation of quantity, they understood it in unconventional and atypical ways. From a Vygotskian perspective, this atypicality is expected and reflective of the incompatibilities between the student's cognitive processing and the mediational tools intended to support an understanding of fractional quantity.

Ineffectiveness and Inaccessibility of the Tutoring Protocol

Atypical ways of understanding representations of fractional quantity may have contributed to the ineffectiveness of the tutoring protocol for Lisa and Emily. Their persistent understandings can be thought of as incompatible with the assumptions upon which the tutoring intervention was based, namely that mathematical representations can be used to represent and manipulate quantities. The tutoring intervention was designed for students to engage with the meaning of these representations by exploring the conventions governing their use. After establishing conventions, new fraction concepts were introduced with heavy reliance upon representations. For the fifth-grade students, these representations did provide students with the needed supports for them to benefit from this kind of intervention. For Lisa and Emily, the representations were ineffective. Understanding MLDs through the Vygotskian explanatory frame suggests that MLDs may be most appropriately conceptualized in terms of inaccessibility of mediational tools, which has direct implications for both identification and remediation approaches.

Identification. The commonality between some of Lisa's and Emily's persistent understandings suggests that the inaccessibility may manifest for students with MLDs in some consistent ways. In addition, given that each of the persistent understandings was evident before the tutoring sessions (on the pretest), it may be possible to identify these kinds of atypicalities in students with a pencil-and-paper assessment. Group-based screening measures that include questions to elicit particular persistent understandings (e.g., Draw a picture of 1/2.), particularly those common to both students, could be developed. This kind of measure would allow for an alternative approach to screen students for MLDs because it involves looking for specific (potential) markers of MLDs rather than simply operationally defining MLDs as low mathematics achievement. Although the detailed case studies reported in this article cannot capture the range of persistent understandings that could occur for students with MLDs, they do provide a starting point for considering alternative screening assessments. It remains an open empirical question what other persistent understandings should be assessed and targeted in this kind of screening assessment.

Remediation. The persistent understandings identified for Lisa and Emily have implications for how one might design remediation for students with MLDs. The failure of standard mathematics instruction, both in the tutoring intervention and in their years of schooling, suggests that students with MLDs do not simply need more instruction but may in fact need *different* kinds of instruction. If standard instructional representations are inaccessible for students with MLDs, it follows that remediation must provide mediational tools that are accessible. Remediation will be just that, a *re*-mediation (Gutiérrez, Morales, & Martinez, 2009), by providing alternative mediational tools that account for and build upon a student's persistent understandings. For example, a re-mediation might build upon Lisa's "taking" understanding and make use of space, movement, and physical weight to help the student maintain focus on the focal fractional quantity. Rather than the standard representation of an area model in which a shape is drawn, partitioned, and shaded to indicate the fractional quantity, the student could begin with a physical whole (e.g., index card), then partition that whole into parts and physically "take" the focal fractional quantity and move it to a measuring space. How might the movement, space, and weight help the student focus on the fractional quantity taken rather than the amount left? Permitting the student to physically move (i.e., take) the desired quantity would allow her persistent understanding of "taking" to be a productive component of her construction of the fraction representation. Designating a space where the quantity is moved to and then measured may enable the student to reorient and focus on the taken quantity as the focal amount. This kind of re-mediation would build upon a student's persistent understandings and repurpose those understandings through alternative mediational tools. Although the Vygotskian framing suggests exciting new directions to pursue in screening and remediation, future research is needed to evaluate the feasibility and effectiveness of these approaches, which are based upon students' persistent understandings.

Limitations and Implications for Future Research

Several limitations of the present study should be acknowledged, and each of these limitations highlights productive avenues to consider in future research. First, the tutoring protocol was evaluated with fifth-grade students before classifying the students with MLDs. The selection of younger students to evaluate the effectiveness of the tutoring protocol, although necessary to avoid ceiling effects, raises questions about the comparison of younger and older students. It is possible that the fifth-grade students who had more recent exposure to introductory fractions content may have been better positioned to benefit from the tutoring intervention than the two adult students with MLDs. It is worth noting, however, that two other students with potential MLDs (one college and one middle school student) also benefited from the tutoring protocol, which suggests that the tutoring protocol was effective not just for younger students. Future research should consider whether the kinds of persistent understandings noted in Lisa and Emily occur with students with MLDs who are younger, particularly because early intervention for MLDs is essential (Gersten et al., 2005).

Second, the persistent understandings that emerged from this analysis were tied to the specific intervention used in this study. The video record of the intervention was the analytic window through which the student's mathematical learning was viewed. Consequently, the intervention itself dramatically shaped the nature of the persistent understandings that emerged. Because the intervention was focused on a part–whole understanding of area model representations, it remains an open question whether these same persistent understandings would occur in the context of a number line model or whether a different intervention would give rise to a different set of persistent understandings for these students. Future research should consider how other representational modalities might address or elicit other kinds of persistent understandings.

Third, this kind of analysis of student thinking invites alternative interpretations. Although a particular instance of a persistent understanding may be explained with an alternative interpretation (e.g., a student was using a ratio model of fractions rather than a part–whole model), these persistent understandings were derived largely from what the student said or explained about her own thinking. In addition to acknowledging the student's own authority, the triangulation of the student's solution process, explanation, gestures, and written work along with the recurrence of the persistent understandings was taken as sufficient warrant to support this interpretation of the data. The analytic machinery of persistent understandings was aligned with the goals of this study and provided some insight into the character of the MLDs for these students. However, the delineation and focus on persistent understandings is just one analytic frame through which to view MLDs. Because this is one of the first studies attempting to recast MLDs from a non-deficit perspective, more research is clearly needed to support a generative dialogue about the nature of MLDs. For example, research that focused on classifying the character of the explanation in terms of process or object (Sfard & Linchevski, 1994) may yield different insights into the nature of the difficulties experienced by the students. Indeed, several of the persistent understandings discussed in this article (e.g., "taking" and "halving") seem processual in nature, but future research is needed to determine if this kind of analysis would challenge or extend the findings presented here.

Conclusion

This study provided an in-depth analysis of two students with MLDs. Individual diagnostic analyses were used to consider how MLDs may manifest across students in similar ways. Two students with MLDs displayed similar persistent understandings that reflected an underlying difficulty conceptualizing fractional quantity. These differences in understanding of fractional quantity, supported by the evidence of the ineffectiveness of the tutoring protocol for Lisa and Emily, suggest that representations of quantity may be inaccessible for students with MLDs. This provides an alternative vantage point for considering the qualitative differences that characterize MLDs. This work represents a first step toward the reconceptualization of MLDs in terms of difference and toward more accurate identification and alternative remediation approaches.

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APPENDIX A

Pretest and Posttest Questions With Administration Stop Rules and Scoring

Questions	Stop rule	Scoring
1. Draw/write the fraction ___. Can you think of another way to draw or write it? *Fraction values: 1/2, 3/4, 2/5, 1 3/8, 5/4*	If student gets more than 2 wrong.	Total possible: 15 points For each fraction value 1 point for correctly producing: • a drawing of the fraction • an equivalent fraction (2/4, 7/14) • a drawing of an equivalent fraction
2. Can you circle all the pictures that you think are the same as ____? (based on Ni, 2001) *Fraction values: 1/2, 1/3, 4/5, 5/3*	If the number of incorrect answers is greater than or equal to the number of correct answers.	Total possible: 15 points Total negative points: 12 1 point for every correct area model or discrete set answer circled. –1 for every incorrect answer circled.
3. Paper folding and cutting comparison task (see Kamii & Clark, 1995)	(no stop rule)	Total possible: 5 points • 1 point for determining the equivalence of differently shaped halves • 1 point for naming the 1/2 piece • 1 point for determining the equivalence of differently shaped quarters • 1 point for naming the 1/4 piece • 1 point for determining how much of the whole both 1/4 pieces are together.

4. Area model fraction comparison problems (based on Armstrong & Larson, 1995) *Fraction values: 1/3 and 1/3; 3/5 and 2/5; 2/3 and 2/4; unequally sized 3/4 and 3/4*	(no stop rule)	Total possible: 8 points 1 point for each correct answer. 1 point for each correct justification.
5. Fraction comparisons: "Which is more ___ or ____?" *Fraction values: 1/6 or 1/8; 2/7 or 2/5; 2/8 or 5/8; 3/6 or 5/10; 3/2 or 7/9; 4/5 or 2/3; 2/6 or 1/2; 2/5 or 3/10; 3/7 or 2/3; 5/8 or 2/3*	After 2 incorrect answers.	Total possible: 10 points 1 point for each correct answer and explanation.
6. Can you come up with a fraction equal to ___? Can you come up with another fraction equal to ___? How many fractions are there equal to ___? *Fraction values: 1/2, 1/3, 2/5, 8/12*	If student is not able to produce an equivalent fraction.	Total possible: 8 points 1 point for each equivalent fraction (up to 2) they can generate for the given fraction.
7. How would you solve the problem:_____ *Fraction operation problems: 1/3 + 1/3 =; 3/4 – 1/4 = ; 3/5 + 4/5 =; 1/2 + 1/4 =; 3/4 – 1/8 =; 1/3 + 1/2 =; 2/5 + 2/3=*	If the student gets 2 or more answers incorrect.	Total possible: 7 points 1 point for each correct answer and explanation.

APPENDIX B

Tutoring Session Protocol Questions
(For full protocol including follow-up questions and prompts, see Lewis, 2011)

Tutoring Session 1

Challenges

- How would you explain to someone else how they labeled each of these fraction pieces?
- I took the 1/8 pieces out of this set. Could you make a replacement 1/8 piece? How many thirds does it take to make a whole?
- I copied some fraction pieces, but I had them upside down. Can you help me figure out what name each should be?
- How can you show what 3/4 would look like with fraction pieces? How would you explain how to make 3/4 to someone else? If someone said this (two 1/10 and one 1/8 pieces) was 3/10, would you agree or disagree?
- This rectangular set of fraction pieces is missing the one whole piece. Figure out how to make a replacement one whole piece.
- For each set of cards put the fractions in order from least to greatest. Set 1: 1/10, 1/3, 1/100, Set 2: 2/5, 2/7, 2/16; Set 3: 3/7, 6/7, 18/7
- How many ways can you find to make 1/2 using other fractions pieces? Come up with your own way to record all the ways you find that work.
- Without using the fraction pieces, can you make each of the fractions close to, but NOT equal to 1/2?
- Put the fractions on these cards in order from least to greatest. Card values, 1 2/5, 2/3, 1/1000, 7/7, 3/2, 0, 9/10, 8/12, 1
- **Game:** Bingo game in which students are asked to create equivalent fraction relationships using foam fraction pieces (e.g., make 1/2 using only 1/8 pieces)

Journal

- Write down something that made sense. How would you explain it to yourself if you forgot?

Tutoring Session 2

Review

- Journal Review: Does this still make sense? Can you explain/give me an example?
- Review Question: Can you come up with a fraction and show me how you would make it with the fraction pieces? (If the student does not volunteer a non-unit fraction, ask the student to construct one, e.g., 2/5, 7/8.)

Challenges

- Without writing down any numbers can, you draw a picture of 3/4 so that other people would know that it's a picture of 3/4?
- Help me guess 5 fractions on the back of the cards. (Student draws card with a fraction value, writes it down, draws a picture, and then hides fraction value so I can "guess." I follow student-generated rules to "guess" the fraction the student drew.) (Note: I intentionally misinterpret rules to highlight ambiguity and help the student further clarify his or her understanding of fraction interpretation).
- Imagine each of these pieces of paper is a cake. Four people want to share the cake evenly. How many different ways can you split the cake into 4 pieces? (student can cut/fold paper)
- If you are given 1 piece of each of the cakes, are they the same amount of cake or different amounts? Show how you would convince someone of your answer.
- If these were both cakes, how can you share each cake among 3 people? Cut out 1 piece of cake. Are they the same amounts or different amounts?
- **Game:** Board game in which students must determine which area model is larger (see Armstrong & Larson, 1995).

Journal

- Write down something that made sense. How would you explain it to yourself if you forgot?

Tutoring Session 3

Review:

- Journal Review: Does this still make sense? Can you explain/give me an example?

- Review Question: Last time we were talking about using pictures to show fractions. How would you draw a picture of 3/5? (draw 5/6) What fraction name would you give this drawing?

Challenges

- Someone has started cutting up these cakes to share, but they haven't finished the job. Finish cutting up the cakes so that they can be shared evenly. How many people could evenly share? How much would each person get?
- Here are some cakes. Some have chocolate frosting and some have vanilla frosting. Your challenge is to figure out how much has chocolate frosting.
- Using a whole sheet of paper, draw 2/3 of a cake. Figure out how you can cut the cake again so you still have even pieces. What has changed and what has stayed the same?
- To complete this challenge you must come up with five equivalent fraction pairs. Draw a card and figure out the fraction of cake shaded. Draw a transparency and use it to help you cut the cake into smaller pieces. Decide how you want to record your progress.
- Come up with 5 different fractions equal to 1/2. Record your answers. What patterns do you notice? (Extend this problem to 1/3 and 3/4.)
- **Game:** Students are given a fraction and using pictures (or other method) must come up with an equivalent fraction. The computer determines if the student got the answer correct or incorrect. (If incorrect the student attempts to figure out her error.) Level 2 of this game involves needing to come up with a fraction with a given denominator.

Journal

- Write down something that made sense. How would you explain it to yourself if you forgot?

Tutoring Session 4

Review:

- Journal Review: Does this still make sense? Can you explain/give me an example?
- Review Question: Last time we were talking about equivalent fractions. What do you remember about equivalent fractions? What fraction is this (area model of 1/3)? Can we divide it into sixths? How? What fraction is this (area model of 1/2)? Can we divide it into 10ths? How?

Challenges

- How much do we have here? (1/2 and 1/3 pieces)
  Can you write it as an addition problem?

  If student solves algorithmically ask the student to represent the problem with fraction pieces and area models. If the student doesn't know how to start the problem, go onto the subproblems below.

  #### Subproblems

  1/3 + 1/3 = (problem with the denominators constant)
  1/2 + 1/4 = (problem where one denominator multiplies into other)
  1/6 + 1/3 = (problem where one denominator multiples into other)

  Questions:

  How would you solve this problem using fraction pieces?
  How would you solve this problem using pictures?

- If you look back at these problems, does that help you solve the original problem 1/2 + 1/3? What other ways can we write 1/2? What other ways can we write 1/3?
- Solve: 2/5 + 3/10 = using fraction pieces and area models. Subproblems: 2/5 + 3/5 = and 2/10 + 3/10 =
- Solve: 7/8 – 3/4 = using fraction pieces and area models.
- 2/3 + 1/4 =
- **Game:** Board game where students solve fraction addition/subtraction problems and reduce the fraction to figure out the next move.

Journal

- Write down something that made sense. How would you explain it to yourself if you forgot?

APPENDIX C

Lisa's Persistent Understandings—Operational Definition and Number of Problem Instances

Operational definition	**Number of instances (% of all problems)**
L1 – **Taking:** Problems were coded as indicative of a "taking" understanding if Lisa (1) used the words *take*, *gone*, or *missing* (or any derivation) to refer to the numerator quantity, (2) used the word *left* to refer to the fractional complement, (3) was gesturing or referring to the fractional complement (represented by the nonshaded area model region or missing fraction pieces) as the focal fractional quantity, (4) used shading to represent the removal of pieces, or (5) interpreted the fractional quantity as the fractional complement.	24 (5.5%)
L2 – **Half as balance:** Problems were coded as indicative of a "half as balance" understanding if Lisa (1) justified her answer by focusing on the balance and similarity between the two quantities (part–part understanding) rather than focusing on the one part out of the total number of parts (part–whole understanding), (2) represented 1/2 by drawing a shape, partitioning it in two, and omitted the shading, or (3) used gestures and gave explanations consistent with 1/2 as a splitting-action rather than a quantity.	15 (3.4%)
L3 – **Discrete set:** Problems were coded as indicative of a "discrete set" understanding if Lisa (1) ignored the intentionally unequal size of the pieces when interpreting fractional representations, (2) understood each part of a continuous model as a discrete entity (e.g., six squares) as opposed to a part of a whole (rectangle partitioned into six pieces), (3) did not account for the difference between sizes of wholes when comparing continuous models of fractions, or (4) operated on fractions as if they were represented by a discrete set (1-out-of-6 plus 1-out-of-5 equals 2-out-of-11).	39 (8.9%)

L4 – **Unit fraction:** Problems were coded as indicative of a "unit fraction" understanding if Lisa (1) judged the size of the fraction based solely on the denominator using an inverse relationship (e.g., 1/3 is bigger than 4/5 because thirds are bigger than fifths), (2) judged the size of the fraction solely on the numerator using an inverse relationship (e.g., 3/5 is smaller than 2/5), (3) asserted that 1/2 was the largest fraction, or (4) represented the unit fraction separate from the numerator value (e.g., 5/10 represented as 5 1/10).	35 (8.0%)
L5 – **Partitioning:** Problems were coded as indicative of a "partitioning" understanding if Lisa (1) attempted to partition a shape into an odd number with a partitioning-by-halving strategy, (2) asserted that it was impossible to partition a shape into an odd number of pieces, (3) drew the number of partitions corresponding to the denominator value (resulting in an extra piece), or (4) involved a nonnormative partitioning of a shape in the final representation.	15 (3.4%)
L6 – **Arbitrary manipulation:** Problems were coded as indicative of an "arbitrary manipulation" if Lisa (1) manipulated symbols and representations based on superficial aspects without respect to their underlying meaning, (2) performed mathematical computations on values without regard to the meaning or relationship of those values ("2/6 equals 1/12 because 2 times 6 is 12"), or (3) interpreted representations by attending to superficial aspects of the form, (e.g., "one-half" can be written as "1.5" because it is "1" ("one") and ".5" ("half")).	16 (3.6%)

APPENDIX D

Emily's Persistent Understandings—Operational Definition and Number of Problem Instances

Operational definition	Number of instances (% of all problems)
E1 – Smaller part: Problems were coded as indicative of a "smaller part" understanding if Emily interpreted a fractional representation based on the part composed of fewer pieces. In addition, because her attention to smaller part created ambiguity with respect to how shading should be interpreted, problems were also coded as "smaller part" if she interpreted the fraction as the fractional complement (corresponding to the nonshaded or missing pieces), irrespective if it was the smaller of the two quantities.	28 (5.5%)
E2 – Quantity as partitions: Problems were coded as indicative of a "quantities as partitions" understanding if Emily (1) understood the partitioning of a shape to represent that unit fraction quantity (e.g., partitioning into two is a representation of 1/2, partitioning into three is a representation of 1/3, etc.) or (2) used gestures and gave explanations consistent with 1/2 as a splitting-action rather than a quantity.	15 (3.0%)
E3 – Part–part: Problems were coded as indicative of a "part–part" understanding if Emily (1) referred to a fractional amount in terms of the numerator and the complement rather than the numerator and the whole, or (2) focused on the balance between the two parts comprising the whole. Instances in which she identified the fraction in part–part terms, but later identified the whole, were excluded, because it is more suggestive of a part–whole understanding.	16 (3.2%)

E4 – More pieces: Problems were coded as indicative of a "more pieces" understanding if Emily (1) focused on the number of pieces in the whole, rather than on the size of the pieces, particularly when size was the relevant dimension, (2) asserted that the larger denominator had the larger sized pieces, (3) asserted that the fraction with the larger denominator was the larger fractional value, or (4) focused on the number of pieces without referencing the size of the pieces in the context of equivalent fractions. In addition, because "more pieces" was refined to be associated with the smaller fraction, problems in which Emily asserted that the fraction with the larger *numerator* was the smaller fraction (e.g., 3/5 is smaller than 2/5 because 3 is larger than 2) were also included.	14 (2.8%)
E5 – Quarters: Problems were coded as indicative of a "quarters" understanding if Emily (1) referred to "fourths" in terms of quarters, (2) used the numeral 25 to represent fourths, or (3) attended to perceptual and figural cues (like perceived right angles) to judge the fractional value as equivalent to 1/4.	16 (3.2%)
E6 – Representation as answer: Problems were coded as indicative of "representation as answer" if Emily (1) treated construction and interpretation of the representation as disconnected acts, in particular by interpreting a representation she had constructed as a different fractional quantity (e.g., draws 5/8 and interprets it as 3/8) or by answering questions about the construction of a representation in terms of interpretation or vice versa (e.g., when asked about constructing a representation she answered with language consistent with interpretation—"count, look at, etc."—or when asked about interpretation she answered with language consistent with construction—"draw, divide, shade") or (2) inappropriately mapped meaning onto the representation or manipulated symbols or a representation in a way that suggested abstracted rules divorced from meaning (e.g., a 1/4 piece and 1/6 piece equals a 1/10 piece).	34 (6.7%)