Student errors are springboards for analyzing, reasoning, and justifying. To induce specific errors and help students learn, choose tasks that might produce mistakes.

Just as failures are stepping-stones to success in life, errors can be launching pads for understanding mathematical concepts. The mathematics education community recognizes the value of student errors, noting that “mistakes are seen not as dead ends but rather as potential avenues for learning” (NCTM 2000, p. 144). Instead of warning students about common errors to avoid, Eggleton and Moldavan (2001) advocated an inquiry-based approach in which mistakes are valued and students are expected to resolve their erroneous thinking.

In a teaching experiment, Borasi (1994) found that using errors as springboards for inquiry offers such learning opportunities as experiencing doubt and conflict, engaging in problem solving, pursuing mathematical explorations, communicating mathematically, and justifying mathematical activity. Such learning opportunities can help students make sense of problems, reason abstractly, and critique the reasoning of others, which refer to three of the eight Standards for Mathematical Practices (CCSSI 2010, p. 6).

By using student errors appropriately in a mathematics classroom, a teacher can stimulate inquiry, discussion, and reflection. The level of control that a teacher has in capitalizing on a student error depends on whether the error is planned, expected, or
Fostering Understanding and Thinking

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unexpected (Borasi 1996). A planned error is incorporated into an activity, for example, when asking students to analyze a list of “definitions” for a concept or several “solutions” to a problem. An expected error is made by a student in a problem-solving situation and is expected by the teacher. An unexpected error is unanticipated and calls for immediate discussion. Although student errors frequently occur, teachers do not always have control over when students make them. How, then, can errors be induced?

Error-eliciting problems are mathematical tasks that are designed specifically to bring forth among students common mistakes pertaining to a particular mathematical concept. This article presents three types of error-eliciting problems:

1. Tasks that elicit a misconception
2. Tasks that elicit a misapplication of a procedure or a formula
3. Tasks that elicit an overgeneralization of a concept

The three types are not mutually exclusive because a misapplication of a procedure or an overgeneralization of a concept may be related to a misconception. Two examples are given for each type. The problems are in the format of multiple-choice items, so that they can be easily implemented.
with classroom voting (Cline 2006; Popelka 2010), either using cards or clickers. The ideas presented in this article are based on the author’s work involving prospective teachers in college and in-service teachers in professional development workshops. Most ideas can also be adapted for use in middle school classrooms.

**ELICITING MISCONCEPTIONS**

Many students, and even some adults, believe that multiplication always “makes bigger” and division always “makes smaller,” in part because of their experience with multipliers and divisors being natural numbers. For example, 78 percent of students ages 12 and 13 circled *less* in this statement: “The answer to 4.6 ÷ 0.6 is *more/less* than 4.6” (Greer 1987). Multiple-choice items can be used to elicit the multiplication makes bigger (MMB) and division makes smaller (DMS) misconceptions (Lim 2011). The items are effective only if students do not perform the actual calculations, which might give correct answers without students understanding the underlying ideas. The purpose here is for students to reason about the effect of multiplying or dividing by a proper fraction. To deter students from performing the computation, the problem in figure 1a uses “ugly” fractions, so that students can think about the effect of dividing by 2884/3717. Figure 1b uses a variable so that students can think about the effect of the multiplier 67/89 on any natural number.

Both items were found in a lesson that uses clickers and visuals with PowerPoint® slides to help prospective teachers address the MMB and DMS misconceptions (Lim 2011). Twenty-two of 35 college students in a math course for prospective teachers initially selected “less than” for the first item, but the correct answer is “greater than.” This mistake revealed a common assumption that the dividend should be divided into equal parts. In this case, the sharing-equally model of division is inappropriate because the divisor is not a whole number. To visualize and understand the meaning of dividing by a proper fraction, these prospective teachers needed the measurement model of division, in which division is modeled by showing the number of times the divisor fits into the dividend.

For the second item (see fig. 1b), only 13 prospective teachers chose the correct answer A. When asked to explain why the inequality is always true, some prospective teachers gave empirical justification; for example, they tried several values of $N$ and found that the inequality was true each time. Some gave a more conceptual explanation; for example, a fraction value of $N$ is going to be less than the number $N$ itself. The two problems in figure 1 give learners an opportunity to make mistakes and discuss their thinking. Such discussions can help them understand the meaning of multiplying by a proper fraction (e.g., part of something or scaling down) and dividing by a proper fraction (e.g., a proper fraction can fit into $N$ units more than $N$ times).

Students have a tendency to treat a graph as a picture when they are asked to interpret graphs (Kaput 1987; Leinhardt, Zaslavsky, and Stein 1990). To elicit the graphs-as-pictures misconception, we can use problems like figure 2, in which a scenario is described and students are supposed to select the distance-time graph that best represents the scenario. Students who select graph B tend to interpret the graph as a picture that represents the hill on which Kula jogged. The correct answer is A, and there are different ways to justify it. Students could reason that the distance traveled in the first 10 minutes (represented by the vertical increase in height) should be less than that for the next 10 minutes.

Alternatively, students could focus on the relative speeds, which are depicted by the steepness of the three segments in the graph. This problem can foster students’ understanding of a Cartesian graph on a plane as a representation to coordinate two covarying quantities. This problem can also be used to promote the habit of attending to meaning and thinking flexibly. For example, the answer choice B
could be correct if the vertical axis represents the vertical distance (i.e., height) between Kula and the base of the hill, and C could be correct if the vertical axis represents the speed at which Kula was jogging.

ELICITING MISAPPLICATIONS

Mathematical learning involves developing both procedural knowledge and conceptual knowledge. However, procedures can be learned and practiced without connections, meanings, or conceptual understanding. The sequential nature of procedure lends itself to rote learning, which “produces knowledge that is notably absent in relationships and is tied closely to the context in which it is learned” (Hiebert and Lefevre 1986, p. 9). Problems can be designed to elicit and address common errors associated with a formula or procedure that is learned superficially. The problem in figure 3 is designed specifically to elicit a misapplication of the speed-equals-distance-over-time formula.

When this question was administered in a pretest, only 18 percent of 307 prospective elementary teachers chose the correct answer choice B; 52 percent chose D because at 20 minutes, the distance from home is 3000 meters and dividing 3000 meters by 20 minutes gives 150 meters/minute. These students were “blindly” applying the $s = \frac{d}{t}$ formula without realizing that how far Gina is away from home after traveling for 20 minutes from her friend’s house will not give us any information about speed.

When used as an error-eliciting problem to initiate discussions in class, this problem presents students with an opportunity to differentiate between $s = \frac{d}{t}$ and $s = \frac{\Delta d}{\Delta t}$. More precisely, the speed should be conceptualized as distance traveled over time taken ($\Delta d/\Delta t$), which means how far one travels per unit time. This problem can be connected to the

Kula was jogging along a road that went up a hill. She jogged at a comfortable, constant speed for the first 10 minutes where the road had a gentle incline (gentle slope). She then jogged at a slightly higher constant speed for the next 10 minutes because the road was flat. She jogged slowly for the last 5 minutes because this part of the road had a steep incline (i.e., a steep slope). Which of the following graphs could represent the distance that Kula had jogged in relation to the number of minutes she had jogged?

Fig. 3 This question is designed specifically to elicit a misapplication of the speed-equals-distance-over-time formula in students who use the concept only superficially.

Gina is traveling home from her friend’s house. The graph represents a portion of Gina’s journey. What is Gina’s speed at the 20th minute?

A. Approximately 3000 meters
B. Approximately 50 meters/minute
C. Approximately 80 meters/minute
D. Approximately 150 meters/minute

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This problem elicits procedural errors associated with the percent sign.

\[ 45\% \times 2\% = \underline{\quad} \]

A. 0.009  
B. 0.09  
C. 0.9  
D. 9

The Enlarge This Figure problem encourages students to think carefully about how to apply proportional reasoning.

Produce a figure that is similar to the one presented below so that the segment that measures 5 cm on the given figure will measure 7 cm on your reproduction. Your reproduction should be composed of 2 pieces in blue, 2 pieces in yellow, and 2 pieces in red.

\[ \text{Fig. 4} \quad \text{Fig. 5} \]

meaning of slope of a line; students can appreciate why slope is defined as \( \Delta y/\Delta x \) and not \( y/x \). In addition to fostering conceptual understanding of speed and slope, this problem can be used to promote sense making in that it highlights the importance of attending to meaning of numerical values (e.g., 3000) and operations (e.g., dividing 3000 by 20).

According to Hiebert and Lefevre (1986), procedural knowledge consists of a representational symbol system (i.e., form and syntax) and rules for symbol manipulation (i.e., algorithms and procedures). Students who have superficial understanding of symbols like \% for percent, \( / \) for ratio, and a superscript for exponent are unlikely to be aware of their procedural errors. The problem in figure 4 is designed to elicit procedural errors associated with the percent sign.

When used as a classroom voting item, 34 percent of 32 prospective middle school math teachers chose the answer choice C. They probably multiplied 45 by 2 and then appended the percent sign to obtain 90 percent, which is equivalent to 0.09. Only 41 percent of 32 prospective middle school math teachers correctly chose A. They could have arrived at the correct answer by converting the percentages to decimals (i.e., \( 0.45 \times 0.02 \)) or to fractions (\( 45/100 \times 2/100 \)). Alternatively, we could treat the percent sign as a “unit” with a value of 0.01 or 1/100. This interpretation allows students to conceptualize \( 45\% \times 2\% \) as \( 45 \times 0.01 \times 2 \times 0.01 \). Such reasoning can be connected to the more familiar concept related to area, such as the notion of square meters. For example, a length of 45 m times a width of 2 m will result in an area of 90 m². This error-eliciting problem can reinforce the habit of attending to the meaning of symbols as well as making connections.

**ELICITING OVERGENERALIZATIONS**

Students may overgeneralize an idea to a new context for which the idea is no longer appropriate. For example, middle school students may continue applying additive reasoning in situations that require multiplicative reasoning. The Enlarge the Figure task (adapted from Brousseau 1997, p. 177) was designed to elicit additive reasoning (see fig. 5a).

My students who reasoned additively enlarged each piece by adding 2 cm to each dimension and produced a distorted figure. Correct strategies produced by my students included the following:

1. Multiplying each dimension by 1.4 or 7/5;
2. Increasing each dimension by 40 percent;
3. Setting up a proportion \( x/D = 7/5 \) and solving for the enlarged length \( x \) for each dimension \( D \); and
4. Dividing each dimension by 5 and then multiplying the quotient by 7.

Students were challenged to explain why these seemingly different solutions work and how they are related to one another. Such discussions served to reinforce their understanding of fraction, decimals, and percentages (\( D + 5 \times 7 = 7/5D = 1.4D = 140\%D \)), the distributive property (\( 1.4D = D + 0.4D \)), and the meaning of a proportion. This activity allows students to discuss what it means to enlarge a figure proportionally, how to enlarge without distortion, why adding 2 cm will distort the figure, and why a multiplicative or proportional strategy is appropriate.

When proportional strategies are overemphasized, students may develop a tendency to apply these strategies even when they are not appropriate. For example, after learning how to set up a proportion to solve missing-value problems—story problems with three known values provided so that students
can determine the fourth value— they tend to use a proportion even when the quantities are not related proportionally (see fig. 5). An important aspect of understanding proportionality is “knowing what it is and when it does not apply” (Lamon 2007, p. 647). Lim (2009) used nonproportional missing-value problems in the context of burning candles to emphasize the importance of analyzing a problem situation, determining the covarying quantities, and identifying the invariant relationship. Overgeneralizations of proportionality may also be elicited through other types of problems, such as the one in figure 6.

When the problem in figure 6 was used as a classroom voting item in a professional development workshop with 27 in-service and 10 preservice secondary math and science teachers, 59 percent selected answer choice A. Underlying this response is an implicit assumption that a percentage increase in productivity covaries proportionally with the number of days, as depicted by the reasoning shown in figure 7a. This incorrect response can also be regarded as a consequence of misapplying division without thinking about the meaning of a percentage increase in productivity. Only 38 percent of the respondents selected the correct answer C. The explanation in figure 7b indicates a correct understanding of productivity. The solution in figure 7c is also correct although it relies on assigning a value of 100 to Jorge’s daily productivity.

**FOSTERING CERTAIN MATHEMATICAL DISPOSITIONS**

The six error-eliciting problems presented here offer students an opportunity to tackle certain errors head-on, discuss the mathematics underlying those errors, learn from their mistakes, and deepen their mathematical understanding. Additionally, these problems can foster certain mathematical dispositions, such as attending to meaning of symbols, numbers, operations, and graphs (problems 1, 2, 3, and 4); making connections (problems 3, 4, and 5); and reasoning quantitatively by attending to quantities and relationships in a story problem (problems 2, 3, 5, 6).
Some of these mathematical dispositions may be an antidote to students’ habitual tendencies of “doing whatever first comes to mind . . . or diving into the first approach that comes to mind” (Watson and Mason 2007, p. 207).

Consequently, error-eliciting problems allow students to become aware of their impulsive disposition and realize the benefits of attending to meaning and reasoning quantitatively. When students notice an improvement in their ability to withhold their immediate response and analyze the problem situation, they are more likely to want to improve their analytical disposition. Error-eliciting problems can therefore help students recognize their misunderstandings in mathematics as well as their impulsivity in responding to problems.

PRACTICAL SUGGESTIONS FOR USING ERROR-ELICITING PROBLEMS

Gojak (2013) offers a list of questions for prompting teachers to reflect on how they respond to student errors. Teachers who wish to optimize the effectiveness of using error-eliciting problems may consider these suggestions:

- Focus on the process (i.e., thinking and analyzing) rather than the result (i.e., the answer).
- Encourage students to think individually before discussing in small groups.
- Allocate time for students to engage in reasoning and sense making.
- Use probing questions to encourage students to reflect on their thinking.
- Be prepared for unexpected errors, and challenge students to analyze them.
- Encourage students to challenge one another’s reasoning.
- Press students for explanations and justifications.
- Ask students to write what they have learned from the mistakes they or their classmates have made.

As mentioned earlier, the problems in this article are presented in multiple-choice format so that they can be used in conjunction with classroom voting. When using voting, capitalize on the opportunity for mathematical thinking and productive discussion. This following series of steps (Lim 2011) is suggested for optimizing student thinking and learning:

- Have students work individually and then vote.
- Display the vote distribution electronically, if available.
- Have students discuss in small groups and then re-vote.
- Pose questions for students to discuss the mathematics underlying the correct answer.
- Have students present their group reasoning.
- Orchestrate a whole-class discussion (see Smith and Stein 2011), and highlight the key mathematical ideas and certain mathematical dispositions.

An advantage of the multiple-choice format is that it draws students’ attention to the “expected” errors. A disadvantage is that the answer choices might cause a student to disregard a solution approach that results in a different answer. We should welcome unexpected errors (Borasi 1996) especially if they offer an opportunity for students to gain insights into their own understanding. Teachers should consider the open-response format if error-eliciting problems are used without classroom voting or for formative assessment.

To create error-eliciting problems on any math topic, we begin by noting students’ errors, analyzing their misconceptions, and then designing items to draw out those errors. We may consider having a personal electronic collection of error-eliciting problems in which we keep adding such problems as we encounter them in print materials, standardized tests, or online resources. We may need to tweak the items to increase the chance of eliciting certain misconceptions, misapplications, and/or overgeneralizations.

We may also need to modify the items, such as rewording or including diagrams, so that they are appropriate for students. Reflecting on students’ responses to an item allows us to understand its strengths and weaknesses and think of ways to refine the task. Over time, we may develop an eye for recognizing opportunities for transforming mathematical tasks into error-eliciting problems, thus enhancing our capacity to create our own problems and use them effectively in the classroom.

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