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The construction of an iterative fractional scheme: the case of Joe

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Abstract

In his paper on *A New Hypothesis Concerning Children's Fractional Knowledge*, Steffe (2002) demonstrated through the case study of Jason and Laura how children might construct their fractional knowledge through reorganization of their number sequences. He described the construction of a new kind of number sequence that we refer to as a *connected number sequence* (CNS). A CNS can result from the application of a child's explicitly nested number sequence, ENS (Steffe, L. P. (1992). *Learning and Individual Differences, 4*(3), 259–309; Steffe, L. P. (1994). Children's multiplying schemes. In: G. Harel, & J. Confrey (Eds.), (pp. 3–40); Steffe, L. P. (2002). *Journal of Mathematical Behavior, 102*, 1–41) in the context of continuous quantities. It requires the child to incorporate a notion of unit length into the abstract unit items of their ENS. Connected numbers were instantiated by the children within the context of making *number-sticks* using the computer tool TIMA: sticks. Steffe conjectured that children who had constructed a CNS might be able to use their multiplying schemes to construct composite unit fractions. (In the context of *number-sticks* a composite unit fraction could be a 3-stick as 1/8 of a 24-stick.) In the case of Jason and Laura, his conjecture was not confirmed. Steffe attributed the constraints that Jason and Laura experienced as possibly stemming from their lack of a *splitting operation for composite units*. In this paper we shall demonstrate, using the case study of Joe, how a child might construct the splitting operation for composite units, and how such a child was able to not only confirm Steffe's conjecture concerning composite unit fractions, but also give support to our reorganization hypothesis by constructing an iterative fractional scheme (and consequently, a *fractional connected number sequence* (*FCNS*)) as a reorganization of his ENS. © 2002 Elsevier Science Inc. All rights reserved.

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1. Two different levels of abstraction with composite units

We began working with Joe (an African-American student) during his third grade year. During the first year of the teaching experiment, Joe worked with two different partners (Tania and Laura) and it became obvious to us that Joe's mathematical abilities were beyond what we expected on the basis of an initial assessment. Our emphasis during the first year was on investigating the children's whole number multiplicative operations. A comparison of Joe's and Laura's operations with composite units during this first year indicates that they ended the year operating at two distinctly different levels of abstraction.

1.1. Joe's initial composite units

A review of Joe's work during the first year confirms that Joe had not only constructed composite units but could use them in units-coordinating, segmenting, and sharing tasks in ways similar to Jason and Laura. While he was able to use two composite units to engage in units-coordinating tasks and unit-segmenting tasks, he initially had difficulties in solving sharing tasks and envisioning a separation of a collection of toys into equal portions. He lacked an *equi-portioning scheme* whereby he could posit and test, for example, a portion of 20 toys that could be put into each of 4 bags. His difficulty seemed to be in envisioning the 20 toys separated into 4 equal, but unknown portions. His strategy was to use the available number, 4 to segment the available total, 20, but he then did not know what the result (5) referred to. By using numbers for which he had symbolized a unit of units of units structure that he could use a priori (such as 10 as five twos or two fives) Joe eventually constructed an equi-portioning scheme that he could use to solve most sharing problems. In doing so, the operations that produced his composite units appeared to have been re-interiorized so that he could work with three levels of units rather than only two.

The act of making written records of his results of sharing situations may have contributed to Joe's re-interiorization of his operations, by symbolizing a structure of composite units on which he could reflect. We encouraged Joe to make records of all possible situations of sharing a collection of toys among a number of friends. Joe eagerly made such records using his own symbolism for the situation. For example, given a total of 8 toys to share, Joe wrote down "$4 = 2, 2 = 4, 8 = 1, 1 = 8$" to represent the following situations: 4 children each get 2 toys, 2 children each get 4 toys, 8 children each get 1 toy, and 1 child gets all 8 toys.

1.2. Laura's composite units

Laura appeared to construct an *equi-portioning scheme* before Joe in that she was able to assimilate the sharing situations using her unit-segmenting activity with a posited portion. She knew that she had to find a portion that used 4 times (say) would exhaust the total of 20 toys. She tested her portions by using them to segment her number sequence from 1 to 20 and keeping track of how many segments she made. Her counting activities, however, were almost always at the level of singleton units. For example, she segmented her number sequence using her concept of four. She counted from 1 to 20 raising 4 fingers at a time and keeping track of how many times she raised her 4 fingers. She knew that she had raised her fingers 5 times rather than 4 so she tried another number with which to segment her number sequence from 1 to 20.

Laura appeared to assimilate the situation using a unit of units structure but was not able to work with composite units in her calculations. She would count on by ones to find out what 6 more than 30 would be, or even 10 more than 30! Joe, on the other hand, used his composite units in his calculations whenever he could. He used known doubles to generate triples by counting on from the double (e.g., 3 nines are 27 because 2 nines are 18 and 18, 19, ... , 26, 27). He also used tens and ones as different levels of units that could be composed without having to count — 30 and 6 more were 36. He could decompose and recompose numbers strategically to find sums. For example, to add 16 onto 24, he added 10 to 20, then added on 4 to get 34, then added on 6 to get 40. He explained that the 10 came from the 16, the 20 came from the 24 as did the 4, and the 6 came from the 16. In contrast, Laura counted 16 onto 24, and kept track of her counting by using her fingers. She was not able to follow Joe's explanation for how he formed the sum of 24 and 16.

1.3. Different levels of abstraction with composite units

Another important difference in the way the two children acted during the teaching episodes of the first year was in their ability to use previous results as input for further operating, and in their abilities to reflect on their prior activity. In a situation where both children had copied some strings of 5 toys under two different covers, they were to determine how many strings were under both covers, how many toys were under each cover, and how many toys total were under both covers. Joe had copied seven strings of 5 under his cover and Laura had copied six strings of 5 under her cover (see Fig. 1).

Both children were able to work out, counting by fives, how many toys were under each cover. To find the total number of toys under both covers, Joe counted by five 13 times while Laura counted on 30 from 35 by ones. Later in this same episode, the teacher added a third cover and copied 15 toys under the cover. He asked the children how many strings of 5 could now be made from all the toys under all three covers. Joe was eventually able to figure out that he could make 3 more strings of 5 with the 15 toys the teacher had added, and he then told the teacher that they would have 16 strings altogether. Joe also added the 15 to his prior result of 65 to arrive at a total of 80 toys under the three covers. Laura also answered that there would be 16 strings total, but could not explain how she arrived at that total. Joe, on the other hand, could explain exactly how he imagined making 3 more strings out of the 15 toys, and added those three strings to the 13 strings he already had. In trying to work out how many toys were now under all three covers, Laura had to reenact her immediate prior activity of adding on 30 to 35 and then adding another 15 to this total. She could not start with her previous result of 65, as did Joe. To check his result, Joe actually continued counting by fives from 65 to get to 80.

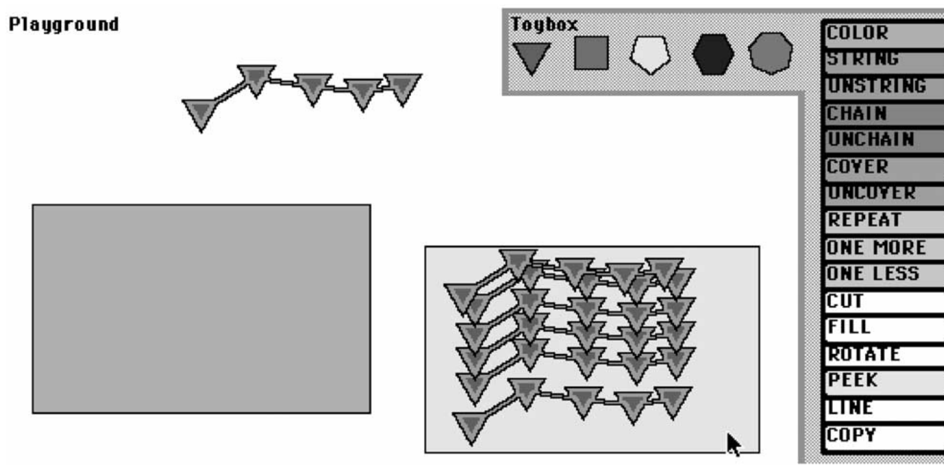

Fig. 1. Screen from TIMA: toys showing 6 strings of 5 toys under Laura's cover.

While counting by fives can be learned as a rote activity by many children, Joe's ability to switch unit levels in counting on by fives indicated that his fives were iterable composite units (Steffe, 1992). This was not so clear for Laura, as she had to calculate her result at the level of singleton units. Later episodes during this first year indicated that Joe had constructed other numbers (besides 5 and 10) as iterable composite units. In a baking and selling game using trays of seven hidden cookies, he was able to use his known fact of 8 times 7 to figure out 7 times 7 by counting down from 56 to 49. For Joe to have conceived of 7 sevens as 7 ones less than 8 sevens indicates that Joe was able to use his composite unit of seven strategically. Such strategic reasoning is indicative of having constructed a unit of units of units and 7 as an iterable unit. "Fifty six" was a multiplicative structure (Steffe, 1994) of 8 sevens that contained 7 sevens. Joe was able to determine that when he had 7 trays of cookies (a total of 49 hidden cookies) he would need to bake three more trays in order to have a total of 70 cookies. He was able to work with the difference of 70 and 49 as three units of seven because he knew that 70 was 10 sevens. This was strong indication that Joe had constructed seven as an iterable composite unit that he could use in numerical reasoning.

The difference in levels of abstractness of Joe's and Laura's composite units became even more evident in the subsequent modification of the baking and selling game, whereby cookies were baked in trays of 10 and sold as single cookies. Joe was able to simply add on tens without counting for each round of baking and then subtract ones from the total after individual cookies were sold. For example, after baking 3 trays and selling 3 cookies he had 27 cookies left in the store. Another round of baking 3 trays gave him 57 cookies (he counted on by tens — 27, 37, 47, 57) and after Laura sold 3 cookies, he simple subtracted 3 from 57 to arrive at a total of 54 cookies. Laura, however, had to re-present for herself how many cookies were left under each cover (tray) in the playground and added these totals together by counting on by ones. She did eventually mimic Joe's counting on by tens but returned to counting on ten ones whenever she had a different result from Joe (due to counting errors). She even explained that she had to count ten ones to be sure she had added ten to her previous total! Laura did not appear to be abstracting the results of her operations as numerical units, but had to establish a figurative record of the situation and then count these figurative records. It seems as if Laura had not yet interiorized her program for making composite units, whereas Joe had. Joe had abstracted a unit of units (he had abstract composite units available to him ahead of action). These abstracted composite units could be composed (into units of units of units) or decomposed (units disembedded from units of units). Laura, on the other hand, had to create her unit of units of units through her actions. She had not yet abstracted this level of composite unit; thus it was not available to her for further operations after she had constructed it.

The difference that such an abstraction makes on the children's available numerical operations and strategies became very apparent during the continuation of the baking and selling game. Laura had difficulty figuring out how many cookies were left in the store after she removed three cookies from the 84 cookies that resulted at the end of the next round of baking 3 trays of 10 cookies. She had to reconstitute the activity of the game by adding first 10 to 54, then 9 to 64 and finally 8 to 73 using her fingers (she had removed one cookie from one tray and two cookies from another tray!). Joe simply subtracted three from their prior result of 84. After three more trays were added, Joe was able to count-on from 81 by tens, going past the 100 mark with no difficulty to reach 111; whereas Laura had to count on from 81 by ones and made a counting error, arriving at 110. She experienced no perturbation in arriving at 110 when adding 30 onto 81. Joe was also able to combine the two activities of the game (baking three trays of ten and selling three cookies) to anticipate results by adding 27 to the prior result.

2. Joe's construction of composite unit fractions

2.1. Creating connected numbers through unit iteration

For the first half of the second year of the project Joe worked with a teacher/researcher without a student partner because of the difficulties he was experiencing in school that often left him with a low self-esteem. The focus during this period was on the children using their whole number concepts and operations in situations that we would regard as involving *continuous quantities*. Through the activities with TIMA: sticks, Joe used his composite units to construct *connected numbers* (Steffe & Wiegel, 1994; Steffe, 2002). A connected number can be constructed by a child in two ways. The first case is where a child mentally projects the units of his/her concept of a whole number (e.g., eight) into an unmarked line segment. This can be enacted in TIMA: sticks by using PARTS to partition a stick simultaneously into eight parts. Children also establish connected numbers by making what we refer to as *number-sticks* — that is, sticks created by joining together copies of an established unit stick.

Fig. 2. A set of number-sticks in TIMA: sticks.

Number-sticks were referred to by their number-name. That is, a stick created from 8 copies of a unit stick would be called an 8-stick. In this way connected numbers carry with them a notion of numerosity as indicating length.

2.2. *Using connected numbers in iteration to establish composite unit fractions*

The following protocol illustrates how Joe (J) and the teacher/researcher, Azita (A) used activities with number-sticks to establish composite unit fractions through iteration of connected numbers as a multiplicative relation. For instance, one-third of a 24-stick is that stick, which repeated three times, will make a 24-stick. This approach to fractions of connected numbers provided Joe with the opportunity to use his multiplying schemes to find the appropriate stick. This teaching episode took place in the fall of Joe's fourth grade year, the second year of the Project. A set of number-sticks had been created and placed at the top of the playground area of the computer screen which was separated by a long thin segment that stretched across the full width of the screen (see Fig. 2). The length of each stick was a multiple of the shortest stick. The shortest stick was designated as the unit stick (the 1-stick) and each of the other sticks were named as an *n*-stick where *n* could be any number from 2 to 10 (e.g., a 5-stick). Sticks created by repeating or joining copies of sticks from this ordered collection of number-sticks were also named in the same manner (e.g., a stick created from 4 repetitions of the 6-stick was a 24-stick).

Protocol I

A: (While Joe has his eyes closed, Azita makes a 24-stick below the separator and erases the marks from her 24-stick.) The stick that I used was one-third of the length of the stick I have right here (pointing to the unmarked 24-stick).

J: (Measures the stick and 24 appears in the number box. He then smiles to himself and counts down the set of number sticks ending on the 8-stick. He copies this stick and repeats it 3 times to make a stick the same length as the 24-stick.)

A: That is right!

J: You said one-third, so what will be . . . three times eight is 24.

A: (Suggests doing more problems with the 24-stick.) Think of a stick you could use to make the 24-stick and tell me what fractional part of the 24-stick it would be, and I will try to tell you what size stick it is and how many times I should use it.

J: Close your eyes.

J: (Trashes the 3-part 24-stick and looks at the set of number sticks.) OK, I didn't have to do nothing... It's umm... It's one-sixth.
A: The stick that you used is one-sixth of the 24-stick?
J: (Nods his head — yes.)
A: So, I want something, I want a stick that when I repeat it six times would give me...
J: No!
A: Would give me the 24...
J: One fourth! (at the same time as Azita is speaking).
A: Oh! You used the one-fourth stick?
J: (Nods his head — yes.)
A: You used one-fourth, so I want a stick that when I repeat it 4 times will give me the 24, and I think that is the 6-stick! What do you think?
J: (Nods his head — yes.)
A: (Copies the 6-stick and repeats it 4 times to make a stick the same as the 24-stick.)

Joe knew to use the 8-stick for 1/3 of the 24-stick because "three times eight is 24." We regard Joe's interpretation of one-third as *something that when multiplied by 3 gave the total number* as a modification of Joe's multiplying scheme because of the way he was able to pose the problem for Azita and the self-correction he made in the process.

Joe hesitated in naming the fraction (*It's umm... It's one-sixth*) when posing his problem for Azita. We suggest that he was trying to figure out both the numerosity of a *hypothetical stick* and the number of times he would have to use it to produce 24. That Joe used the numerosity of the hypothetical stick to generate the fraction rather than the number of times he would have to use that stick, indicates that he was aware of the two numerosities. This awareness was confirmed when Joe realized his mistake as soon as Azita voiced her interpretation of 1/6. At this point he made a self-correction rather than accepting Azita's actions. This indicates that Joe was aware of the operation of iterating as well as what was being iterated prior to action. It is in this sense that Joe was constructing meaning for composite unit fractions through *iteration of his connected numbers*. He was aware of the 6-stick as 1/4 of the 24-stick because he knew that the 6-stick iterated four times would produce a 24-stick.

Joe's self-correction in the above protocol is in stark contrast to the consistent problems that Jason and Laura experienced with this same task. Steffe (2002) describes the attempt to have Jason and Laura establish composite unit fractions using their multiplying schemes with connected numbers in the same context of comparing number-sticks in TIMA: sticks. The attempt failed because of what Steffe described as *necessary errors* arising from the structure of the results of their *equi-portioning* schemes. In a similar task to the one described in Protocol I, above, both Jason and Laura named the 3-stick (that they had iterated 8 times to make a stick the same length as a 24-stick) as 3/8 of the 24-stick, rather than 1/8, and named the 6-stick (that they had iterated 4 times to make a 24-stick) 6/4 rather than 1/4 of the 24-stick. Steffe commented that neither Laura nor Jason seemed aware of the structural relations among the three levels of units that they had produced. Joe's realization that he did not have to do anything in order to pose the problem for Azita (as the unmarked 24-stick was still visible on the screen) indicates that he was aware of the structure of the three levels of units *prior* to action. Joe's ways of operating indicate that he had constructed at least the iterating operations of an *equi-partitioning scheme for composite units* in the context of connected numbers.

3. Joe's construction of a partitive fractional scheme

While generating composite unit fractions through iteration of connected numbers was productive for Joe; he was yet to use these composite unit fractions as parts of a whole. For example, in the continuation of the teaching episode in Protocol I, Joe successfully identified the 4-stick as 1/7 of the 28-stick, but when asked by Azita what fraction of the 28-stick two 4-sticks joined together would be, Joe responded with "One-fourteenth... because you add one-seventh and another seventh it makes 14." Streefland (1993) referred to such errors as N-distractors (miss-application of whole-number arithmetic to a fraction situation). However, Joe was yet to interpret the 4-stick as one out of seven equal parts of the 28-stick, or as four out of 28 parts. So, rather than consider his multiplying scheme as interfering with his construction of fractional knowledge, we consider it enabling a rather powerful conception of unit fraction that was yet to be incorporated into (and thus transform) his more general fractional knowledge.

3.1. Connecting partitioning and iterating operations

As mentioned above, Joe had more difficulty than Laura did in establishing an equi-portioning scheme during the first year of the project. He also had difficulty with separating a stick into equal parts in sharing situations. His visual estimates for sharing a stick into three or four equal parts were not very accurate. Rather than making a mental separation of the stick into four equal parts, and indicating this partition by placing three marks on a stick (what Steffe has termed *simultaneous partitioning*), he would draw an estimate for 1/4 of the stick and repeat this estimate 4 times to see if it matched the stick. In the teaching episode following the one above, we attempted to provoke Joe's partitioning operations by asking him to make fractions of a stick that was drawn free-hand (e.g., make a stick that is 1/7 of an unmarked stick). Without a known multiplication fact to solve the problem, we hypothesized that Joe would need to mentally separate the unmarked stick into seven equal parts in order to make a reasonable estimate for 1/7 of the stick. We hoped to provoke a simultaneous partitioning operation.

The screen display consisted of an ordered collection of number sticks (from 1 to 10) as in Fig. 2. A stick had been drawn free hand below the separator. Joe was to choose an estimate for a stick that would be 1/7 of this mystery stick. He chose a 2-stick from the set of number sticks and repeated it 7 times directly below the mystery stick. His resulting stick was approximately 2/3 of the mystery stick. He made a second estimate (without acting) that the 3-stick would be 1/7 of the mystery stick because "It's about 27/21, I mean." Rather than asking Joe to justify this second (verbal) estimate, Azita continued using the 2-stick Joe had originally chosen, creating 10 repetitions to make a stick that was about one unit stick short of the target stick.

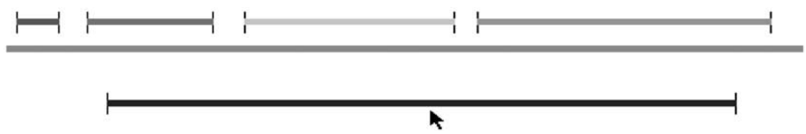

Fig. 3. Which stick is 1/5 of the bottom stick?

Joe confirmed that he was thinking of the mystery stick as a 21-stick when he then exclaimed "21! That's what I said!" and that the 3-stick would be 1/7 of the 21-stick because "If you use 3 seven times you might get 21."

The preceding task may have provoked Joe's mental partitioning operations because he could imagine seven of something being equal to the mystery stick. This relation had to be based on the results of iterating the estimate 7 times in visualized imagination, and a comparison of this imagined result with the mystery stick. By means of the comparison, Joe could project units into the mystery stick. Joe had, in fact, produced a way of operating that would enable him to use the results of iterating a stick in partitioning. His multiplicative operations with connected numbers provided him with a way of positing the numerosity of an unknown stick as a result of the imagined partition.

Confirmation for this possible connection between Joe's iterative strategies and his partitioning operations came in the next teaching episode. Joe accurately chose a stick that was 1/5 of a target stick where no numerical values were known. Four sticks and the unknown target stick were arranged on the screen as shown in Fig. 3.

Joe copied the second (from the left) of the four upper sticks below the lower target stick and repeated it five times to make a stick that was the same length as the target stick. In order to have quickly chosen the correct stick, we hypothesize that he engaged the results of mentally iterating the chosen stick 5 times in comparison to the target stick. Later in this episode, Joe found 1/3 of the target stick (the third upper stick from the left). Azita asked him to make a stick twice as long as this 1/3-stick. Joe did so by repeating the 1/3-stick. When asked what fraction of the target stick this repeated 1/3-stick would be, Joe at first compared it to the longest stick in the top row of sticks. He said this last stick was a bit shorter than his stick. When pressed for a fraction name he called his stick "two thirds" of the target stick. He also called three repetitions of the 1/3-stick a "whole stick" and "three thirds." He had established an iterative relation between a part and a whole that paralleled his multiplicative approach: what stick iterated three times will give me the whole? Further along in this same episode Joe was able to link a part-in-whole view of 5/11 with his multiplicative view of a stick that was "five times as long as the 1/11-stick." He justified calling his five repetitions of the 1/11-stick "Five elevenths!" "Because its 5 and its part of 11." He was able to project the stick that was the same as 5 parts out of the 11 (a disembedded stick) back into the 11/11-stick. He was now able to combine unit fractions into composite parts of a whole through iteration. Joe was beginning to connect partitioning of a unit whole with the generation of both unit and non-unit (common) fractions. For this reason, we refer to Joe's operations for generating unit and common fractions as a *partitive fractional scheme*.

Fig. 4. Marking 4/5 of a unit stick.

Fig. 5. Making 6/5 of a unit stick.

4. Fractional numbers and the iterative unit fractional scheme

4.1. Construction of fractions greater than the whole

The teaching episode described above (in which Joe named the 5/11-stick) occurred just before the winter break in December of Joe's fourth grade year. In the next teaching episode (that took place after the winter break in February) Joe extended his partitive fractional scheme into a scheme to generate fractions greater than the whole (such as 6/5 and 9/7) by using a new operation that Steffe (2002) calls *splitting*.

Protocol II begins after Joe had created first 3/5 and then 4/5 of an unmarked unit-stick by making a mark at the appropriate point, using a copy of the unit stick partitioned into 5 parts as a guide. The 5-part unit stick and a unit stick with one mark 4/5 the way along the stick were visible on the screen (see Fig. 4).

Protocol II

A: That's really neat! Now I'm really hungry. I want you to make me another one. I want you to make me 6/5 of that candy (meaning the unit stick).
J: Can't!
A: Why not?
J: You only got 5 of them.
A: Five what?
J: Fifths.
A: You only got 5 fifths. So is there any way of making one, do you think?
J: Make a bigger stick.
A: Make a bigger stick. How much bigger do you think it should be?
J: One more fifth.
A: OK, Do you want to show me?
J: (Pulls the end part out of the original stick that has a mark at the 4/5 position only, and joins this one piece to the original stick to make a stick one-fifth larger than the original — see Fig. 5).

What is important about the above protocol is how Joe was able to interpret Azita's request for 6/5 of the candy as being one more fifth than the whole bar. The creative production for Joe was to envision a *bigger stick* that would *include* the whole stick. We regard this as a modification of his partitive fractional scheme (for which a fraction had to be included in the whole of which it was part). The modification was a possibility for Joe because his partitive fractional scheme included unit fractions as iterable units. Thus, 6/5 was six of one-fifth, which was one more fifth than the whole stick. TIMA: sticks enabled Joe to enact this novel idea by simply pulling out the 1/5 from his unit stick and joining it to the end of the unit stick. This was a novel use of 1/5 for Joe, as 1/5 was *freed from the constraint of being part of a whole*. This novelty was an indication of Steffe's *splitting operation*. Steffe (2002) regards a split as a *composition* of partitioning and iterating. This requires the operations of partitioning and iterating to be implemented *simultaneously* rather than *sequentially*. For Joe to have generated the 6/5-stick by adding a 1/5-stick to the end of the original stick, he would have had to have simultaneously conceived of the whole as containing 5 equal parts (for one of those parts to be constituted as 1/5 of the whole) and the 6/5-stick as being six times any one of those parts.

Steffe (2002), in his account of Jason and Laura, hypothesized that the constraints these children experienced in the construction of what he has called *fractional numbers* stemmed from their lack of this splitting operation. Steffe defined a fractional number as a connected number that took "its fractional meaning from the part of which it is a multiple. The relation to the whole of which it is a potential part would be inferential in that it could be established by means of reasoning" (p. 31).

Later in the same teaching episode, Joe confirmed that he could now create such fractional numbers. Joe had successfully estimated 1/7 of a candy stick and had used his estimate to mark off all seven 7ths on the original candy stick. He had then pulled out 4/7 of the candy stick to give to an observer. Azita asked him to make a stick that was 9 times as long as the 1/7-stick. Joe cut off the first part of the 4/7-stick. He then repeated this 1/7-stick 9 times to make a 9-part stick (see Fig. 6).

The following conversation ensued in the continuation of protocol II:

A: How long is that stick?
J: (Thinks for 3 s). Nine sevenths.
A: Why?
J: (Thinks for 15 s.)
A: You are right. It is 9/7 but why do you think it is 9/7?

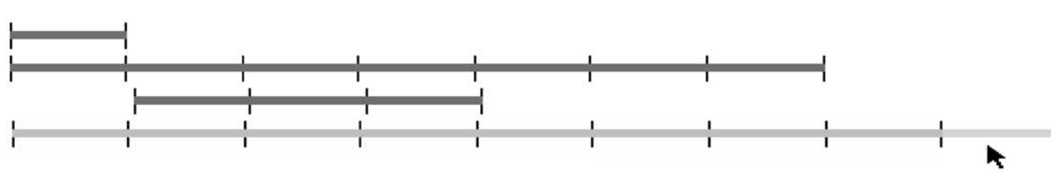

Fig. 6. Making a stick 9 times as long as the 1/7-stick.

J: Because it was . . . you were making these, the sevenths (pointing to the parts of the 9-stick) so each of these would be one-seventh.

In this episode, Joe was able to work with a fraction as both a part of a whole (the 4/7) and a unit part *freed from the whole*. In repeating a 1/7-stick 9 times to make a stick 9 times as long as 1/7 of the original whole stick, Joe was able to go beyond the whole, and name the resulting stick as nine sevenths "because . . . you were making the sevenths, so each of these would be one-seventh." In this sense Joe had taken the 1/7-stick as an *iterable one* to generate a composite unit of 9, each one of the 9 units being 1/7 of his 7/7-stick. This suggests that Joe had constructed 9/7 as a *fractional number*. Joe gave 9/7 its fractional meaning from the part (1/7) of which it was a multiple.

4.2. A splitting operation for composite units

Confirming evidence emerged in Joe's next teaching episode on February 22 of his fourth grade year, that he had constructed a splitting operation. In a very similar task to the one posed to Jason and Laura in Protocol VIII (Steffe, 2002, p. 21) Joe and his new partner, Patricia were asked to produce a stick such that a given 9-stick was 9 times longer than their stick. Patricia attempted to make a stick by iterating a copy of the 9-stick 9 times (only part of the resulting stick could be seen, as it was too long to fit on the screen). After a pause of 20 s, Joe pointed to one part of the 9-stick and said "We've already got it!" He then pulled one part out of the 9-stick as his solution.

For Joe to realize that the 9-stick was 9 times as long as any one of its parts, and he could therefore use any one of its parts to solve the task, indicates a simultaneity of his partitioning and iterating operations. Furthermore, in the situation of posing a problem for Azita in Protocol I above, Joe was aware that he didn't have to do anything because an unmarked 24-stick was present on the screen. He was aware of the possible structure of four 6-sticks being the same as a 24-stick without having to act. This awareness could be an indication of a *splitting operation for composite units* in the context of connected numbers that Steffe (2002) interprets as a *split of a split* in Confrey's (1994) definition of a split. It is certainly evidence that Joe (unlike Jason and Laura) had constructed a multiplicative relation between the whole, unpartitioned stick and one of its hypothetical parts.

Evidence for this more complex splitting operation came later in the February 22 teaching episode, when Joe and Patricia were asked to make a stick such that the 9-stick would be three times as long as it. In response to Azita's request to make a stick that would be 10/18 of the 9-stick, Patricia had repeated a copy of the 9-stick to make an 18-stick, and filled 10 of those 18 parts. In response to Azita's request to Joe to make a stick such that the 9-stick would be three times as long as his stick, Joe thought for 10 s then smiled. He erased the first two marks from Patricia's 18-stick and pulled out the piece so formed. He moved this piece under the original 9-stick and repeated it three times to make a stick the same length as the original stick. When asked by Azita how long was the stick that he used, Joe responded with 1/3. Patricia called it 3/9 and Joe eventually agreed that it could be 3/9.

For Joe to have solved the task of making a stick such that the 9-stick would be three times as long as his stick in the way that he did, he must have re-processed the request as finding 1/3 of the given stick. The fact that he established this 1/3-stick by erasing marks from Patricia's 18-stick, indicates that he regarded each part of the 18-stick as being the same as any one of the parts of the 9-stick, and that three of these parts, repeated three times would produce the 9-stick. Joe was able to free the parts of Patricia's stick from the whole stick that they constituted, and use them as fractional parts of the 9-stick. In agreeing with Patricia's name for his unmarked 1/3-stick as 3/9 of the original stick, Joe was probably projecting the three parts back into his stick to reconstitute the three one-ninths. Confirming evidence for this conjecture came towards the end of the same episode. Azita asked the children to find a stick that was 1/18 as long as the 9-stick. Joe attempted to solve this task by marking one part of the 3/3-stick (that was the same length as the original 9-stick) into six parts. Joe was able to interpret the task as finding a stick, 18 of which would make the 9-stick. Using his multiplicative knowledge (3 times 6 is 18) he projected 6 parts into one of the three parts of the stick that was the same length as the 9-stick. Furthermore, he connected this task (of finding a stick 1/18 as long as the 9-stick) with the prior task of finding a stick such that the 9-stick would be 9 times as long as it. Patricia solved the task by erasing all marks from Joe's 3/3-stick and putting 18 parts in the blank stick, using PARTS. She then pulled one of these 18 parts out of the stick. Joe agreed that this stick was 1/18 of the 9-stick, saying: "Because when you said make a stick that this is nine times bigger than it, that's what I did." Joe's statement indicates that he had, indeed, established a *splitting operation* — a composition of his partitioning and iterating operations, and that he could apply this operation to the results of a prior split, as he did when attempting to partition one part of his 3-part 9-stick into 6 parts in order to make 1/18 of the original 9-stick.

4.3. Eliminating an important perturbation

The teaching episodes during March of Joe's and Patricia's fourth grade year focused on provoking recursive partitioning and reversible operations. Joe was able to figure out that 1/2 of 1/5 was 1/10. He also spontaneously posed a reverse situation in a guess my stick game when he created a stick on the screen and told Patricia that the stick he just drew was 1/2 of the stick he was thinking of. In the same episode he posed problems starting with a stick that was 1/4 of his mystery stick and one that was 1/5 of his mystery stick. In every case he verified Patricia's correct responses and corrected her initial incorrect response (to make half of the stick Joe had drawn on the screen). Both situations (making a fraction of a fraction, and making the whole from a fractional part) provided confirming evidence for Joe's splitting operations. They also reinforced the partitive aspect of a fraction (the part-to-whole relation).

Following these teaching episodes there was one episode in which Joe experienced a severe perturbation regarding improper fractions. The following protocol begins after the children had been setting "make my stick" problems for each other (e.g., this stick is 1/5 of my stick, make my stick). Azita had drawn a stick and asked the children to make a stick that was 4/7 of her stick. Joe and Patricia had solved this task easily by putting 7 parts into Azita's stick and pulling out 4 of those parts.

Protocol III

A: (To Joe) I would like you to make me a stick that is two times as long as the 4/7 (pointing to the 4/7-stick).
J: (Appears to count to himself. He appears to be saying "8." He then repeats the 4/7-stick to make an 8-part stick.)
A: How long is that stick, Joe?
J: (Erases the marks from the 8-part stick.)
A: Good! How long is that stick in terms of the red stick, the stick we started out with?
J: (After 5 s) Eight.
A: Eight what?
J: Sevenths. No.
(Joe seems perturbed. He is moving his stick around.)
J: I don't know.
A: You are right. It is 8/7.
J: How can it be EIGHT sevenths?
A: How can it be 8/7? Good question! (To P) How can it be 8/7?
P: You want me to tell you?
A: Yes.
P: Because there's 7 in there (pointing to the partitioned 7-stick) and you used the same little pieces as in that 7-stick except that that stick is bigger and you used the same little pieces and there's 8 in there (pointing to the unmarked 8/7-stick).
A: (Asks Patricia to show Joe what she means.)
P: (Pulls out 4 parts from the 7-part stick and explains to Joe that he repeated that stick to make a stick with 8 parts "because 2 times 4 is 8," and then he erased the marks. The 8 parts were the same as the little pieces in the 7-stick, so its 8/7.)

After producing a stick that was twice as long as the 4/7-stick, Joe knew that he had 8 something. His perturbation came in naming those things "sevenths." He had not appeared to experience such a perturbation in the episode following Protocol II above in which he named a stick 9 times as long as a 1/7-stick "9/7." He expressed his perturbation when he said, "How can it be EIGHT sevenths?" (Placing emphasis on the "eight"). Patricia provided, what to us as observers, would appear to be a very clear and helpful explanation, very similar to Joe's explanation for why 9 times 1/7 was 9/7 in the prior episode.

The perturbation itself is evidence that Joe is reflecting on his own actions. This self-reflection could be indicative of Joe being in the process of modifying his fractional schemes. During the continuation of this particular teaching episode, Azita attempted to help Joe use his iterable unit fraction and his splitting operation to go beyond the whole.

Continuation of Protocol III

A: Using one-seventh, can you make 8/7?

J: (Shakes his head, "No.")
A: (Waits for 25 s.) Using 1/7 can you make me 3/7?
J: (Repeats the 1/7 three times.)
A: Very good. You are absolutely right. Can you make me 7/7 using 1/7?
J: (Goes to repeat the 3/7-stick but Azita stops him. She asks him to use the 1/7-stick. Joe does so, repeating a 1/7-stick 7 times.)
A: Very good. (To P) Do you like that?
P: (Nods "Yes.")
A: Using 1/7, can you make me 10/7?
J: (Shakes his head "No.")
A: (Asks P to try.)
P: (Goes to repeat the 1/7-stick but REPEAT is still active from Joe's 7/7 so one more part is added to Joe's stick.)
A: (Asks Joe what he would call his stick now that one piece has been added.)
J: (Checks number of pieces in his stick using the menu item. The number 8 appears in the number box.) Eight.
A: Eight what?
J: Eight of those (pointing to one part of his 8-part stick).
A: (Asks Joe to repeat what he said because she didn't hear.)
J: Eight of those (pointing to the 1/7-stick).
A: And how long is that stick?
J: One-seventh.

Even after Patricia's explanation for why twice 4/7 was 8/7, Joe refused to make 8/7 or 10/7 using a 1/7-stick. Joe's refusal would suggest that his 1/7 was no longer freed from the whole, as it had appeared to be in the prior episode. After Joe had established that the 8-part stick was 1/7 longer than the 7/7-stick, Azita repeated her request of Joe to make 10/7.

Second continuation of Protocol III

A: Now, what I would like you to do is make me 10/7.
A: (After waiting 10 s) Is it going to be larger than 7/7 or smaller than 7/7?
Both: Longer.
A: How much larger, or longer, is it going to be?
J: (After 5 s) Three sevenths.
A: That's absolutely right! Would you like to make it?
J: (Very slowly copies the 1/7-stick and then repeats it very deliberately 10 times.)

Joe knew that 10/7 was going to be 3/7 longer than 7/7. He must, therefore had conceived of the 7/7-stick as being embedded in a 10/7-stick with three extra 1/7-parts left over. Such a realization indicates a reactivation of his splitting operation. He was then able to generate the 10/7-stick through 10 iterations of the 1/7-stick. The perturbation, however, had not been entirely eliminated as the last task of this teaching episode indicates. Patricia made a small stick and posed the problem to Joe that the stick was 1/99 of the stick she was thinking of.

Joe proceeded to make Patricia's stick by iterating a copy of the small stick (intending to iterate it 99 times). Azita stopped him after he had made 14 iterations, asking if there was a quicker way to make the 99/99-stick using the 14/99 that Joe had created. Joe eventually multiplied 14 by 7 using the on-screen calculator to get the number 98. When asked what the 98 referred to he responded with 98/99. The third continuation of Protocol III picks up at this point in the teaching episode.

Third continuation of Protocol III

A: The stick that I am thinking of is 10 times as long as this one (pointing to the 14-part stick), without using the calculator.
J: As this one (pointing to the 14-part stick)?
J: (Makes several inaudible guesses, which could be 70/99, 99, or 99/99.)
P: (Uses the calculator to find out 10 times 14.)
A: (With 140 showing in the calculator, to P) You stop now. (To Joe) You tell me the measure of the stick, tell me how long the stick is, that I'm thinking of.
J: 99? No! 140/99?
A: Very good.
J: I still don't understand how you could do it. *How can a fraction be bigger than itself*?
A: That's a really good question. Think about that for next time.

Even though Joe had named the length of the hypothetical stick that would be ten times as long as the 14/99-stick as 140/99, he still indicated a perturbation when he asked "How can a fraction be bigger than itself?" The question does indicates reflection on the partitive aspect of a fraction (How can a fraction be bigger than the whole of which it is a part?) and suggests, again, that Joe may be doubting his iterative notion of a fractional number. Our conjecture for this apparent doubt of his iterative fractional number concept is that our focus on recursive partitioning and reversible operations brought the part-in-whole relation to the fore for Joe. He certainly maintained the iterative nature of his unit fractions but was unwilling to iterate them more times than would constitute the whole of which it was part. Another possibility (for which we have no evidence) is that classroom instruction on fractions (that was taking place during this period) may have been reinforcing the part-of-a-whole conception of a fraction. Whatever the reason for the apparent lapse in Joe's construction of iterative fractional numbers, they reappeared in very solid form in the next series of episodes that followed on from this one in April of his fourth grade year.

4.4. A "realistic" context for generating fractional numbers

In the teaching episode on April 19, a new Pizza Baking and Selling Game was introduced. An oven had been created on the screen using three covers. A stick inside the oven represented a pizza. The pizzas could only be cut into 8 slices (pieces). One person was to tell how many people came to the Pizza Restaurant and how many slices (pieces) each person wanted. The baker then had to bake (copy) enough pizzas to feed the group and show one person's share. Both children then had to say how much of one pizza each person received and how much pizza was eaten altogether. Joe set the first problem — 6 people each had two slices. Patricia copied one stick out of the oven and used PARTS to put in 8 parts. She then wanted to use PULLPARTS to make the share of one person but Azita asked how many pizzas they needed.

Protocol IV

A: How many pizzas does she need to bake?
Both: Two
A: Why?
J: Because there's only 8 and she'll need four [more] of those [slices] and there'll be 4 left over.
P: (Copies a second stick out of the oven and partitions it into 8 parts. She breaks this stick up into its 8 parts. She goes to break the second stick and the computer freezes.)

(They continue talking about the problem while Olive reboots the computer and resets the sticks screen with an oven and a stick in the oven.)

A: So how much of the whole pizza does each person get?
P: 2/16.
A: (Restates the question.) How much of ONE pizza does each person get?
P: What?
A: Joe, what do you think? How much of one pizza does each person get?
J: Two slices.
A: Two slices, but how many slices did we have in the pizza to begin with?
J: Eight. And then you had to get another pizza and you'll need four out of that pizza.
A: So, if you had to give me a fraction to tell me how much all people, how many slices, how much of the pizza all people had together...
J: 12/8. 12/8.
P: (Agrees with Joe.)

Joe had no problem using improper fractions to describe how much pizza six friends ate altogether. Patricia agreed with Joe most of the time, but Joe was usually first with the improper fraction. Later in this teaching episode the number of slices in a pizza was changed to 12. Joe had posed the problem of 12 people, each of whom ordered 2 slices. Joe knew immediately that they only needed two pizzas. Patricia copied two pizza sticks out of the oven and put 12 parts in each following Joe's directions. Protocol V begins at this point in the episode.

Protocol V

A: Show me the share of one person, Joe.
J: (Cuts off the first 2 parts from one stick.)
A: Yeah, how much of the whole pizza is that?
P: 2/12

J: 2/24
A: No, of just one.
J: 2/8
P: 2/12
J: Oh yeah! 2/12.
A: And how much pizza are they getting, altogether?
Both: 2/24.
A: No, how many, how much pizza ALL people are getting, put together?
P: Oh! 24. Is that what you meant?
A: Yeah. 24 what?
J: Pieces.
P: 24/8.
J: 24/12!
P: Oh yeah! 24/12.
A: How many pizzas would that be?
Both: Two.

There was some confusion in what Azita was referring to as the fractional whole in the above interchange. Each person did get 2/24 of ALL the pizza. This amount was also 2/12 of one pizza. The ability of the children to change the fractional whole appropriately in this manner indicates a flexibility in their fractional schemes and a strong sense of the part-whole relation indicated by a fractional number. Joe's willingness to use 24/12 as the amount of pizza eaten by all the people indicates that he has re-established his iterative fractional number concept. The continuation of Protocol V also indicates that Joe is able to use his fractional numbers in additive as well as multiplicative situations.

Continuation of Protocol V

A: Another person walks in, so instead of 12 now we have 13. And say each person wants 3 slices of pizza.
P: You said another. . .
J: One person? He wants three slices of pizza?
A: Yeah. How many pizzas does she need to put in the oven?
Both: One.
J: (Copies the stick from the oven and puts 12 parts in it. He then cuts off the first 3 parts.)
A: What's that Joe
J: One person's, uh the person's pizza.
A: One person's share?
Both: Yeah.
J: You said that another person comes in and he wants three.
A: So how much of the whole pizza would that person get?
P: How much of the whole pizza.
Both: Three, uh twelfths.

A: Very good. Now let me ask you one question, and you can choose not to answer (to Joe). If we put the share of that last person together with the share of the other people, together, how much would that be?
J: 27... twelfths?
A: Why?
J: Because you added 3 to 24 and that's 27, and these (pointing to the parts of one stick) are twelfths, and that (pointing to the partitioned extra stick) is twelve, so it'll be 24, 27/12.
A: Gimme five! (Azita and Joe slap each other's hand.)
At this point in the episode an observer (O) asked a follow-up question.
O: In terms of pizzas, can you tell me how much you sold?
P: The whole thing?
O: Yes, the whole thing.
J: How many pizzas I sold?
O: Yes, how much pizza did you sell?
J: 27 pieces.
O: Can you tell me in terms of pizzas?
Both: Three.
O: Did you sell all of...
P: Two and three pieces.
J (At the same time as P) Two and 3/12.
O: Two and 3/12. Thank you.

Joe had added 3/12 to 24/12 to get 27/12 as the amount of pizza eaten by 12 people, each of who ordered 2 slices, and one person who ordered 3 slices. He had performed *progressive integration operations* with his unit fractions. The children eventually realized that this amount was the same as two whole pizzas and 3/12 of a pizza.

Providing a "realistic" situation (baking and selling pizza by the slice) apparently provided Joe with a context in which he could make sense of fractional quantities greater than one whole pizza. This episode reinforced our hypothesis that Joe had constructed an *iterative unit fractional scheme* with which he could produce any fractional amount through iteration of a unit fraction and establish it's multiplicative relation to the fractional whole as a result of his splitting operation. The continuation of Protocol V also suggests that a possible result of an iterative fractional scheme could be a connected number sequence (CNS) in which the units of the connected numbers are unit fractions. Steffe (2002) refers to such a number sequence as a *fractional connected number sequence* (*FCNS*).

5. The construction of a fractional connected number sequence

It became evident that Joe had, in fact constructed a CNS for unit fractions in the teaching episodes that took place during the next school year, when Joe was in fifth grade. Because of the children's other school activities, we had to make a change in Joe's partner for the third and final year of the teaching experiment. Joe worked with Melissa during his fifth grade year. Melissa had worked with two different partners during the first 2 years of the teaching experiment. The following protocol took place in January of the final year of the teaching experiment (approximately 9 months after the episode reported above in Protocol V). In Protocol VI, the children were working with the TIMA: bars software in which they could make rectangular bars and partition them similarly to the way they had operated in TIMA: sticks. The protocol begins at a point after they had made a long thin bar partitioned vertically into 11 parts and pulled out several fractional parts among which were a 2/11-bar, a 4/11-bar and a 6/11-bar. T stands for the teacher, M for Melissa, and J for Joe.

Protocol VI

J: (Pulls a 5/11-bar from the 11/11-bar in such a way that it is the complement of the 6/11-bar that Melissa pulled.)
T: If you had two pieces like that, how much of the bar would you have?
M: Ten elevenths! (Joe also answered "ten elevenths").
T: Would that be the whole bar?
J&M: Less than the whole bar.
T: How much less?
J&M: One-eleventh.
T: OK, now put out the six elevenths.
M: (Drags the 6/11-bar from the side of the screen and places it and the 5/11-bar end-to-end as in Fig. 7).
T: If each of you have a bar like that (the 6/11-bar)...
M: It wouldn't be any more of the bar (the 6/11-bar and the 5/11-bar together made a bar commensurate with the 11/11-bar).
T: What if you both had six elevenths?
M: It would be one more than the bar.
J: Twelve elevenths.
T: Twelve elevenths! How much more than a bar would that be?
J&M: One-eleventh.
T: If you took eight of these two elevenths (Melissa dragged a 2/11-bar under the 11/11-bar).

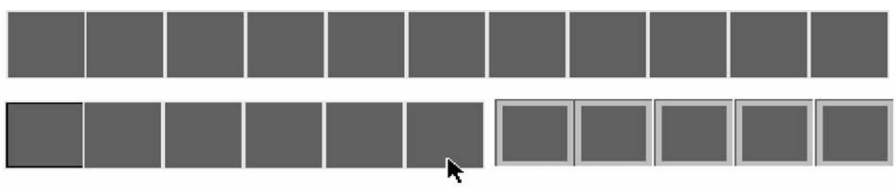

Fig. 7. Placing a 6/11-bar next to the 5/11-bar.

M: Sixteen elevenths!
T: And how much of the bar would that be?
M: That would be. . .
J: Five elevenths.
M: Five elevenths more. . .
T: Five elevenths more than. . .
J&M: The whole bar!!
T: (Referring to a 4/11-bar) If you took eleven pieces like this, can you tell me how much you would have?
M: Forty-four!
T: Forty-four what?
M: Forty-four elevenths!
T: How many whole bars would that be?
J: Four!
T: Why would forty-four elevenths be four bars?
M: Because four times eleven is forty-four!

Both children convincingly demonstrated that they understood that a 12/11-bar contains the 11/11-bar and that it is 1/11 more than the 11/11-bar. That is, they demonstrated a reversal in relation between the fractional part and the fractional whole in that what was before a fractional part now contained the fractional whole as a part. We can, therefore infer that Melissa could operate with fractional numbers in a way similar to Joe.

Because the children knew that the composite unit of numerosity 12 is one more than the composite unit of numerosity 11, it would seem that it would be rather straightforward for them to understand that the *fractional* connected number 12/11 is 1/11 more than the fractional connected number 11/11. However, this relation between the connected number 12 and the connected number 11 had to be constructed anew in the case of fractions. The children's construction of the splitting operation opened the way for their construction of the relation between 12/11 and 11/11 as well as a more general iterative fractional scheme for composite fractions.

Even though the children had constructed an iterative fractional scheme for unit fractions, it was still remarkable that they both knew that if they each had a 6/11-bar, then together they would have a 12/11-bar and that that bar would be 1/11 more than the bar. It was remarkable because they seemed explicitly aware of their reasoning and of the elements on which they operated. This inference of the children's awareness is based on Melissa saying that "It would be one more than the bar." in reply to the teacher's question, "What if you both had six elevenths?" and on Joe knowing that it would be twelve elevenths as well. The children seemed aware of performing progressive integration operations (an operation of adding) using two composite units each containing six 1/11-bars. In that one of these composite units was not in the children's visual field, the children worked in re-presentation when performing progressive integration operations. That is, the children operated on elements in visualized imagination, which supports the inference of awareness. Joe had made progressive integrations with fractional parts of pizza sticks in the previous protocol, 9 months prior to this episode. Applying these operations to the decontextualized fraction-bars in this episode strongly suggests that he had constructed a fractional number sequence and was aware of the operations he could perform using this number sequence.

Another indication of their explicit awareness of how they operated is that they mentally produced a 1/11-bar that was a part of the twelve elevenths bar they mentally established but was not contained in the 11/11-unit bar. Mentally producing this 1/11-bar was a major achievement for the children and saying that the 12/11-bar was one-eleventh more than the 11/11-bar stood in for actually producing this 1/11-bar. Our judgment is that the children's comment symbolized producing the 1/11-bar, which would entail an awareness of the involved operations. Joe had first exhibited this awareness in the previous year when he stated that a 6/5-stick would be one more fifth than a unit stick.

The inference that the children were explicitly aware of the numerical whole-to-part relation between 12/11 and 11/11 as well as of the status of each 1/11-bar contained in the 12/11-bar as a unit fractional part of the 11/11-bar is corroborated by their knowing that sixteen elevenths was five elevenths more than the bar (the 11/11-bar). Producing five elevenths further indicates that one eleventh had become an iterable unit for the children that was on a par with their iterable unit of one. That is, the children could use one eleventh as they used the unit of one in the case of their explicitly nested number sequence. The children had constructed a CNS of which one eleventh was the basic unit element that was analogous to their explicitly nested number sequence for one, and they could operate with it in a way that was also analogous to how they operated with their explicitly nested number sequence involving the unit of one. It is in this sense that we find confirmation for Joe (and Melissa) in Protocol VI for their construction of a *FCNS*.

The claim that the children could operate with their FCNS for one-eleventh in a way that was analogous to how they operated with their explicitly nested number sequence for one, finds corroboration in how the children operated after the teacher asked them the incomplete question, "If you took eight of these two elevenths. . . ?" Melissa almost immediately replied "Sixteen elevenths!" and both children knew that this result was five elevenths more than the unit bar. The children's way of operating indicates not only that two elevenths was an iterable composite fractional unit for the children, it also indicates that the children had made an accommodation of their units-coordinating scheme for composite units. They could now iterate two elevenths eight times and produce sixteen elevenths as the result.

It is quite impressive that the children also knew that eleven 4/11-bars would yield forty-four elevenths. This knowledge is another corroboration of the inference that the children had made an accommodation of their units coordinating scheme for composite units. It is especially impressive that both Melissa and Joe knew that there were four unit bars in a 44/11-bar, "Because four times eleven is forty-four!" They were aware that if they iterated a 4/11-bar eleven times, they would produce a 44/11-bar. They were also aware that they could produce this 44/11-bar by iterating the 11/11-unit bar four times, "Because four times eleven is forty-four!" Although this knowledge may have been based on a functional interchange of the number of iterations and the number of elements in the composite unit being iterated (rather than on operations that produce an awareness of commutativity) it still indicates an accommodation of their units-coordinating scheme for composite units. This accommodation, together with their progressive integration operations with fractional numbers, and their ability to iterate composite unit fractions, indicate that they have constructed a *FCNS* that is on a par with their explicitly nested number sequence.

6. Final word

In reviewing the sequence of advances that Joe made during the 2-year period covered by this paper, and contrasting these advances with the constraints that Jason and Laura experienced, it becomes clear that Joe's operations with composite units were critical building blocks for his construction of fractional numbers. Having abstract composite units available prior to activity which he could use as if they were units of one in strategic reasoning made it possible for him to compose his iterating and partitioning operations and thus achieve a splitting operation for composite units as indicated by his use of 8 times 7 to figure out 7 times 7. Moreover, he was able to apply this splitting operation for composite units in the context of connected numbers, thus generating a splitting operation for connected numbers.

These splitting operations provided Joe with the means by which he could establish a unit fraction both as a part of a partitioned whole and as a part *freed from the whole*. Unit fractions had achieved the same status as the iterable ones of his ENS. He could thus apply the iterating and progressive integration operations of his ENS to these new iterable unit fractions, to produce composite unit fractions that were not constrained to the fractional whole. Joe was able to reflect on the results of these operations with unit fractions and so became aware of the composite units and their multiplicative relations both to the unit fraction and to the fractional whole. It is this awareness that made possible Joe's construction of an iterative fractional scheme and, consequently, to the construction of a FCNS. Joe's case, we believe, provides confirmation for our reorganization hypothesis, whereby children's fractional schemes can emerge from a reorganization of their number sequences (Olive, 1999; Steffe, 2002).

The following diagram (Fig. 8) illustrates a possible progression of fractional operations and schemes, along with the results of those schemes, emanating from children's explicitly nested number sequence. The different fill patterns for each group of icons represent major stages in this progression of constructive activity. While this diagram culminates in the production of the FCNS, children's productive activity with fractions does not end there. For example, during the remainder of Joe's fifth grade year (the final year of the teaching experiment) Joe went on to construct schemes for composing fractions (fractions of fractions) and for making multiplicative comparisons among fractional quantities (e.g., finding how many 1/144 would make 1/48 of a unit bar). A detailed analysis of similar constructions made by Arthur and Nathan (another pair of children in the teaching experiment) can be found in Olive(1999).

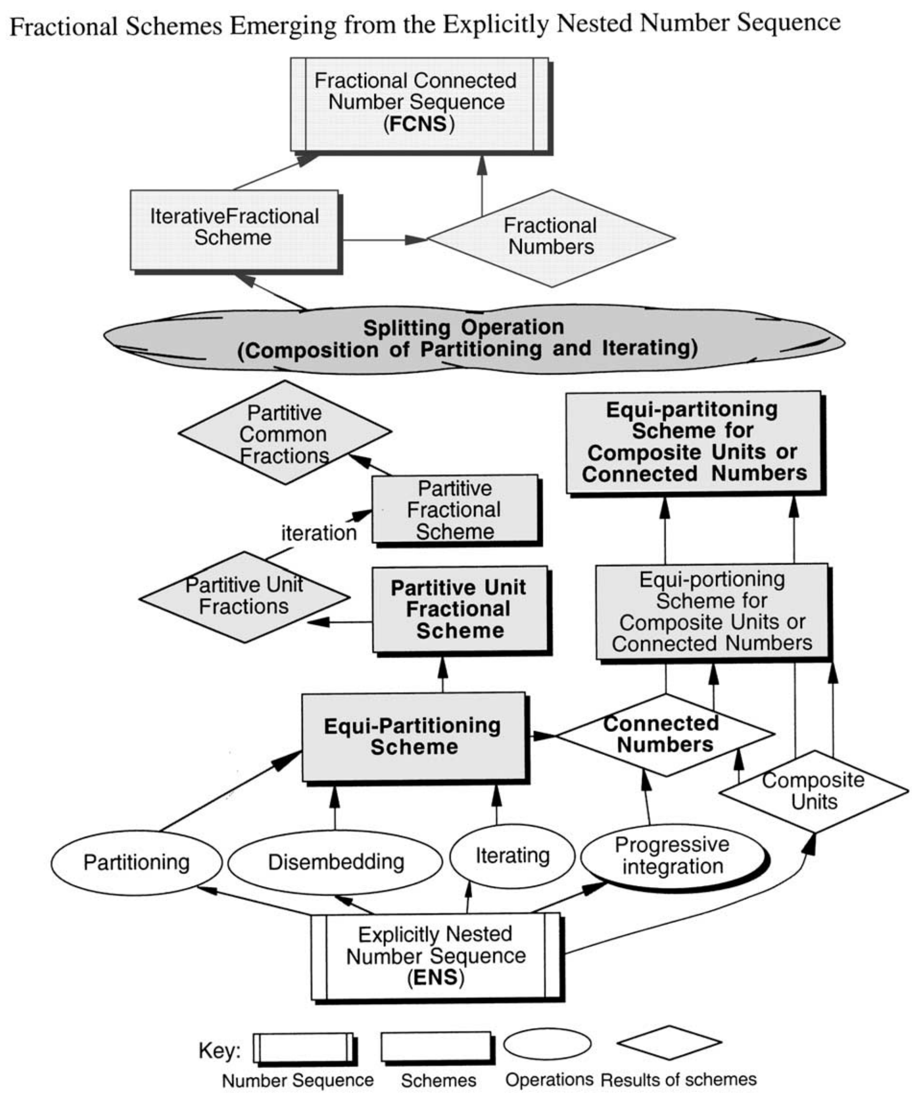

Fig. 8. A constructive itinerary from the ENS to a FCNS.

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