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# Explicating a Mechanism for Conceptual Learning: Elaborating the Construct of Reflective Abstraction

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We articulate and explicate a mechanism for mathematics conceptual learning that can serve as a basis for the design of mathematics lessons. The mechanism, reflection on activity-effect relationships, addresses the learning paradox (Pascual-Leone, 1976), a paradox that derives from careful attention to the construct of assimilation (Piaget, 1970). The mechanism is an elaboration of Piaget's (2001) reflective abstraction and is potentially useful for addressing some of the more intractable problems in teaching mathematics. Implications of the mechanism for lesson design are discussed and exemplified.

*Key words:* Cognitive development; Conceptual knowledge; Constructivism; Epistemology; Learning theories; Piaget

> The difficulty of studying learning—and teaching—lies, in my view, in the fact that it demands the study of the processes by which children come to know in a short time basic principles (in mathematics, but also in other scientific disciplines) that took humanity thousands of years to construct. (Sinclair, 1990, p. 19)

The current mathematics education reform in the United States, heralded by the publication of the *Curriculum and Evaluation Standards for School Mathematics* (National Council Teachers of Mathematics [NCTM], 1989), has resulted in a large-scale movement away from direct instruction, leaving the field of mathematics

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education with the challenge of articulating new approaches to mathematics teaching. Steffe and Wiegel (1994) raised the question, "Is there a model of mathematical learning that is powerful enough to be useful to mathematics teachers?" (p. 117). In this article, we elaborate a mechanism for mathematical concept development. In so doing, we offer further elaboration of Piaget's (2001) reflective abstraction and Simon's (1995) hypothetical learning trajectory. Our inquiry into conceptual learning has been guided by the explicit goal of identifying constructs that can provide a basis for reconceptualizing aspects of mathematics teaching. The mechanism for conceptual development that we articulate in this article is a result of our interpretation, synthesis, and extension of existing literature and our longitudinal studies of student learning (cf. Simon & Blume, 1994; Tzur, 1999).

## OVERARCHING THEORETICAL FRAMEWORK

In this article, we articulate theoretical constructs that are built on radical constructivist interpretations of knowing and learning (e.g., von Glasersfeld, 1995). We highlight three key principles of radical constructivism that are basic assumptions of our work:

1. Mathematics is created through human activity. Humans have no access to a mathematics that is independent of their ways of knowing.
2. What individuals currently know (i.e., current conceptions[1]) affords and constrains what they can assimilate—perceive, understand.
3. Learning mathematics is a process of transforming one's ways of knowing (conceptions) and acting.

The theoretical discussion that follows is grounded in the assumption that mathematics educators can make significant use of the ground-breaking research done by Jean Piaget and his associates. Piaget was not focused on the challenges of mathematics education. However, careful selection from, interpretation of, and building on Piaget's work can contribute to the theoretical foundation for mathematics teaching. To what extent we are selecting from and explicating aspects of Piaget's work and to what extent we are building on this work will be viewed differently by different readers. Our intention is that the result of our efforts supports mathematics educators to think productively about mathematics learning and teaching.

## BUILDING AN EXPLANATION OF CONCEPTUAL LEARNING: UNDERSTANDING THE PROBLEM

Promoting the transformation of learners' current understandings toward the development of more advanced understandings is an inherently problematic aspect

[1] We use the term "conceptions" to refer to cognitive entities that, in the constructivist literature, have been termed "schemes," "conceptual structures and operations," and "mental objects."

of mathematics teaching. Better understanding learning processes, in particular how learners develop new conceptual entities (e.g., number, division, ratio, function, variable), could enhance progress in this area. Thompson (1985) argued, "Little attention has been given to the issue of the development of mathematical objects in people's thinking" (p. 232). Dubinsky (1995) noted: "Our curriculum generally asks students to understand logical arguments establishing relations between mathematical objects that are not, for the students, objects at all" (p. 2). The work reported here is motivated by the challenge of explicating the development of new conceptual entities. The key question is, *How can learners construct mathematical conceptions beyond those that are currently available to them?*

We clarify the theoretical (and practical) problem that we are addressing in two ways. First we discuss a common pedagogical adaptation of constructivism, engendering cognitive conflict, and consider its limitations and the inadequacy of the explanation of learning underlying it. Then we focus on the theoretical challenge presented by the learning paradox.

### *Attempts at Using Constructivist Theory: Engendering Cognitive Conflict*

One construct that is frequently highlighted in constructivist theory is that learning—conceptual transformation—is triggered by a disequilibrating experience (perturbation) for the learner. The focus on perturbations seems to be based on interpretations of Piaget's equilibration theory, such as the one offered by von Glasersfeld (1995):

> The learning theory that emerges from Piaget's work can be summarized by saying that cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or reestablishes equilibrium. (p. 68)

Perturbation is commonly understood as cognitive conflict, that is, learners' experiences of an event not fitting with their current conceptions or lack of fit among the conceptions they hold. Based on this idea of cognitive conflict as the trigger for learning, it seems to be a logical extension to think about the teacher's role as including intentional actions aimed at provoking cognitive conflict with respect to particular student conceptions (Gruber & Voneche, 1977). Let us look more closely at this idea.

If a teacher attempts to provoke cognitive conflict, three results are possible: (a) the students experience the intended conflict, and it results in the learning intended by the teacher; (b) the students experience the intended conflict, but the conflict does not result in learning or at least not the learning that was intended by the teacher[2]; (c) the students do not experience the intended conflict. These three possibilities are in line with Piaget's (1977) assertion that "A device is not a disturber in itself, but, on the contrary, is conceived as a disturbance or is not one according to the

[2] "Learning by the teacher" should not be interpreted as learning the teacher's conception. Rather, it indicates that the students develop a compatible conception.

elements that have been acquired by the structure in formation" (p. 39). In order for the first result to occur—that is, the students experience the intended conflict, and it results in the learning intended by the teacher—the learner must have schemes that allow for a compatible interpretation (assimilation) of the situation to that of the teacher (i.e., a recognition of the conflict) *and* schemes that can be (and are) accommodated to construct a new understanding compatible with that of the teacher. Vinner (1990) concluded:

> If at first it appears that inconsistencies can be very helpful in the learning of mathematics, . . . this is not necessarily the case. It is true that a student will try to accommodate a recognized contradiction. But this will happen only if the student is convinced there is a contradiction. Secondly, even if the student recognizes a contradiction and tries to accommodate, there is no guarantee that the accommodation will be in the desired direction. (pp. 91–92, cited in Steffe & D'Ambrosio, 1995, pp. 148–149)

Engendering cognitive conflict is a useful teaching approach when it works, because not only do the learners make a conceptual advance, they do so relatively autonomously; they determine how to deal with the conflict. However, as noted, attempts to provoke cognitive conflict do not necessarily result in the intended learning. (An example of this is described in Simon, 1995.) Inhelder, Sinclair, and Bovet (1974) reviewed experiments in which children were able to experimentally determine that their predictions were incorrect. They concluded:

> Such experiments . . . clearly showed that while readings of experimental situations can to a certain extent facilitate the understanding by means of a simple abstraction of some of the physical properties of objects, such readings do not ipso facto lead to the formation of operatory structures. As Piaget hypothesized, these structures, particularly in the case of logical and mathematical operations appear to be the product of the subject's own coordination of actions, which is carried out by means of a process of reflective abstraction. (p. 13)

In subsequent sections of this article, we discuss how reflective abstraction is a powerful explanation of conceptual learning and elaborate on it in an attempt to make it useful to mathematics education.

Disequilibrium, as an explanation for learning, is not a sufficient basis for elaborating theories of teaching for two reasons. First, as already indicated, disequilibrium does not necessarily foster learning in the desired direction. Second, it does not explain how the change in the learner's conceptions takes place, how the accommodation is made by the learner (Smith, diSessa, & Roschelle, 1993). Explanation of the process by which changes in conceptual structures take place would guide mathematics educators in designing situations to foster specific conceptual changes. This is the goal of the work reported in this article.

If equilibration theory (without reflective abstraction) does not offer mathematics education sufficient explanation of conceptual learning, why did it figure so prominently in Piaget's early work? We offer two (overlapping) reasons. First, Piaget (1952) developed his theory based on a biological model:

> My aim of discovering a sort of embryology of intelligence fit in with my biological training; from the start of my theoretical thinking I was certain that the problem of the

> relation between the organism and the environment extended also into the realm of knowledge. (p. 245)

Piaget's equilibration theory is a specific solution to "the problem of the relation between the organism and the environment."

Second, Piaget initially was interested in broad intellectual development characterized by his stages of development. Equilibration theory was of a grain size useful for initial characterization of development from one stage to another. Sinclair (1987) pointed out:

> When Piaget and his collaborators devised their well-known conservation tasks, their main concern was the formation of cognitive structures.... The mechanisms by which the structures change did not become a topic of research and theorizing until much later. (p. 12)

Campbell (2001) reported:

> Piaget and his collaborators did not get around to a focused investigation of reflecting abstraction[3] until he had been elaborating and testing his theories of human development for over 50 years. Reflecting abstraction did not even appear as a theoretical concept until 30 years into this program of research. (p. 2).... Even a fully elaborated treatment of equilibration leaves something out. . . . This is where *abstraction réfléchissante,* reflecting abstraction, comes into Piaget's theory. (p. 4)

For Piaget (2001), reflective abstraction was a characterization of the mechanism of equilibration that is "constructive, not merely inductive or extensional" (p. 315).

The theoretical work that follows builds particularly on two key elements of Piagetian theory—assimilation and reflective abstraction. Our theoretical enterprise rests in part on two issues: identifying those aspects of Piagetian theory on which a theory of mathematics learning can be built; and determining a level of detail for the elaboration of a mechanism to explain conceptual learning that is sufficient to inform mathematics teaching.

### *Identifying the Pedagogical Challenge: The Learning Paradox*

Building on constructivism, we eschew the notion that learners can take in new (to them), more powerful concepts and embrace the idea of learning as an internal process of construction. This quest, however, puts us face-to-face with what has been called the *learning paradox* (Pascual-Leone, 1976)—the need to explain how learners "get from a conceptually impoverished to a conceptually richer system by anything like a process of learning" (Fodor, 1980, p. 149 cited in Bereiter, 1985). This is conceived of as a paradox for the following reason. Piaget's (1970) idea of assimilation, a core idea of constructivism, suggests that one needs to have

[3] The French term *abstraction réfléchissante* used by Piaget, is translated as "reflective abstraction," "reflecting abstraction," and "reflexive abstraction." Here we use the term *reflective abstraction*, except in cases where we are quoting or paraphrasing the work of others who used one of the equivalent terms.

concept X in order to make sense of one's experience in terms of concept X. That is, among the vast set of sensory-motor signals that bombards the organism's senses, only those that can be incorporated into structures and operations already available in the mental system evoke particular cognitive responses. Building on Piaget's work, von Glasersfeld (1995) contended that assimilation functions like a card-sorting mechanism that is structured to detect a certain arrangement of holes. Assimilatory conceptions afford and constrain (a) what the mental system can recognize, and (b) the activities it can trigger to accomplish a goal. If a certain conception is not available, the learner cannot recognize a situation in the way someone who has that conception can. Thus, a child who has no conception of multiplication will not perceive multiplicative relationships in any situation, including those considered by the teacher to transparently display multiplication (e.g., an array).

This understanding of assimilation seems to imply a vicious cycle (i.e., the learning paradox). In order to experience a new concept in the world, one must already have that concept available to organize that experience. But if one cannot experience a new concept, how can one acquire a concept that is not already a part of the mental system? In other words, how can learning of new conceptions be explained *without* attributing to learners prior assimilatory conceptions that are as advanced as those to be learned? Bereiter (1985) articulated the importance of the learning paradox as a problem for education:

> The learning paradox descends with full force on those kinds of learning of central concern to educators, learning that extends the range and complexity of relationships that people are able to take account of in their thought and action—the kinds of learning that lead to understanding core concepts of a discipline, mastering more powerful intellectual tools and being able to use knowledge critically and creatively. . . . The practical payoff in taking the learning paradox seriously is that it may lead to the development of educational strategies that are commensurate with the complexity of the task that learners face. (p. 202). . . . The areas in which instruction has proved most uncertain of success have been those areas in which the objective was to replace a simpler system by a more complex one. (p. 217)

### *Attempts at Addressing the Learning Paradox*

Chomsky (1975) and Fodor (1975), finding no answer to the learning paradox in the work of Piaget, argued that cognitive structures are innate. Bickhard (1991) refuted this *radical innatism* by arguing that if the emergence of new representations can emerge through evolution, then they must be able to emerge through development. Sfard (2001) found a constructivist perspective to be useful, but suggested that the unanswered questions might be addressed by complementing it with a "thinking-as-communicating" (p. 13) perspective. Bereiter (1985) found no answer to the learning paradox either in constructivism or sociocultural theory:

> This "constructivist" view of learning and development . . . is in trouble theoretically. It seems to be generally agreed that there is no adequate cognitive theory of learning—that is, no adequate theory to explain how new organizations of concepts and how new and more complex cognitive procedures are acquired (p. 202). . . . The whole paradox hides in the word "internalizes." How does internalization take place? It is evident from

> Luria's first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved. (p. 206)

However, Bereiter did not find the argument for innate mental structures to be compelling either. He pointed out that from the perspective of education, the key question is the role of experience in the development of cognitive structures. Seeing no answers on the horizon, he (Bereiter, 1985) identified ten human resources that might be implicated in an answer to the learning paradox.

Bickhard (1991) argued that the solution to the learning paradox "is intrinsic in the nature of interactive constructivism. . . . The *constructive* relationship between one level and the next higher level is essentially that of Piaget's notion of reflective abstraction (Campbell & Bickhard, 1986)" (p. 24). Smith, diSessa, and Roschelle (1993) asserted, "Like Piaget, we accept that a major task for a constructivist theory of learning is to present a psychologically plausible resolution of the learning paradox" (p. 124).

Following Bickhard and Smith et al., we believe that Piaget indicated the direction for a solution to the learning paradox. We began with the following observations that overlap with those made by Bereiter and others:

1. While a small group of educational researchers continue to work within a Piagetian theoretical framework, many researchers remain unconvinced that Piagetian theory offers a solution to the learning paradox and a useful basis for the design of educational interventions.
2. Piagetian theory has not been used as a basis for an adequate scientific approach to teaching mathematics. By *scientific* we mean a theory-based approach to designing lessons that can promote the development of particular new (to the learner) conceptual entities.

In the discussion that follows, we suggest an elaboration of Piagetian constructs that can contribute to a theoretical explanation for how humans construct more advanced conceptions from the conceptions that they already have, an explanation that can serve as a basis for articulating a role for pedagogy in promoting such learning processes.

## A STARTING POINT FOR EXPLAINING MATHEMATICS CONCEPTUAL LEARNING: PIAGET'S REFLECTIVE ABSTRACTION

### *The Challenge of Building on Piagetian Theory*

The work we present in this article builds on Piagetian theory in major ways and also elaborates constructs not represented in Piagetian theory. Although we try throughout this article, it is difficult to make clear where Piagetian theory leaves off in the ideas we are presenting. There are several reasons for this difficulty. First, Piaget's ideas are spread out over time and a plethora of publications. His ideas and emphases changed as he worked on different problems. Kuhn (1979) argued, "Uncertainties that one encounters in endeavoring to apply [Piaget's theory of cognitive development] to education are very revealing of the ambiguities that exist within the theory itself" (p. 340). Second, researchers and scholars have significantly different interpretations of Piaget's work. Thus, there is no shared interpretation on which to build. Third, Piaget's purposes and emphases were different from ours. His focus was on demonstrating the self-regulatory nature of intellectual development, thus the biological model. He worked at a time in which constructivist ideas had no real foothold in the field of child development; he was challenging and challenged by maturationists, empiricists, and behaviorists. His work offered a foundation for thinking about children's thinking as being qualitatively different from the thinking of adults. Piaget (2001) reminded us that his research "was not done with any pedagogical purpose in mind" (p. 32). Our purpose is to contribute to an explanation of conceptual learning in mathematics that can serve as a basis for mathematics teaching.

We believe that the ideas we are presenting are both compatible with many of Piaget's ideas and impart some new theoretical possibilities. Nonetheless, we are aware that we risk responses from scholars that range from "this is not consistent with Piaget at all" to "there is nothing new here—Piaget covered all of this."

### *Reflective Abstraction: A Key Concept*

Piaget (2001) distinguished among three types of abstraction: empirical, pseudo-empirical, and reflective. Empirical abstraction refers to the generalization of properties of objects. Because empirical abstraction is only indirectly related to the development of mathematical concepts, it is not the focus of our discussion. Piaget (2001) indicated that "[Pseudo-empirical abstraction] is really a special case of reflecting abstraction" (p. 303). In our discussion of reflective abstraction, we do not distinguish the special case of pseudo-empirical abstraction.

Piaget (1980) postulated reflective abstraction as the process by which new, more advanced conceptions develop out of existing conceptions:

> All new knowledge presupposes an abstraction, since, despite the reorganization it involves, new knowledge draws its elements from some preexisting reality, and thus never constitutes an absolute beginning. (p. 89) . . . [Reflective abstraction] alone supports and animates the immense edifice of logico-mathematical construction. (p. 92)

Many researchers and theorists have recognized reflective abstraction as an essential focus for educators. Steffe (1991) considered reflective abstraction "to be incredibly useful as a guiding heuristic in a search for insight into mathematical learning" (p. 43). Reflective abstraction is central to Thompson's (2000) use of didactical objects. Gallagher and Reid (1981) maintained that "seeking ways to facilitate reflexive abstraction is the key to fostering growth" (p. 175). Dubinsky (1991) asserted, "When properly understood, reflective abstraction appears as a description of the mechanism of the development of intellectual thought" (p. 99). Brun (1975) considered the goal of teaching to be promoting reflexive abstraction.

By introducing the construct of reflective abstraction, Piaget established the important idea that a reflective process is key to the development of new conceptions. He described reflective abstraction as having at least two components and categorized the results of reflective abstraction. It is our contention that the mechanism itself is underspecified for guiding the design of instructional interventions intended to address challenging learning problems in mathematics. Here we seem to be in agreement with Cohen (1986): "The problem with Piaget's two part description of reflective abstraction is that it may not capture the process sufficiently to be useful for either research or educational purposes" (p. 6).

Piaget (2001) postulated reflective abstraction, the process by which higher level mental structures could be developed from lower level structures, and described it as having two phases: a projection phase in which the actions at one level become the objects of reflection at the next and a reflection phase in which a reorganization takes place[4]:

> "Reflecting" abstraction ranges over . . . all of the subject's cognitive activities (schemes or coordinations of actions, operations, cognitive structures, etc.). Reflecting abstraction separates out certain characteristics of those cognitive activities and uses them for other ends (new adaptations, new problems, etc.). It is "reflecting" in two complementary senses. First, it transposes onto a higher plane what it borrows from the lower level (for instance, in conceptualizing an action).
>
> We call this transfer or projection a *réfléchissement*. Second, it must therefore reconstruct on the new level *B* what was taken from the previous level *A*, or establish a relationship between the elements extracted from *A* and those already situated in *B*. This reorganization that is forced by the projection will be called a reflection [*réflexion*] in the strict sense. (Piaget, 2001, p. 30)

Further, Piaget (2001) indicated that reflective abstraction is not necessarily a conscious process.

In his volume on reflective abstraction, Piaget (2001) presented research in the areas of logico-arithmetical and algebraic abstraction, the abstraction of order, and the abstraction of spatial relationships. In each case, he described 5 substages (IA, IB, IIA, IIB, III) based on his interviews with a range of learners. For each substage, he distinguished the particular abstractions characteristic of that substage. By specifying distinctions in the learners' abstractions, Piaget and his colleagues offered considerable precision with respect to the endpoints of conceptual development processes. However, this research did not explain the transitions from the abstractions of one substage to the next.

Inhelder et al. (1974), in summing up contributions of Piaget's genetic epistemology wrote, "Little is as yet known about the mechanisms of transition from one major stage to the next and about the passage between two successive substages" (p. 14). They laid out the challenge of answering the question, "What are the laws of learning that at each developmental level account for the acquisition and modification of knowledge?" (p. 14). Piaget's empirical work did not involve longitu-

[4] Piaget (2001) also defined "reflected abstraction" and "metareflection" as higher levels in which the process of reflective abstraction operates on the product of a prior reflective abstraction. This distinction is not essential to the focus of this article.

dinal studies of children. He was able to investigate different levels of conceptualization by studying different learners at different points of development. This methodology did not enable and was not focused on studying the transition process in detail.

The postulation of reflective abstraction was a significant contribution for it sketched out a solution to the learning paradox, describing the kind of process that can derive more advanced structures from those at a lower level. However, the problem that motivated our work is, *What elaboration of conceptual development in general and reflective abstraction in particular would allow this contribution to genetic epistemology to serve as a basis for the design of situations for mathematics learning?* We sought an explanation that could guide the process by which a mathematics educator (teacher, curriculum designer, researcher) begins with a particular concept to be taught and designs a set of mathematical tasks, to be carried out by the learners, leading them to reflectively abstract the concept.

Recent efforts to ground mathematics pedagogy in Piagetian theory have been hampered, in our estimation, by the lack of a sufficiently elaborated mechanism for explaining mathematics conceptual development. We cite two well-known and valued works as examples: Gallagher and Reid (1981) and Dubinsky (1991). Gallagher and Reid's (1981) purpose was "to highlight those aspects of [Piaget's] theory that might prove most relevant to professionals concerned with applying Piaget's ideas in order to better understand and guide children's learning" (p. 172). They recognized that "The important question in learning is *how* something occurs. Thus, our major concern has been with the mechanisms of transition" (p. 172). A significant part of Gallagher and Reid's discussion focused on equilibration in general. Their treatment of reflective abstraction was limited to a description of the two phases of reflective abstraction and categories for the abstractions that result.

Dubinsky (1991) made a case for reflective abstraction as a "theoretical basis . . . [for] how we can help students develop [advanced mathematical thinking]" (p. 95). Like Gallagher and Reid, he explained the two phases of reflective abstraction and the categories of constructions produced by reflective abstraction. Dubinsky went on to describe the three kinds of abstraction discussed by Piaget—empirical, pseudo-empirical, and reflective—and the relationship among them, and quoted Piaget in referring to reflective abstraction as *general coordinations* of actions. Although Dubinsky's instructional approach is organized to "induce students to make specific reflective abstractions" (p. 123), we argue that a further elaboration of the mechanism of reflective abstraction is needed in order to guide the generation of instructional situations.

The problem of what would constitute a useful elaboration of reflective abstraction is a difficult one, because it involves determining a level of specificity appropriate to the work of mathematics educators. Piaget was interested in explaining the transition within and between major developmental stages. Mathematics educators are interested in promoting particular mathematical understandings, a challenge that makes different demands on an elaboration of a mechanism of learning.

## POSTULATING A MECHANISM FOR LEARNING A NEW CONCEPT: ELABORATING REFLECTIVE ABSTRACTION

In this section, we introduce a way of explaining conceptual learning that addresses the learning paradox and can contribute to a basis for the design of mathematics instruction. We introduce these ideas in the context of an example and then consider the theoretical issues in greater detail.

### *Developing a Concept of Fraction as a Quantity*

In this example, we focus on *part* of the learning process—a part that is sufficient to illustrate the mechanism that we are proposing. Our own work and the work of Piaget and others suggest that the development of a concept is a multistage process. We are currently working on a stage account of conceptual learning that uses the mechanism that we explicate here to explain transitions between stages.

Our example focuses on development of understanding that a fraction specifies a particular unit relative to the whole (Simon, 2002; Tzur, 1999). To create an image of a student who does not yet have this concept, consider the following description of a conversation with Micki (pseudonym), age 9 years.

Micki is shown two identical square pieces of paper and told that they are identical cookies. She agrees that they are the same size and responds that it would not matter which one she chose to eat. Each piece of paper is then cut in half. The first cookie, labeled as Sam's cookie, is cut vertically, the second, labeled as Anne's cookie is cut diagonally. Representations of these cookies appear in Figure 1. Micki is first asked, "If you like these cookies, which of the pieces of Sam's cookie would you rather have or wouldn't it matter?" Micki determines, through superimposing one part of Sam's cookie on the other, that they are exactly the same size, that it would not matter which part she eats, and refers to each of the two parts as "half." She does the same for Anne's cookie. She is then asked, "If you like these cookies, would you rather eat one of the pieces of Sam's cookie, one of the pieces of Anne's cookie, or wouldn't it matter?" She says that she would rather eat one of the pieces of Anne's cookie because "it is bigger." She indicates what aspects of her perception of the two shapes tell her that one of the pieces of Anne's cookie is larger than one of the pieces of Sam's cookie.

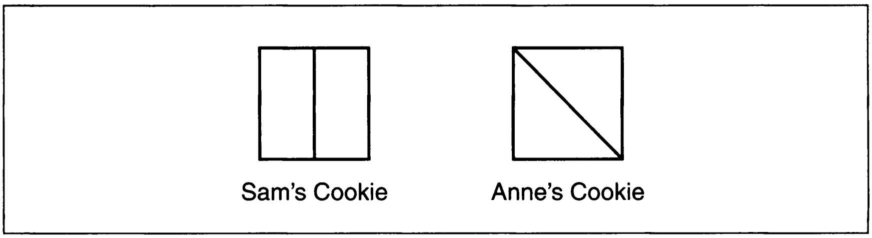


*Figure 1.* The Cookies problem

This description is based on an interview with an elementary school student conducted by the second author. The claims made by Micki are identical to those we have collected from many fourth- and fifth-grade students. These are students who understand conservation of area. Not only do they claim that the triangular half or the rectangular half is larger, but they usually cannot be talked out of their position by other students who defend the position that all of the halves must be the same size.

We understand the fraction conceptions of these students in the following way: Halves are produced when an object is partitioned into two *congruent* parts. That is, halves are parts of an arrangement in which twin subsections are created. We refer to these students' conception as a "fraction as an arrangement," because a fraction is seen as the arrangement in which an object is subdivided into identical parts.[5] What these students do not yet understand is that partitioning a unit into equal-sized portions, a special case of subdividing, creates a new unit of quantity that has a specific size relative to the original unit.

The mathematical teaching example that follows is not a report of data per se. Rather, it is a composite of data sources. Micki, the child interviewed by the second author, was not one of the students in the teaching experiment (see Tzur, 1996) on which the example is based. By combining the learning activity of several students in the teaching experiment with the interview of Micki, we have created an example that we believe clearly illustrates the mechanism for conceptual learning that we are elaborating.

We use this description of a pedagogical intervention to examine the mechanism of the student's learning of an aspect of the concept of a fraction as a quantity, that is the *development of an understanding that equal partitioning produces a particular unit (relative to the whole)*. The intervention involves mathematical tasks in a computer microworld. Micki has a set of whole-number concepts that can be thought of as a basis for beginning to construct this understanding. She understands that if one combines two or more lengths or areas (i.e., iterates a length or an area), that the iteration produces a new unit of a particular size relative to the original unit, which is what researchers call a *composite unit* (e.g., she knows that 10 little cubes can be combined to make a rod that is 1 unit of 10). However, as described above, she does not yet understand that equal partitioning produces a particular unit relative to the whole.

In addition to the elements of Micki's knowledge already noted, aspects of Micki's prior experience are also essential contributors to her learning. Micki has experience using a computer microworld program called Sticks (Steffe, 1993; Tzur, 1999) that allows her to create sticks of various lengths (line segments with marked endpoints), copy them, and iterate them multiple times end to end to make a longer stick (with the original endpoints still visible). From her previous work with

[5] Thompson and Saldanha (2003) described this limited view of fractions as an additive rather than a multiplicative relationship.

Sticks, Micki has an activity sequence for producing a stick of $x$ number of equal parts; she makes a small stick and iterates it $x$ times. We refer to this as the *repeat strategy*.

Micki is asked to draw a large stick. She is then given the task of cutting it into 5 equal parts and approaches the task in the following way.[6] She makes a small stick under the large stick. She iterates the small stick 5 times and notices that the stick she has just composed is shorter than the original stick, as shown in Figure 2. Micki makes a new small stick, longer than the stick she iterated the first time, and iterates it 5 times. She compares the resulting stick to the reference stick, the original. The adjustment continues until she is satisfied that she has produced a composite stick exactly the size of the original. She is then given additional tasks in which she makes a large stick and is asked to partition it into a particular number of parts (e.g., 7 and 11).

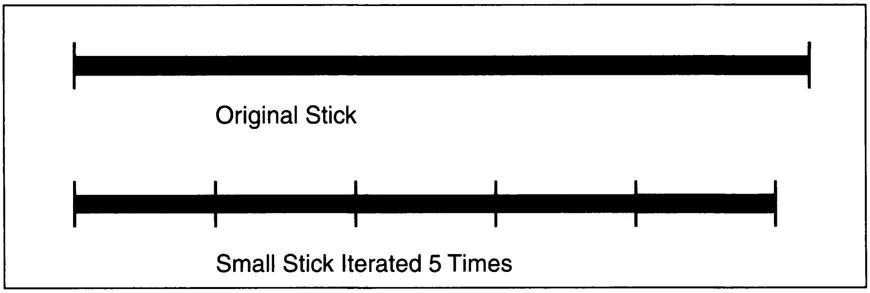


*Figure 2*. Problem from the Sticks microworld

We amplify the description of Micki's activity by inferring aspects of Micki's thinking. When Micki first attempts to subdivide the stick by iterating a smaller stick, she does so because she is able to anticipate that she needs to create a stick, identical to the original, that is made up of a smaller stick iterated 5 times. This anticipation is based on her conception of a fraction as an arrangement and her conceptions of iterating parts in the context of the Sticks microworld. What she cannot anticipate at this point is that there is a unique size for the part that when iterated 5 times produces the target length, and more important, there is no need for her to have this anticipation in order to successfully complete the task.

[6] Micki's strategy is not the initial approach of most learners when they first use Sticks. If the strategy of using iteration of a part does not occur spontaneously, the teacher can intervene to promote this strategy. Promoting this strategy has proved to be unproblematic. In this example, we assume that Micki already has the strategy available.

Micki begins by creating a small stick and iterating it 5 times. She creates a stick that is shorter than the stick she was trying to subdivide. Her prior conception of composite unit allows her to deduce that the length of the large stick that she created was determined by the length of the small stick and the number of times that she iterated it. Because the stick she made was too short, Micki lengthens the small stick and repeats the iteration and comparison steps. She continues to make such adjustments until she creates a composite stick equal to the original. In subsequent tasks, this process is repeated with different size sticks and different numbers of parts. Through reflection on her activity and the effects of her activity, Micki comes to develop the (perhaps implicit) understanding that partitioning a stick into a given number of parts produces a part of a particular length relative to that whole. We now explicate the mechanism that accounts for this last claim, drawing examples from this description of Micki's work in the Sticks software environment.

### *The Mechanism*

Following Piaget, we attribute development of a new conception to a process involving learners' goal-directed activity and natural processes of reflection. Development of a new conception begins with the learners setting a goal. During instruction, the learners' goal setting is often related to a task that has been posed by the teacher. The goals that learners can set are a function of their current conceptions. The *learners' goal*[7] (which might be implicit) should not be confused with the *teacher's goal* for student learning. For example, the teacher might introduce a game to foster a particular mathematical concept. The learners' goal would be to win the game; the teacher's goal is to foster development of the mathematical concept. In our example, Micki's goal is to subdivide the stick into the appropriate number of equal parts.

Having set a goal, the learners call on one of their available activities (or a set of activities) in an effort to meet the goal. The activity embodies a set of the learners' current conceptions. Micki's activity sequence can be summarized as follows:

1. Create a small stick to be one part.
2. Iterate the part creating a composite length.
3. Compare the composite stick to the original (the whole).

As they engage in these activities, the learners attend to the *results* of their goal-directed activity, distinguishing between positive results of their activity (closer to their goal) and negative results (farther from their goal). The ability to set the goal subsumes an ability to judge the results (e.g., having the goal of winning a game subsumes an ability to consider whether particular results are leading toward victory). The claim here is that learners can make distinctions of this type

[7] We use "goal" in the singular to refer to the goal that is most relevant to the learning. Certainly learners' actions are governed by multiple goals at any point in time.

consciously or unconsciously. However, the distinctions they make might be different from or less sophisticated than those made by a more knowledgeable observer. Micki distinguishes between positive results, closer to the length of the target stick, and negative results, farther from the length of the target stick.

As learners employ their activity sequence, they make goal-directed adjustments on the basis of the results they are noticing. These adjustments, like the original activities, embody current conceptions. We refer to these adjustments as the *effects* of the activities. For Micki, each effect of her activity is a composite-unit based adjustment. That is, she uses her conception of composite unit to adjust the size of the estimated (iterated) part.

To this point, we have identified the following components of the mechanism: the learners' goal, the activity sequence they employ to try to attain their goal, the result of each attempt (positive or negative), and the effect of each attempt (a conception-based adjustment). Each attempt to reach their goal is preserved as a mental record of experience (von Glasersfeld, 1995). We offer the following physical metaphor to promote an image of the records of experience and how they are used in reflective abstraction. Each record of experience can be thought of as being stored in a jar. Inside of each jar is a particular instance of the activity and the effect of that activity. Each jar is labeled as to whether the record of experience inside was associated with a positive result or a negative result.

In the first phase of Piaget's reflective abstraction, the projection phase, jars are sorted according to their labels (i.e., learners mentally—though not necessarily consciously—compare /sort records based on the results). In the second phase, the reflection phase, the contents of the jars that have been grouped together are compared and patterns observed. Thus, within each subset of the records of experience (positive versus negative results), the learners' mental comparison of the records allows for recognition of patterns, that is, abstraction of the relationship between activity and effect. Because both the activity and the effect are embodiments of available conceptions, the abstracted activity-effect relationship involves a coordination of conceptions (Piaget, 2001). An abstracted activity-effect relationship is the first stage in the development of a new conception. The notion of anticipation, highlighted by Piaget (1971), is key to the framework presented here. A conception can be thought of as the ability to anticipate the effect of one's activity without mentally or physically running that activity. The two phases of reflective abstraction should not be thought of as distinct in time, but rather as a description of the two reflective processes that occur.

In postulating this elaboration of reflective abstraction, we build on von Glasersfeld's (1995) claim that each of the components—creating records of experience, sorting and comparing records, and identifying patterns in those records—is an inborn mental ability and tendency of human learners. Von Glasersfeld (1990) argued that a key adaptive mechanism of humans and animals is the *abstraction of regularities* in their records of experience. We suggest that the records of experience from which this abstraction derives are records of activity associated with the effects of that activity. We stress that the regularities abstracted by the learners are not inherent in the situation but rather a result of the learners' structuring of their anticipation-based observations in relation to their goals and related (existing) assimilatory structures. According to Inhelder et al. (1974), "Whatever the degree of regularity [from the observer's perspective, the observations] are always organized by the human learner" (p. 12). The process described is a process that may take place in one or two iterations of the activity sequence or over the course of repeated experience. Note that the conceptual advance is not motivated by the children's desire to make such an advance, but rather by the goals of the children's activity (e.g., the goal of partitioning the stick).

To summarize, learners enact available activities in service of their goal. They distinguish among those attempts that move them closer to their goal and those that do not. Through reflection on the set of attempts that yielded positive results, they abstract a relationship between their activity and its effect. Micki uses the repeat strategy to reach her goal of subdividing a stick into the requested number of parts. The *effect* of her activity is a *thought*, a mental activity based on her conception of composite units, for example: "The stick I want to make requires that I start with a larger small stick than the one I just used." Through reflection on her activity (sequence) and its effects across a number of tasks of this type, she distinguishes a regularity: Partitioning a stick into a particular number of parts results in a part of a particular (unique) size. That is, she develops the anticipation that when she creates the part to be iterated, she is looking for a part of particular length. This learned anticipation is an important part of understanding that a fraction specifies a particular unit relative to the whole.

We emphasize here that the process described is *not* inductive, but constructive, a distinction made by Piaget (1980), Steffe (1991), Thompson (1985), and others—the distinction being between empirical and reflective abstraction). Micki's learning is not a result of reflection on a pattern in the outcomes (i.e., a unique size is found). Rather, it is a reflection on a pattern in the activity-effect relationship that leads to the new conception. Note the activity and the effect are conception-based mental activities, our interpretation of Piaget's (2001) notion of *coordination of actions*.

We clarify a few of the constructs that we have been using:

- *Activity* refers to mental activity. Even when the learner is engaged in relevant physical activity, it is the associated mental activity that is the basis for abstracting new activity-effect relationships. A focus on mental activity is key to generating an explanation of learning that builds on the construct of assimilation. It is the mental activity (related to the learner's goals) that is governed by the learner's assimilatory conceptions. Therefore, it is the mental activity that provides the raw material for the construction of a new conception.
- *Activity sequence* refers to a set of actions used in an attempt to meet a goal.
- *Learners' goals* are not necessarily conscious. For example, learners may have an unformulated goal of doing a particular task more efficiently, that is, with less time or effort. Although they are not aware of the goal, it structures what they notice, the comparisons that they make, and the relationships that they abstract.

- *Effects,* in order to contribute to reflective abstraction, are not the output of a "black-box" experience. Rather, they are structured by assimilatory conceptions that the learner brings to the situation. In the fraction example, the learner did not just know that the composite stick was too short; her composite-unit conception allowed her to know that the size of the part affected the length of the composite stick. It is also important to know that there are many effects to which a learner might attend. However, we are only interested in those effects that result in conceptual learning.

Reflection on activity-effect relationships is our elaboration of Piaget's reflective abstraction for the purpose of describing a basic mechanism for pedagogical theory. Piaget (2001) distinguished conscious *reflected abstraction* and *metareflection* or *reflective thinking* from *reflecting abstraction*, which is not necessarily conscious. Our description of reflective abstraction is meant to identify a basic mechanism common to both conscious and nonconscious reflection.

## IMPLICATIONS FOR LESSON DESIGN

The usefulness of the explanation of conceptual development above rests on the extent to which it offers specificity at a level that can guide pedagogical decision making, particularly the planning and use of mathematical tasks. In this section, we discuss implications for lesson design. We refer to the lesson "designer," aware that contributions to lesson design are made by both curriculum developers and teachers. Below, we identify steps in the lesson design process. We illustrate these steps with references to the instructional intervention used with Micki.

The lesson design process outlined below focuses on the question, What mathematical tasks[8] should be used? The first two steps of lesson design discussed below, specifying students' current knowledge and specifying the pedagogical goal, are already well accepted in mathematics education. However, due to the variation in how those steps are conceptualized, we emphasize the particular framework within which we understand them. In Steps 3 and 4, identifying an activity sequence and selecting a task, one can see the particular impact of the mechanism of conceptual learning outlined above.

*Step 1: Specifying students' current knowledge*. What learners know affords and constrains what they can learn. As Hoyles (1991) suggested, "[Instructional tasks] must connect with initial pupils conceptions and ways of working" (p. 154). It is important to understand the students' conceptions in order to determine appropriate learning goals for the students and to anticipate interpretations that the students can make of proposed tasks, goals that they can set, and activity sequences in which they can engage to work toward their goals. In the example of Micki, it was impor-

[8] The consideration of *task* may or may not include the specifications of the context (e.g., the Sticks microworld).

tant to specify that Micki understood fractions as an arrangement (not a quantity) and that she understood composite units. This specification established a basis for thinking about what Micki might learn and the conceptual tools that Micki had available for building a more sophisticated understanding. It was also necessary to make sure that Micki had the expected activity sequence available for the context (microworld) used.

Note that the steps outlined are not entirely sequential because of interdependent aspects of these steps. For example, in Step 1, specifying Micki's understanding of a fraction as an arrangement is an appropriate beginning to the planning process. However, attention to Micki's conception of composite units and the availability of a particular activity sequence in Sticks become important only in the context of Steps 3 and 4. Thus, there is a back-and-forth movement between steps.

*Step 2: Specifying the pedagogical goal.* Specifying the conceptual advance intended is a difficult undertaking. Not only is it insufficient to specify what the student will be able to do (the traditionally employed behavioral objective), it is insufficient to identify the mathematics that the student will know (e.g., "The student will understand the distributive property."). Specifying understandings involves articulating developmental (conceptual) distinctions as opposed to mathematical distinctions. (This point is further elaborated in Simon, 2002.) In order to promote conceptual transformation effectively, it is necessary to specify the nature of that transformation. To do so, the designer must consider at least two states of student understanding, a current state and a goal state, and the differences between them. In the example of Micki's fraction learning, we specified that, at the outset, Micki understood a fraction as an arrangement and understood numbers and lengths as composite units (resulting from iterating a unit of one). We specified the instructional goal as understanding that equal partitioning produces a particular unit (relative to the whole).

*Step 3: Identifying an activity sequence.* Once the designer has a useful specification of the conceptual advance that she wants to promote (i.e., a current state and a goal state), her problem is to identify an activity sequence that can lead to the intended advance. That is, she must identify an activity that the learner can initiate on the basis of extant conceptions that might lead to an abstracted activity-effect relationship corresponding to the pedagogical goal. Often, the activity sequence is considered in conjunction with the mathematical task. In this way, Steps 3 and 4 (identifying an activity sequence and specifying a task) are interdependent. In this discussion, we separate them to focus on particular aspects of each.

Key to the identification of an activity sequence is that it is available to the students at their *current* conceptual level. Generally this means that the students already have the activity sequence available. In some cases, it may be useful to help them develop a new sequence (e.g., the playing of a new game). This is not a problem if the new sequence can be assimilated into their current conceptions, that is, the new activity sequence does not require conceptions more sophisticated than those that students already have. It is essential that the students be sufficiently competent with the activity sequence so that they can call it up in service of a goal that they set and can engage in it during the lesson without having to work on how to do it. In the fraction example, the designer hypothesized that the activity sequence that we described (repeat strategy) was available to Micki and was likely to lead to the desired abstraction.

*Step 4: Selecting a task.* The designer selects a task that she conjectures will result in the students setting a goal and engaging in the intended activity sequence to accomplish that goal. In the case of Micki's fraction learning, the task involved subdividing sticks of a given size into a given number of equal parts in the context of the Sticks microworld.

We see Steps 2–4 as an elaboration of Simon's (1995) hypothetical learning trajectory. The original description of the trajectory indicated that the trajectory was made up of the learning goal, the learning tasks, and the hypothetical learning process.[9] Using the framework developed in this article, the hypothetical learning process is articulated in terms of the student setting a goal, initiating an activity sequence in pursuit of the goal, noticing the effects of this activity, creating records of experience (iterations of the activity associated with the particular effects that ensued), and reflecting on those records of experience resulting in the identification of invariant aspects of the activity-effect relationship. This elaboration of the hypothetical learning process results in greater specification of the requirements of learning tasks.

## DISCUSSION

The theoretical work that we describe in this article is part of an ongoing effort to understand and explain mathematics learning in powerful ways. The work has been guided by two assumptions. First, the mechanisms of learning (and teaching) that underlie successful lessons can be understood. Second, better understanding of mechanisms of learning can lead to a more methodical approach to lesson design, more consistent generation of successful lessons, and greater effectiveness in the modification of unsuccessful lessons.

### *A Response to the Learning Paradox*

The mechanism for conceptual learning that we have specified explains the process of transformation from less advanced to more advanced conceptions. One can examine fraction example and note that no conceptions were attributed to the learners that were more advanced than their assimilatory structures at the outset. Piaget (2001) made the major contribution to the mechanism that we described. He

[9] Step 1 (Specifying students' current knowledge) was also part of Simon's (1995) Mathematics Teaching Cycle as was the hypothetical learning trajectory.

postulated reflective abstraction as the type of process that can account for the development of a new concept. Our work has taken Piaget's broad notion and specified particular workings of this process at a level of detail that can be useful for pedagogical theory and practice.

To be useful for the design of pedagogical tasks, the mechanism of learning that we have articulated must both address the learning paradox in a theoretically defensible way and provide sites within the mechanism for pedagogical intervention. We have described the mechanism, an elaboration of reflective abstraction, in the following way:

- The learner sets a goal based on that individual's current assimilatory conceptions. That person cannot have a goal to learn the new conception, because that conception is not yet part of the conceptual universe.
- The learner enacts an activity sequence (part of existing assimilatory conceptions) to reach the goal.
- The learner monitors the successfulness of individual attempts and creates records of experience of each attempt with its effect (those effects that the learner notices relative to the goal and current knowledge).
- Through an innate and not necessarily conscious process of comparison, the learner compares these records of experience, identifying invariant relationships between that person's activity and its effects. These invariant relationships constitute a new level of anticipation (abstraction).

This articulation of the mechanism builds on the principle of reflective abstraction and provides a response to the learning paradox, that is, how advanced conceptions can emerge from less advanced ones. In this mechanism for conceptual learning, we specify the interrelationship of the student's goal, the task, the activity, the effect, and the reflection on the records of activities and their effects. This mechanism is potentially useful, because it affords a way of thinking about learning that indicates sites for pedagogical interventions, particularly in the design of instructional tasks. That is, based on understanding of the learners' conceptions and an analysis of the concepts to be learned, tasks can be designed to promote the learners' setting of particular goals and engagement in particular activities, enhancing the possibility that particular activity-effect relationships will be abstracted.

### *Not Ignoring the Complexity of Learning*

This framework is *not* needed for every instructional situation involving conceptual learning. Indeed, students learn some things spontaneously and other things through relatively unstructured inquiry lessons. Rather, the mechanism can be useful for thinking about the more intractable problems in teaching mathematics. However, mathematics learning is a complex process. Multiple conceptions are involved in the development of a new conception. Often a single concept is not the goal but rather a "knowledge system" (Smith et al., 1993, p. 131). Engagement in a variety of activities may be necessary for a particular conceptual advance. How then might this relatively simple mechanism support effective mathematics teaching?

Understanding mathematics learning as reflection on activity-effect relationships allows the teacher, curriculum designer, or researcher to generate useful conjectures as to the types of experiences, and therefore the types of tasks, that might contribute to the learners' construction of new conceptual entities. The process will remain uncertain. Conjectures may prove wholly or partially unsuccessful. However, this conceptualization of the learning process can structure the educator's subsequent interventions. The goal is *not* a step-by-step outline of how to foster a new conception. Rather, it is a theoretical foundation for intentional interventions and modification of those interventions (e.g., Tzur, 2002).

This approach to teaching is what Bereiter (1985) calls *indirect* in that the cognitive advance cannot be directly brought about; rather, the teacher promotes specific experiences for the development of the intended cognitive structure. Thus, based on her understandings of the students' available mathematical conceptions and activities, the teacher anticipates a developmental process in the context of particular learning activities, which is what Simon (1995) called a "hypothetical learning trajectory" (p. 133).

### *Theoretical Distinctions Inherent in this Mechanism*

*Reinvention rather than discovery.* The theoretical constructs that we have elaborated suggest that learning be understood as reinvention (Freudenthal, 1973) rather than discovery. Guided discovery has its roots in a different understanding of mathematical knowledge and learning. The distinguishing issue is the epistemological assumption about what determines the mathematical relationships that learners perceive. Following Piaget (1985), Dewey (1933), and von Glasersfeld (1995), we argue that it is the state of the learners' conceptions that determines what they can notice; mathematical relationships are not simply picked up (discovered) from universally accessible situations. Thus, the question for the teacher is how to foster the students' reinvention of particular mathematical ideas. This question is quite different from "What situation will allow the students to discover the mathematics?" The former involves inquiring into and hypothesizing about the cognitive processes of the learner; the latter requires only consideration of which situations (from the teacher's perspective) make the mathematical idea apparent. Underlying the notion of learning as discovery is the assumption that the mathematical relationships exist in the situation (e.g., place value relationships are in base 10 blocks). Underlying the notion of reinvention is the assumption that learners impose mathematical relationships on the situation based on their available conceptions. From our perspective as mathematics educators, conceptual advance is a process of reinvention. Nonetheless, students experience their mathematical advances, as do mathematicians, as if they are discovering preexisting mathematical truths (Cobb, 1989).

*Specifying the role of activity.* Like the notion of guided discovery, a common understanding of activity has its roots in what we have called a perception-based (Simon, Tzur, Heinz, Kinzel, & Smith, 2000) epistemological stance. In a perception-based perspective, activity is understood as an engagement in concrete experience as a way of seeing that which is more difficult to see in the abstract. From a perception-based perspective, the role of students using an activity sequence during a lesson is for *that* activity sequence to be learned, first at the concrete level and then at an abstract level. Take, for instance, the ubiquitous place value mats (mats divided into a "tens" area and a "ones" area). Students are encouraged to trade chips or arrange sets of cubes on their mats according to a set of rules (e.g., "No place should have more than 9 items in it."). This activity sequence is meant to mimic paper-and-pencil addition and subtraction with regrouping. Students, by learning this new activity sequence, are expected to "see" and therefore understand why the actions of regrouping are as they are. Using the construct of assimilation, we would question the expectation that the learner who needs the lesson would be able to attend to the conceptual underpinnings of regrouping.

The framework that we have elaborated emphasizes a different understanding of the role of activity. The successful learning of the activity sequence is *not* the goal of a conceptual lesson. Rather, the teacher identifies an activity sequence already available to the students that they can employ to work toward a particular goal. The purpose of the activity is for students to begin to differentiate regularities in variations of their activity and the effects those variations produce. The students distinguish among effects—the basis for students observing regularities—based on how the effects contribute to (or fail to contribute to) the students' goals.

## POTENTIAL INTERACTION WITH OTHER THEORIES

The mechanism that we have described can be related to two categories of theoretical work. The first are descriptions of stages of concept development (cf. Pirie & Kieren, 1994; Sfard, 1991), and the second are pedagogical theories that specify pedagogical situations or stages such as the French theory of didactical situations (Brousseau, 1997) and the Dutch Realistic Mathematics Education (Gravemeijer, 1994). We suggest that not only are the constructs presented in this article compatible with these theories, but that these constructs provide a way of understanding the mechanisms by which students progress from one stage to the next or from one situation to the next. For example, reflection on activity-effect relationships can potentially be used to explain how a learner progresses from a process conception to an object conception (Dubinsky, 1991; Sfard, 1991). Similarly, it might be used to explain how the *situation adidactique* functions in the *situation didactique* (Brousseau, 1997) and how students progress from realistic situation, to model-of, to model-for, to formal mathematics (Gravemeijer, 1994).

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