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The "MOST" Productive Student Mathematical Thinking

Instruction that meaningfully incorporates students' mathematical thinking is widely valued within the mathematics education community (NCTM 2000; Sherin, Louis, and Mendez 2000; Stein et al. 2008). Although being responsive to student thinking is important, not all student thinking has the same potential to support mathematical learning. Thus, teachers must make choices about which student contributions should or should not be incorporated into the whole-class discussion. In this article, we provide a framework to help teachers make these choices.

Consider the following vignette:

> During a geometry lesson focused on right-triangle trigonometry, students are grading a homework problem that asked them to find an angle of depression given a situation in context. After the teacher gives the correct answer, a student raises her hand and says, "I solved it by finding the angle at the bottom, and that gave me the right answer."

Many mathematics teachers would recognize the student's comment in the vignette as one that contains rich mathematics and has considerable potential to support student mathematical learning; thus, these teachers would likely choose to incorporate this comment into the lesson. Some student questions or comments—such as correct answers with no explanation or justification or clarification questions—are more routine. Thus, teachers may choose not to act on such instances.

Choosing whether to incorporate student thinking should depend on the nature of the student thinking as well as how the teacher intends to incorporate that thinking. Productive incorporation occurs when a teacher engages the class in reasoning about a student's contribution and, as a result, builds on student thinking.

We are interested in how teachers go about deciding which instances of student thinking have the most potential for productive incorporation. We identified potential characteristics of such instances from the literature (e.g., Davis 1997; Walshaw and Anthony 2008; Schoenfeld 2008) and refined these characteristics by applying them to classroom video data (Leatham et al., in press). We describe these characteristics and their associated criteria to demonstrate how they might be used to identify instances of student mathematical thinking that have considerable potential for advancing students' mathematical understanding. We refer to such instances as mathematically significant pedagogical opportunities to build on student thinking or, more succinctly, Mathematical Opportunities in Student Thinking (MOSTs).

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"MOST" CHARACTERISTICS

We identified three characteristics that are critical for identifying MOSTs: (1) student mathematical thinking; (2) mathematically significant; and (3) pedagogical opportunity (see **fig. 1**). Although in the midst of a lesson teachers make decisions so quickly that they may think about these three characteristics nearly simultaneously, the characteristics are defined sequentially to build on one another. We focus first on identifying student mathematical thinking. We then focus on determining whether the student mathematical thinking is mathematically significant, a process that broadens the teacher's attention from an individual student's contribution to how that contribution relates to the mathematics curriculum and goals for the class. Finally, we consider whether the student mathematical thinking provides a pedagogical opportunity at that moment, thus broadening the teacher's perspective in another way—to how an individual student's contribution might be used to engage the whole class in mathematical reasoning.

Analysis of each characteristic is based on key questions about two related criteria. In the following sections, we define each characteristic and its associated criteria. We also give examples of instances that satisfy a particular criterion and of those that fall short. Some level of subjectivity is inherent in the process. The strength of the MOST framework is that it provides a language and structured process for analyzing the mathematical and pedagogical potential of instances of student thinking that surface during a lesson (see **fig. 1**).

Student Mathematical Thinking

The student mathematical thinking characteristic consists of two criteria: student mathematics and a mathematical point. *Student mathematics* refers to any instance in which an observable student action allows a reasonable inference about what the student is thinking mathematically. In the classroom, student mathematics may be evident in students' utterances, board work, gestures, or individual written work, so we must evaluate each observable student action within its context. In the vignette, the

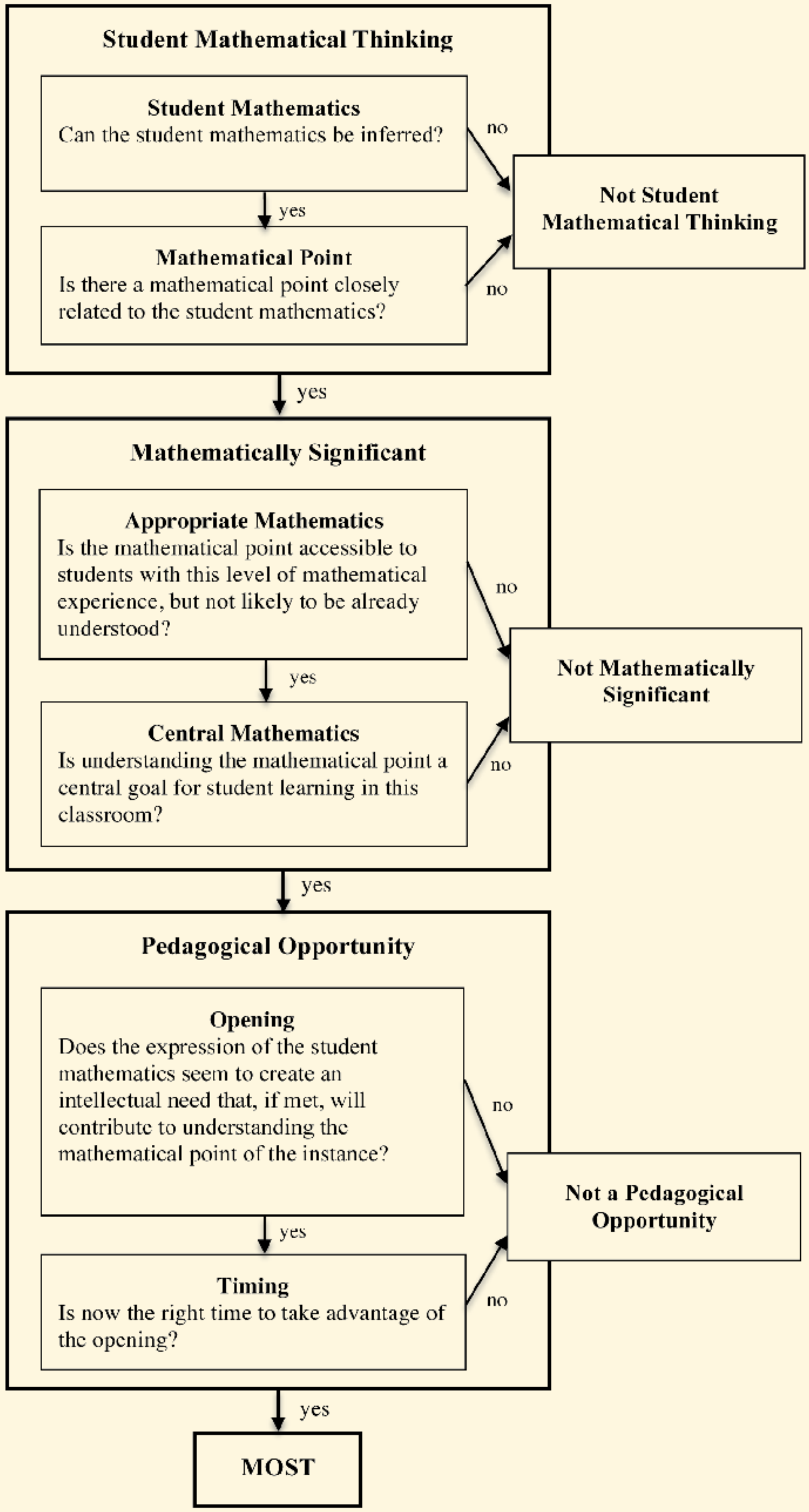

Fig. 1 This flowchart outlines the characteristics, criteria, and key questions used to determine whether a student's contribution is a MOST.

context allows us to infer the student mathematics to be, "I found the angle of elevation instead of the angle of depression, and it gave me the right answer."

Other student utterances are not as straightforward—for example, the simple student answer "no." As a response to the question "Do you understand?" this answer would not provide insight into what the student is thinking mathematically. However, if "no" answers "Do you agree with Jim's solution?" we can infer the student mathematics to be, "Jim's solution is mathematically incorrect." In meeting the student mathematics criterion, we must be able to make an evidence-based inference about what the student is thinking mathematically. That said, we emphasize that student ideas need not be (and often are not) clearly articulated; we need only be able to infer the student mathematics.

Once the student mathematics is identified, we determine whether there is a mathematical idea that is closely related to it—a *mathematical point*. To be "closely related" to student mathematics, the idea must be one that learners could better understand by considering the student mathematics. In our vignette, the mathematical point would be, "When a higher vantage point A and a lower vantage point B are viewed in relation to one another, the angle of elevation from B to A and the angle of depression from A to B are congruent." Other mathematical points in different classrooms and situations might be as follows:

- Addition and subtraction are inverse operations.
- Fractions with a common denominator can be added by adding their numerators and keeping the common denominator.
- A single counterexample is sufficient to disprove a mathematical statement.

Not all student mathematics, however, has a mathematical point. For example, in our trigonometry vignette, suppose a student had said, "I got 89 degrees, which doesn't make sense, but I don't know where I messed up." Here we infer the student mathematics to be, "I know that 89 degrees is an unreasonable answer, but I don't know where I made an error." However, we have no means for inferring what the student did mathematically, so there is no related mathematical point.

Mathematical Significance

An instance is mathematically significant if the mathematical point is both appropriate for the mathematical development level of the students and central to mathematical goals for their learning. Appropriate mathematics must be neither too complex nor too simple. In particular, the mathematical point must be accessible to students given their prior mathematical experiences while not being a point that most students at this mathematical level would already understand. Determining appropriateness is based on experience with students, publications such as the Common Core State Standards for Mathematics (CCSSI 2010), and research on learning trajectories (e.g., Maloney, Confrey, and Nguyen 2014; Stylianides 2008).

To illustrate this criterion, consider the mathematical point, "When a higher vantage point A and a lower vantage point B are viewed in relation to one another, the angle of elevation from B to A and the angle of depression from A to B are congruent." Because learning to solve problems using right-triangle trigonometry and understanding relationships between angles formed by parallel lines and a transversal are both part of the geometry curriculum, the mathematical point, which involves putting these ideas together, is accessible to these students while not being something that they would already understand. Thus, this instance satisfies the appropriate mathematics criterion.

To understand how this criterion might not be satisfied, consider the mathematical point, "Limits can be used to find the exact area of a shape." This point would be mathematically appropriate in a calculus course in which the formal definition of a limit was being studied, even if limits had not yet been used to find the exact area of a shape. However, this same mathematical point arising in a prealgebra class of students without sufficient background knowledge would thus fail the appropriate mathematics criterion. Or consider the mathematical point, "Adding multidigit numbers where the sum in a given place value is greater than ten requires regrouping with the next-higher place value." When children are learning the addition algorithm, this mathematical point would be appropriate. Were this same point to surface in an algebra class, the instance would not be appropriate because such students would likely already understand the idea and discussing it would not help them move forward in their learning.

Once the mathematical point is deemed to be appropriate, it meets the central mathematics criterion if it is closely related to a lesson goal, a unit goal, or a broader mathematical goal considered high priority for students' overall mathematical understanding.

The mathematical point in the trigonometry vignette is closely related to the lesson goals because it provides a method for solving some of the homework problems; it is also related to broader course goals because it connects to theorems about parallel lines that are central to students' learning in a geometry class. Thus, this instance would satisfy the central mathematics criterion. Because this instance satisfies both the appropriateness and centrality criteria, it would be mathematically significant.

Not all mathematical points that are appropriate, however, necessarily meet the central mathematics criterion. Suppose that during an algebra lesson about the Pythagorean theorem, a student asks how to find the square root of a number without a calculator. The mathematical point—"We can find the square root of a number without technology using an algorithm similar to the long division algorithm"—is appropriate (that is, accessible but not likely already understood) but not related to a central learning goal for students, so it does not meet the central mathematics criterion.

Pedagogical Opportunity

Instances of student mathematical thinking that are determined to be mathematically significant are not necessarily MOSTs. Teachers must still determine whether such an instance creates a pedagogical opportunity requiring two key criteria: opening and timing.

An *opening* is an instance in which the expression of a student's mathematical thinking creates, or has the potential to create, an *intellectual need* (Harel 2013) for students to make sense of the student mathematics. Common situations we have identified that tend to create intellectual need include—

- a correct answer with novel reasoning;
- an incorrect answer that involves a common or mathematically rich misconception;
- a mathematical contradiction;
- incomplete or incorrect reasoning; and
- "why" or generalizing questions.

To illustrate the opening criterion, we return once again to our trigonometry example. The student has proposed an alternative method for making sense of a problem that students in the class have spent time working on. It is likely that many would be interested to find out whether and why the proposed alternative (and possibly more efficient) method works here or in general. Because we can infer a potential intellectual need among many students, an opening has been created.

To illustrate how this criterion may not be met, consider a lesson in which a teacher is demonstrating a method for solving systems of equations. A student asks the teacher to repeat the explanation of the step wherein two equations were added together to eliminate the variable x. The student mathematics is related to the mathematical point, "the addition property of equality," which is appropriate and central to this lesson; thus, the instance is mathematically significant. Here, the question does not clearly focus on making sense of the procedure, so it does not set up a situation for students in the class to do so. Hence, this question does not create an intellectual need for students to make sense of the mathematics and thus does not create a pedagogical opening to build on the student's thinking.

Timing of the instance is important. Pedagogical opportunities occur at times that are opportune—when taking advantage of the opening at that moment is likely to further students' understanding of the mathematical point of the instance. We again return to the trigonometry vignette, in which we have an instance of mathematically significant student thinking that has created a pedagogical opening. Is now the right time to pursue that thinking? Because the students have already engaged with problems in the homework that are directly related to the mathematical point of the instance, they are likely ready for fruitful discussion about this student's mathematics, so the timing criterion is satisfied and the instance creates a pedagogical opportunity. The instance in the vignette embodies all three characteristics—it is a MOST.

To illustrate how an instance can fail the timing criterion, let's suppose that a student proposes an interesting strategy to solve a task whose purpose is to explore relationships among multiple solution strategies. Because of the focus on sense

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making, this instance (assuming that it is judged to be student mathematical thinking that is mathematically significant) creates an intellectual need and thus an opening. But if the student's proposition is stated just as students have begun to work on the task, the timing is not yet right for a class discussion of an individual solution strategy, because it would likely undermine the intent of the task.

MOST: A FRAMEWORK FOR ANALYZING STUDENT THINKING

Building on student thinking can be a means for improving mathematics instruction and has been advocated in the field (e.g., NCTM 2000). However, not all student thinking has the same potential for contributing to students' mathematical understanding and, consequently, not all should be pursued in similar ways. As teachers, we need to listen carefully to all student ideas but follow with thoughtful consideration of whether a particular idea or comment is worth pursuing in the limited amount of instructional time that is available. The MOST framework highlights three critical characteristics to help teachers identify which instances might be most productive to pursue for mathematical purposes.

Teachers may choose to discuss other student ideas for different purposes. For example, we may choose to discuss a rather routine solution of a student who rarely participates in class as a means of engaging that student in the lesson, even though the instance is unlikely to further the mathematical understanding of the class as a whole. It is important, however, that teachers consider their purpose for incorporating student ideas into a lesson, understanding that some instances of student thinking directly support the learning of mathematics whereas others may not.

The MOST framework has the potential to become a tool for making sense of classroom interactions and for learning how to build more productively on student thinking during a lesson. It could, for example, be used by an individual teacher as a tool to reflect on the student thinking that emerges during a lesson, by pairs of teachers to facilitate mutual peer observation, or by groups of teachers as a framework for analyzing classroom artifacts in professional development settings or in professional learning communities. We have found that determining the mathematical point often results in rich mathematical discussions. Further, by focusing on what student thinking could be incorporated into a lesson and why that incorporation might be valuable, the framework provides a common language for teachers to discuss the practice of building on student mathematical thinking. Better understanding of this practice has considerable potential to enhance the teaching and learning of mathematics.

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REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Davis, Brent. 1997. "Listening for Differences: An Evolving Conception of Mathematics Teaching." *Journal for Research in Mathematics Education* 28 (3): 355–76.

Harel, Guershon. 2013. "Intellectual Need." In *Vital Directions for Research in Mathematics Education*, edited by Keith R. Leatham, pp. 119–51. Dordrecht, The Netherlands: Springer.

Leatham, Keith R., Blake E. Peterson, Shari L. Stockero, and Laura R. Van Zoest. "Conceptualizing Mathematically Significant Pedagogical Opportunities to Build on Student Thinking." *Journal for Research in Mathematics Education* (in press).

Maloney, Alan P., Jere Confrey, and Kenny H. Nguyen, eds. 2014. *Learning over Time: Learning Trajectories in Mathematics Education*. Charlotte, NC: Information Age.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Schoenfeld, Alan H. 2008. "On Modeling Teachers' In-the-Moment Decision Making." In *A Study of Teaching: Multiple Lenses, Multiple Views, JRME* monograph no. 14, edited by Alan H. Schoenfeld, pp. 45–96. Reston, VA: National Council of Teachers of Mathematics.

Sherin, Miriam G., David Louis, and Edith P. Mendez. 2000. "Students' Building on One Another's Mathematical Ideas." *Mathematics Teaching in the Middle School* 6 (3): 186–90.

Stein, Mary Kay, Randi A. Engle, Margaret S. Smith, and Elizabeth K. Hughes. 2008. "Orchestrating Productive Mathematical Discussions: Five Practices for Helping Teachers Move beyond Show and Tell." *Mathematical Thinking and Learning* 10 (4): 313–40.

Stylianides, Gabriel J. 2008. "An Analytic Framework of Reasoning-and-Proof." *For the Learning of Mathematics* 28 (1): 9–16.

Walshaw, Margaret, and Glenda Anthony. 2008. "The Teacher's Role in Classroom Discourse: A Review of Recent Research into Mathematics Classrooms." *Review of Educational Research* 78 (3): 516–51. doi:10.3102/0034654308320292




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