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Teaching Algebra to Students with Learning Difficulties: An Investigation of an Explicit Instruction Model

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Abstract. Thirty-four matched pairs of sixth- and seventh-grade students were selected from 358 participants in a comparison of an explicit concrete-to-representational-to-abstract (CRA) sequence of instruction with traditional instruction for teaching algebraic transformation equations. Each pair of students had been previously labeled with a specific learning disability or as at risk for difficulties in algebra. Students were matched according to achievement score, age, pretest score, and class performance. The same math teacher taught both members of each matched pair, but in different classes. All students were taught in inclusive settings under the instruction of a middle school mathematics teacher. Results indicated that students who learned how to solve algebra transformation equations through CRA outperformed peers receiving traditional instruction on both postinstruction and follow-up tests. Additionally, error pattern analysis indicated that students who used the CRA sequence of instruction performed fewer procedural errors when solving for variables.

Abstract thinking requires a person to work with information that is not readily represented at the concrete or pictorial level (Hawker & Cowley, 1997). To work with abstract information is to understand theoretical properties and think beyond what a person can touch or see. On a practical level, an ability to work with abstract concepts allows one to work with predictions of what may happen and expectations about what is happening elsewhere. Abstract knowledge may also be considered a conscious awareness that a symbol stands for some number of things. That symbol represents things known and even unknown. In an abstract algebraic equation, mathematical parts can be manipulated and adjusted, and thus equations change upon each manipulation. In mathematics, abstract symbols are used for numbers (e.g., 5), sets of numbers ($\{X{:}1,2,3\}$), and properties of a statement or solution ($X = 2Y + 5$).

Algebra is considered a gateway to abstract thought. The need for students to successfully complete algebra has been become increasingly apparent over the last decade. Several states now require students to pass end-of-year or graduation tests that show knowledge in algebraic understanding (Ysseldyke et al., 1998). Although the intention of improving math standards appears justified, end-of-year exams have not helped students graduate. Now that states require all students to adhere to the same graduation standards, introducing high-stakes assessments is a great concern for students with disabilities. With historically poor graduation rates and low overall success rates in secondary school, increased graduation requirements have lessened the opportunities for students with poor performance and students with disabilities to graduate.

Due to its abstract nature, educators have struggled to help students comprehend initial algebra instruction. Devlin (2000) stated that for students to understand abstract concepts more easily, it is important for them to learn precursor concepts in a concrete manner first. One way to simplify students' understanding of abstract concepts is to transform such complex concepts into concrete manipulations and pictorial representations. Such suggestions, while they appear logical, go against the very definition of abstract. Often, when educators attempt to break down abstract mathematics into concrete and representational steps, they may alter the concept by eliminating parts of equations, thus making generalized learning inappropriate, if not impossible.

To improve the opportunities for success in algebra and possibly improve graduation rates, teachers

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and researchers need to develop means for teaching secondary math concepts to more students (Witzel, Smith, & Brownell, 2001). One approach that shows promise of making algebra instruction more accessible to students with difficulties involves the use of concrete materials that develop into representational and eventually abstract thought. Such instruction is known as the concrete-to-representational-to-abstract sequence of instruction (CRA). CRA instruction starts with a student using manipulative objects to display and solve math problems. Once students understand a topic concretely, they work with the same concept using pictorial representations. Representational knowledge is a very practical step for students unwilling to carry objects out of their classroom to solve algebraic problems.

Although much research on CRA has focused on the effectiveness with arithmetic instruction (Miller & Mercer, 1993), recently more researchers have attempted to design CRA models for algebra instruction (Borensen, 1997; Maccini & Hughes, 2000). The difficulty of creating an effective CRA model in algebra revolves around the core issue of abstractness. Building a representation of an algebra expression or equation in a way that maintains the abstract nature of the concept has been difficult for educators.

The algebraic representations developed by Maccini and Hughes (2000) and Borenson (1997) interfere with algebraic learning. In the concrete and representational steps used in these approaches, the materials did not adequately represent algebraic variables and coefficients. For example, equations such as $X + 3 = 5$ and $5X = 15$ appear easily represented in these models, but the representations used by these authors in both instances did not differentiate coefficients from exponents. This may lead to possible future confusion. Specifically, by asking students to represent the variable X with a cube, the coefficient is misrepresented. Instead of thinking five cubes is 5X, mathematically, five cubes should be X^5 when working with exponents. In a practical sense, when the student who learned that five yellow cubes represents the abstract concept of 5X, how is the student going to represent X^5 or even X^2? By creating incorrect models for representational and concrete stages, it is likely that students will perform well at simple inverse operations and reducing expression, but they will be at a disadvantage when confronted with more complex equations beyond the immediate lesson. Thus, the models used by Maccini and Hughes and Borenson may help build confidence for solving "algebra"; however, the failure of their manipulative objects to generalize to more difficult algebraic concepts represents a weakness in current CRA models, and thus in their long-term effectiveness for students.

The purpose of this research was to test the effectiveness of a new explicit CRA algebra model that was designed to represent more complex equations. The difference between this model and other hands-on algebra curricula is that this model displays the conceptual components in its concrete and pictorial representations in a manner that prepares the student to succeed in more advanced algebra concepts. Although other programs were effective as short-term interventions, this model provides representational processes and procedures that are appropriate for both beginning and advanced algebra topics by allowing students to work with every algebra component within an equation.

Effectiveness of the CRA model for students with learning disabilities and students who were at risk for failure in secondary mathematics was evaluated according to a posttest and a three-week follow-up measure. The scores of the students who were taught using CRA instruction were compared to scores of matched peers taught using abstract forms of instruction. To reduce error and increase power, students with an at-risk concern for difficulties in algebra or a disability label were matched according to grade level and teacher, standardized math achievement score, pretest score, class performance as rated by the teacher, and age.

METHODOLOGY

Study Participants

Teachers

Twelve classrooms and 10 teachers in a southeastern United States urban county participated in this research. Four teachers individually taught a total of eight mathematics classes for sixth graders, and two sets of teachers team-taught four mathematics classes for seventh graders. Every class included students with and without disabilities. One sixth-grade teacher was certified to teach students with learning disabilities, and one other sixth-grade teacher had taken courses on students with learning disabilities. Both seventh-grade team-taught classes contained a math teacher and one teacher certified in teaching students with high-incidence disabilities. In these team-taught classes, the regular education math teacher took the lead in planning and implementing lessons. The special educators monitored student behavior and worked to keep students on-task. The six sets of teachers were trained individually by the senior author in a one-day session with several follow-up meetings. Additional training was provided through demonstrations and guided practice until the teachers reported that they felt capable of delivering the instructional model.

Students

Approximately 358 sixth- and seventh-grade students participated in the instruction. Of these students, 34 students with disabilities or at risk for algebra difficulty in the treatment group were matched with 34 students with similar characteristics across the same

TABLE 1
Study Participants by Teacher

	Grade Level	Students Using Abstract Method	Students Using CRA Model	Number of Matched Pairs
Teacher 1	7	53	64	13
Teacher 2	7	32	27	8
Teacher 3	6	24	21	6
Teacher 4	6	26	21	2
Teacher 5	6	30	18	2
Teacher 6	6	23	17	3

teacher's classes in the comparison group. Table 1 contains descriptive information about these students. All students were included in the general education classroom for sixth- or seventh-grade math. The at-risk category was developed by the researchers because many students who are not labeled as having a learning disability do not fare well academically, especially in more abstract topics such as algebra. To qualify as at risk for algebra difficulties, the student had to meet the following criteria: (1) performed below average in the classroom according to the teacher, (2) scored below the 50th percentile in mathematics in their most recent statewide achievement test, and (3) was from a low socioeconomic status (SES) background, as indicated through district data regarding students who received free or reduced-price lunch. Since arithmetic achievement alone is not a guarantee for algebra success, teacher input was integral to identifying students who showed signs of difficulty with abstract math such as algebra. The student's SES was considered relevant since low SES has been shown to be an effective predictor of low academic achievement (Hobbs, 1990), specifically in math (Albedi & Lord, 2000; Jimerson, Egeland, & Teo, 1999).

Students identified as having a learning disability were identified through school services as those who needed additional support and who evidenced a 1.5 standard deviation discrepancy between ability and achievement. The students with learning disabilities who participated in the present study had math goals listed in their individualized education plans. Those students with learning disabilities who had only reading disabilities or other disabilities were excluded from the learning disabilities group for the purpose of this study.

Matching

The 34 pairs of students were matched according to previous math course, a statewide achievement test math score within one stanine, age within one year, grade level, at-risk or disability label, pretest accuracy within one item, and teacher. Also, teachers helped develop student matches by discussing who was performing at an equivalent level in class. Table 2 contains a summary of the characteristics of the matched pairs. Matching increased the power of the study by reducing the error in results due to previous learning, age, grade, disability label, teacher-rated class performance, and pretest. Each teacher who participated in the implementation of the study taught two equivalent algebra classes. In the first class, the CRA model was implemented; in the second class repeated abstract instruction was implemented.

To account for differences between teachers, students were matched across teachers. To determine the effects of different instruction on different students, students were also matched across type of instruction. Because participants were matched in pairs, some students had to be eliminated from the data analyses. Subjects were eliminated if they missed a lesson and were unable to make up the lesson, if they missed an assessment and were unable to make up the assessment, if they moved to a different class, or if parents or the student did not sign the permission forms. Several students had to be eliminated from the data analysis because they had no county documentation regarding their achievement stanine score, disability label, or economic status. In such cases both matched students' data were eliminated.

Assessment Tools

The lead researcher developed a test instrument to measure the acquisition and maintenance of knowledge on single-variable equations and solving for a single variable in multiple-variable equations. From the curriculum material, the researchers developed a pool of 70 questions designed to assess the final step of the sequence of lessons taught (i.e., transformation equations). The pool of questions was then distributed to four algebra teachers for expert review. Reviewers accepted the question, rejected the question, or provided improvements for each question. The revised 63-item pool of questions was then distributed to 32 students who had successfully completed their first year of pre-algebra. After completion of the pool of items in an untimed assessment, each answer was recorded as correct (1) or incorrect (0). The percent correct from all the students on each individual item marked the difficulty of the item. The 27 items that had a medium difficulty level between 0.375 and 0.625 correct were selected for the assessment instrument. Because of the length of time between tests and the expected low pretest scores, a single-test form was used for pretest, posttest, and follow-up assessment. Sample test items are displayed in Figure 1. Pretest measures were obtained one week prior to implementation of the treatment. Posttest measures were obtained upon completion of the last day of the treatment, and follow-up measures were obtained three weeks after treatment ended. Students were not instructed on any of the equations between the posttest and the follow-up measures.

TABLE 2
Student Matching Demographics

			Age			*Math Stanine*		*Diagnostic Label*	
Student Match	*School*	*Teacher*	*Abstract*	*CRA*	*Grade*	*Abstract*	*CRA*	*Abstract*	*CRA*
1	1	1	12	12	7	3	3	sld	sld
2	1	1	13	12	7	4	3	sld	sld
3	1	1	12	13	7	7	6	sld	sld
4	1	1	12	13	7	3	3	sld	sld
5	1	1	12	13	7	2	2	sld	sld
6	1	1	12	12	7	6	7	sld	sld
7	1	1	13	12	7	4	5	sld	sld
8	1	1	13	12	7	7	6	sld	sld
9	1	1	12	12	7	3	3	sld	sld
10	1	1	13	13	7	4	4	at-risk	at-risk
11	1	1	13	13	7	4	3	at-risk	at-risk
12	1	1	12	12	7	3	3	at-risk	at-risk
13	1	1	12	12	7	4	4	at-risk	at-risk
14	1	2	13	12	6	7	6	sld	sld
15	1	2	11	12	6	6	6	sld	sld
16*	1	2	12	11	6	6	5	sld	at-risk
17	1	2	12	12	6	4	4	at-risk	at-risk
18	1	2	11	12	6	4	5	at-risk	at-risk
19	1	2	12	11	6	2	3	at-risk	at-risk
20	2	3	12	12	6	5	5	at-risk	at-risk
21*	2	3	12	11	6	4	4	at-risk	sld
22	2	4	12	12	6	5	5	at-risk	at-risk
23	2	4	11	12	6	4	4	at-risk	at-risk
24	3	5	12	12	6	3	4	at-risk	at-risk
25*	3	5	12	13	6	4	4	sld	at-risk
26	3	5	12	11	6	5	5	at-risk	at-risk
27	4	6	13	13	7	2	2	sld	sld
28	4	6	13	12	7	4	4	sld	sld
29	4	6	14	13	7	6	5	sld	sld
30	4	6	13	13	7	5	4	sld	sld
31	4	6	13	14	7	3	3	sld	sld
32	4	6	13	13	7	2	2	sld	sld
33	4	6	12	13	7	3	3	sld	sld
34	4	6	13	13	7	3	3	sld	sld

*Despite differing diagnostic labels, the teacher reported equal overall performance between the two students in these pairs.

Algebra Construct

The present algebra model is designed to take students from reducing simple two-statement expressions to solving more complex equations. The algebraic concept of transforming equations with single variables was selected as the final algebra skill to teach. To have students effectively solve transformations, a five-step 19-lesson sequence of algebra equations was used. Skill areas included reducing expressions, solving inverse operations, solving inverse operations with negative and divisor variables, performing and solving transformations with multiple variables on one side of the equal sign, and, finally, performing and solving transformations with multiple variables on both sides of the equal sign. This sequence of lessons follows that of major algebra textbooks by Houghton Mifflin (Brown, Smith, & Dolciani, 1988), Scott Foresman-Addison Wesley (Charles, Dossey, Leinwand, Seeley, & Embse, 1998), and Saxon (Saxon, 1997, 2000). However, for the purpose of the present study, the total number of lessons was reduced to 19.

Reducing Expressions

Before a student can solve for unknowns and perform transformations on equations, the student needs to be able to effectively reduce expressions. Reducing expressions allows a student to see that unknowns and numbers cannot be added or subtracted together while the unknown exists. Additionally, unlike variables cannot be added or subtracted. For example, the expression $4X + 3B + X$ can be reduced to $5X + 3B$, which is its most basic form. Both groups received three lessons on reducing expressions.

Name: ______________________________ Date: ______________________________

Year in School : ________________________ Present Math Class: ___________________

Teacher's last name: ____________

Solve for the variable. Circle your answer and write legibly.

2 - 2X = 17 + X   7X – 10 = 2X + 5   -3 + 9X = 5X + 13

6Y – 2 = 18 – 2Y   -8X + 7= 22 - 3X   6 – 2X = X - 6

FIGURE 1 Assessment format and item examples.

Inverse Operations

Solving for inverse operations, often called single-variable equations, involves solving for an unknown number or set of unknown numbers. To help establish the concept of a variable, all variables in this unit represented one specific number. The usefulness of solving for single unknowns using inverse operations becomes apparent when students work with practical math problems to find a missing quantity. For example, students learned how to determine how much money a child had borrowed when he bought a movie ticket for \$6 and received \$4 in change, $X - 6 = 4$. To isolate the needed variable in inverse operations, students used subtraction, addition, multiplication, and division. In this example of an inverse operation, $X = 10$. Other examples of inverse operation questions are: $4Y = 16$; $M - 2 = 7$; and $K - 4 = 9$. Both groups received four lessons on solving inverse operations.

Negative and Divisor Variables

Solving for variables when they appear as negative or as the denominator of a fraction involves an additional step when solving inverse operations. Students must first transform the variable to make a positive sign or to position the variable in the numerator. For example, when solving for $6/X = 3$, the X first should be multiplied to each side to produce $6X/X = 3X$. After dividing X by X, the student is left with a simple inverse operation $6 = 3X$, which the student has already been instructed how to solve. An example for negative variables is $18 - W = 8$. Both groups received four lessons on solving for negative and divisor variables.

Transformations on One Side of the Equal Sign

One of the more complex concepts that develops from solving for a single variable is using the same techniques with multiple variables. Transforming equations allows the student to experience the ambiguity and abstractness associated with algebra (i.e., that variables need to be combined before solving). When multiple variables are on the same side of the equal sign (i.e., $2X + 6X = 64$), then variables must be combined through addition or subtraction. Combining one side of an equation is simply reducing an expression. Once combined, the student is left with an inverse operation (i.e., $8X = 64$). Both groups received four lessons on solving for single variables distributed on the same side of the equal sign.

Transformations Across the Equal Sign

Transformations become more complex when variables are on both sides of the equal sign. The usefulness of such a task becomes more evident in courses of study such as chemistry, physics, and personal finance, and students may use this technique to solve logic problems when they are not given sufficient information to precisely answer the question. For example, in physics, to determine the power supplied to an electric motor, people may use the equation $VI = I^2R + EI$ where V = voltage, I = current, R = resistance, and E = electrical energy. By transforming the equation, one can solve for V, I, R, or E. For example, using the equation $25 - 2N = 5 + 2N$, students must add or subtract variables on one side of the equation in order to place all variables on the same side of the equal sign (i.e., $25 - 2N - 25$

$-2N = 5 + 2N - 25 - 2N$). Once the variables are on one side (i.e., $-4N = -20$), students perform the same steps as they did previously (i.e., $\frac{-4N}{4} = \frac{-20}{-4}$) to determine that $N = 5$. Both groups received four lessons on solving for single variables distributed on both sides of the equal sign.

Fidelity

To ensure that the sequence of instruction components was used consistently throughout the treatment and comparison groups, a fidelity checklist was used during an observation of each teacher four times during instruction. The teacher was observed on delivery of instructional components, use of an advance organizer, description of activity, and implementing modeling, guided practice, and independent practice. Every teacher who participated in the study completed every required scripted component during the observations. Each teacher was observed teaching a concrete lesson, a representational lesson, and two abstract lessons.

Research Design

A pre-post-follow-up design with random assignment of clusters was employed for this study. Students were clustered by classroom and divided into two groups, a treatment group and a comparison group. The objective for both groups was to improve pre-algebra skills. The teachers taught the comparison groups according to explicit instruction, following modeling, guided-practice, and independent-practice strategies. The teachers also taught the treatment group using explicit instruction techniques but with the addition of CRA components of instruction. Since one-to-one matching was used in this analysis, the instruction was a within-subject factor.

The dependent measure, number of correct answers out of 27 possible on the algebra assessment, was analyzed for both instruction groups for pretreatment, posttreatment, and maintenance. A repeated measures analysis of variance was used to determine if any significant differences existed between the instructional groups on posttreatment and maintenance measures. Because students were matched on pretest assessment, the pretreatment scores did not significantly differ. Follow-up univariate analyses of variance and t-tests were computed for acquisition and maintenance.

Using a coin flip, the lead researcher randomly chose one of the two math classes for the teacher to teach using CRA instruction and the other class to be taught using the abstract-only traditional methods. The students who learned through CRA were labeled the treatment group, and the students who learned through repeated abstract instruction were called the comparison group.

Treatment Group

Students in the treatment group worked in the same classroom setting that they had throughout the year, but their teacher used the CRA model. Since the students had minimal prior experiences with algebra, they were introduced to algebraic thinking through CRA. Each treatment lesson had four steps: introduce the lesson; model the new procedure; guide students through procedures; and begin students working at the independent level. These four steps were used for instruction at the concrete, representational, and abstract stages of each concept. Teachers taught the concrete lessons using manipulative objects and the representational lessons using pictures. Figure 2 displays a sample problem layout for an inverse operation, using concrete, representational, and abstract steps. Teachers used the same instruction for each class during the abstract lessons. Students performed the algebra using the same method as the teacher.

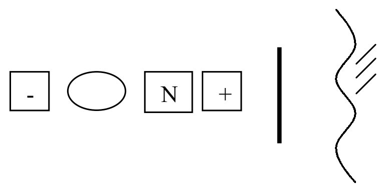
Comparison Group

The only difference between the comparison group and the treatment group was that the comparison group used repeated abstract lessons rather than concrete objects and pictorial representations. The four instructional steps matched those of the treatment group: introduce the lesson; model the new procedure; guide students through procedures; and begin students working at the independent level. Since the students had minimal prior experiences with algebra, the abstract approach of instruction introduced them to the abstract thinking associated with algebraic concepts. The teacher instructed each lesson using examples of abstract equations. For example, each inverse operation lesson sheet displayed equations in Arabic symbols, such as $-N + 10 = 3$. The teachers covered the same content in both groups, and they used the same length of class time for each instruction—50 minutes.

Materials

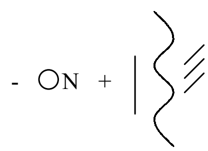
The materials used in the study were developed to determine the difference in acquisition and maintenance of algebraic understanding for students with learning disabilities in math. Students in both the treatment and the comparison groups received the same assessment instruments throughout the study. Treatment and comparison groups received the same questions and equations on daily learning sheets to guide them and their teachers through instruction. The only difference was that the treatment group used manipulative objects for concrete instruction.

A concrete representation of –N + 10 = 3 uses manipulative objects. For this problem the objects would appear in the order indicated by the drawing below, i.e., a minus sign, one coefficient marker, an N, a plus symbol, a large stick, an equals line, and three small sticks. It is important to note that physical objects, not drawings were involved.



To solve a concrete problem, students manipulate objects at each step towards the solution.

A pictorial representation would closely resemble the concrete objects but could be drawn exactly as it appears here.



To solve a representational problem, students draw each step towards the solution.

An abstract problem is written using Arabic symbols as displayed in most textbooks and standardized exams.

- 1N + 10 = 3

To solve an abstract problem students write each step in Arabic symbols.

FIGURE 2 Concrete, representational, and abstract examples of an inverse operation.

RESULTS

Repeated measures analysis of variance was performed on two levels of instruction (CRA vs. abstract) and three levels of occasions (pretest, posttest, and follow-up). See Table 3 for a summary of student performance on the assessments as a function of condition and time. Both groups showed significant improvements in answering single-variable algebraic equations from the pretest to the posttest and from the pretest to the follow-up. However, the students who participated in the CRA instruction outperformed their traditional abstract instruction peers on both the posttest and follow-up test.

Instructional Differences

The interaction between test occasion and treatment condition yielded a significant difference ($F(2, 66) = 13.89$, $p < 0.01$). Calculation of a point biserial ($r^2_{pb}(33)=0.56$) showed that 56.27 percent of the variance of the posttest scores was accounted for by the type of instruction. Since three follow-up tests were used,

TABLE 3
Mean and Standard Deviations for Each Test Within Each Instructional Group

	Descriptive Statistics			
	N	*Maximum*	*M*	*SD*
PRETEST CRA	34	2	0.12	0.41
PRETEST Abstract	34	2	0.06	0.34
POST CRA	34	23	7.32	5.48
POST Abstract	34	17	3.06	4.37
FOLLOW CRA	34	22	6.68	6.32
FOLLOW Abstract	34	21	3.71	5.21

Note. Total possible score is 27 for each measure.

the Bonferroni correction procedure ($\alpha = 0.05/3 = 0.017$) was used to maintain a 95 percent confidence level.

Follow-up analyses on group means indicated that the students who received CRA instruction over the four-week intervention outperformed matched students who received traditional instruction during the same time period with the same teacher. Although there was no significant difference between the pretest scores ($t(33) = 0.63$, $p = 0.27$) of the two groups, there were significant differences at posttest and follow-up. On the posttest, the group receiving CRA instruction ($M = 7.32$; $SD = 5.48$) outperformed the group who received abstract instruction ($M = 3.03$; $SD = 4.39$), $t(33) = 6.52$, $p < 0.01$. The group who received CRA instruction ($M = 6.68$; $SD = 6.32$) also outperformed the abstract group ($M = 3.71$; $SD = 5.21$) on the three-week follow-up test ($t(33) = 3.28$, $p < 0.01$).

Change Over Time

The initial instruction in the treatment and comparison groups combined showed significant improvement in students' ability to solve single-variable multiple-step algebra equations ($F(1, 33) = 31.98$, $p < 0.01$). Post hoc analysis of variance within each instructional group showed that the abstract group improved from pretest to posttest and pretest to follow-up ($F(2, 99) = 8.34$, $p < 0.01$), as did the CRA group ($F(2, 99) = 23.10$, $p < 0.01$). The abstract group increased from pretest ($M = 0.06$; $SD = 0.34$), to posttest ($M = 3.06$; $SD = 4.37$) and then scored similarly on the follow-up exam ($X = 3.71$; $SD = 5.21$). The CRA group increased from pretest ($M = 0.12$; $SD = 0.41$) to posttest ($M = 7.32$; $SD = 5.48$) and scored similarly on the follow-up exam ($M = 6.68$; $SD = 6.21$). Although both groups showed a significant increase from pretest to posttest and pretest to follow-up exam, neither group showed any reliable change from their respective posttest to follow-up exams.

Practical Significance

Although the posttest scores appear low compared to the overall possible score of 27, the students' growth was remarkable. As stated earlier, the assessment instruments included only items that were answered correctly by 37.5–62.5 percent of the group of students who had completed Algebra 1 with a C or better. This means that a large number of average- to above-average-achieving students who passed algebra were unable to correctly answer all or most questions on this assessment. Also, students practiced each of the five skills for only three to four lessons, totaling 19 lessons, while in the typical classroom a teacher might spend a few weeks on each skill when students show difficulties. The assessment design anticipated that although students learned many parts to solving algebra equations, a few would be unable to answer posttest and follow-up questions on transformations. In actuality, 15 students in the group who received traditional abstract instruction scored 0 correct on the posttest, while only three students who received CRA instruction scored 0 on the posttest. Therefore, students who answered 10 questions correctly in the present study achieved near the average of students who had successfully completed an entire Algebra 1 course. For this reason, it is encouraging that the students who learned how to process algebra equations through the CRA model averaged nearly 7 correct answers out of 27 relatively difficult equations.

Although the reliability of group differences was of importance to this research, the high variability within each group was also important. Standard deviations were fairly high relative to mean scores for the pretest and posttest. In addition, standard deviations increased from the posttest to the follow-up test. Although students selected for the research had learning disability or at-risk labels, they were still highly variable in their performance following instruction. This variability is evident in Table 4, which displays raw scores on each assessment for each student.

Error Pattern Analyses

Examining answers on tests and daily lessons not only allows for inspection of more than right or wrong answers, but also provides some indication of common error patterns for the two groups. This information helps us understand why a student might have solved for a variable incorrectly. Table 5 contains a summary of the noted error patterns. Although both groups made errors solving with negative numbers and adding opposites to both sides, the abstract group made more errors attempting to combine variables and numerals. Students in the CRA instructional group may have made fewer computational errors because the materials in the hands-on and pictorial instruction reinforced arithmetic concepts.

Summary

Both groups of students showed significant learning from the pretest to the posttest. However, on the posttest and follow-up exams, the students who participated in CRA instruction outperformed the students who participated in traditional abstract instruction. In addition, inspection of error patterns in the two groups of students indicated that the types of errors may match the type of instruction.

DISCUSSION

Teachers need to use concrete and pictorial representations that are appropriate to the age level and developmental level of the students. Howard, Perry, and Conroy (1995) noted that many teachers in secondary settings

TABLE 4
Individual Matching Data Across Teachers

	Pretest Scores		*Posttest Scores*		*Follow-Up Scores*	
Student Match	*Abstract*	*CRA*	*Abstract*	*CRA*	*Abstract*	*CRA*
1	0	0	2	0	8	0
2	0	2	0	3	7	3
3	0	0	0	4	0	18
4	0	0	5	16	4	14
5	0	0	2	4	8	5
6	0	0	9	5	16	17
7	0	0	7	5	9	5
8	0	2	17	21	21	22
9	0	0	0	5	0	9
10	0	0	0	6	4	6
11	0	0	1	9	1	4
12	0	0	0	5	2	9
13	0	0	1	12	0	10
14	0	0	9	11	3	16
15	2	0	9	11	4	7
16	0	0	14	23	15	21
17	0	0	0	14	1	4
18	0	0	4	9	2	11
19	0	0	0	4	0	2
20	0	0	0	3	2	1
21	0	0	0	4	2	1
22	0	1	1	6	1	3
23	0	0	6	13	9	10
24	0	0	0	0	0	3
25	0	0	2	7	0	2
26	0	0	2	9	0	9
27	0	0	0	2	0	0
28	0	0	5	6	3	5
29	0	0	7	11	4	4
30	0	0	1	7	0	0
31	0	0	0	3	0	0
32	0	0	0	8	0	6
33	0	0	0	0	0	0
34	0	0	0	3	0	0

Note. These scores are based on total correct answers out of a maximum test score of 27.

do not use manipulative objects with their students. Some secondary teachers may not trust the usefulness or efficiency of manipulative objects for higher-level algebra. For this reason, some popular algebra models market more to elementary teachers than secondary educators. The present algebra model, however, maintains the algebraic concept while allowing students to view equations in a different manner than by abstract alone. This algebra model shows promise to enable teachers to teach higher-level concepts in a manner that actively involves students, generalizes to complex equations, and adapts to individual learning styles.

The power of the CRA sequence of instruction is also supported by this research. The CRA sequence of instruction has been beneficial to students with disabilities and academic difficulty in the learning of basic facts (Harris, Miller, & Mercer, 1995; Mercer & Miller, 1992) and initial fractions (Jordan, Miller, & Mercer, 1999). Research even supports the CRA sequence to represent word problems in simple algebraic inverse operations (Maccini & Hughes, 2000). However, prior to the present research study, there has not been a published examination of a manipulative and pictorial method that translates into more complex equations beyond simply solving for single inverse operations. Not only was the present CRA sequence of instruction model applicable to equations with coefficients other than one, but also the students were able to make significant gains from pretest to posttest in solving algebraic transformations.

Some programs for students have been effective in one-on-one or small-group instruction (Maccini & Hughes, 2000), but it is unclear how teachers can use the program for an entire class. Because the CRA model in the present study was taught to diverse learners in mainstream classrooms, the CRA sequence appears to be effective within whole-class settings characteristic of inclusion models. Although students with disabilities have not always been academically successful in mainstreamed settings (Zigmond & Baker, 1995), this CRA algebra model proved effective for students with identified disabilities and those at risk for math difficulties in a setting with normally achieving peers.

Limitations and Future Direction

There were some limitations to this study. First, the assessment instrument used for pretest, posttest, and

TABLE 5
Error Patterns Common in Incorrect Answers

	Frequency of Error			
Error Type	*CRA Group*	*Abstract Group*	*Sample Problem*	*Example of Error*
Negative numbers	22 students and 73 errors	18 students and 56 errors	$-2y = -14$	Students see a negative and place it in the answer without knowledge of the rules of negatives.
Add opposites to both sides	17 students and 66 errors	12 students and 37 errors	$5x = 2x + 6$	Instead of subtracting, the student added 2x to the left side making the wrong equation $7x = 6$.
Adding variables and numerals together	1 student and 3 errors	9 students and 57 errors	$3x - 4 = 8$	The student would combine the $3x - 4$ to be $-1x$ thus making the equation $-1x = 8$.

Note. More students in the CRA instructional group attempted solving the transformation equations. Specifically, students in the CRA group made 652 attempts to answer questions, and 535 of these answers showed their work. Students in the comparison group made 459 attempts to answer questions, and only 348 of these showed their steps to solving the problem.

follow-up in this study was designed specifically for this study and has not been fully evaluated. To measure the differences between traditional instruction and CRA instruction, the assessment was designed to be difficult enough to reduce the chance of ceiling effects while measuring the end result of students advancing through all 19 lessons. To develop the appropriate questions, the assessment included only items that approximately half of those students who had passed Algebra 1 could answer correctly. The questions on the posttest and follow-up tests did not cover the spectrum of the five lessons taught, but rather the final, most difficult step. Although this increases the power of the statistics, it may cause confusion if the scores are inspected through simple percentages. Since most students do not learn the algebraic concepts taught during this project until they are in ninth grade, the sample of students used to develop the assessment differed from the targeted sample of students with learning disabilities in the study. Additionally, the assessment was not standardized to the entire district but merely reflected the knowledge of 32 students mixed between public and private school who had completed Algebra 1 with an A or B grade. Only eight of the students who participated in the design of the assessment had been diagnosed with learning disabilities. The same assessment items were used in the pretest, posttest, and follow-up tests. Although there may be a concern with history effects, most teachers claimed that it would be uncharacteristic of their students to remember the specific items from one test to the next.

A second and related limitation relates to the score of the assessments. Two teachers commented that the hands-on program resulted in student success that was not addressed in the assessment provided. One teacher remarked that the assessment covered only the most difficult material and not the other four types of equations and expressions learned. Although the teachers noted a limitation within the assessment development, their comments supported the new algebra model for teaching students at the sixth- and seventh-grade levels. There were clear differences in performance between students who participated in the CRA model and those who participated in traditional instruction, but an encompassing study of the components with algebra for these age levels needs to be examined.

A third limitation relates to the lesson sequence. As previously discussed, this study employed the sequence of lessons found in typical algebra textbooks (i.e., Houghton Mifflin (Brown et al., 1988), Scott Foresman-Addison Wesley (Charles et al., 1998), and Saxon (Saxon, 1997, 2000)). Much of the low student performance for these grade levels may be due to the untested sequence of lessons listed in different textbooks. Matching reduced the possibility that the curriculum lesson sequence influenced the comparison of the treatment and comparison group. However, the curriculum does influence how well students perform overall.

The nature of the assessment and the sequence of lessons may have led to a reduction of perceived student performance on the posttest and follow-up test. However, these limitations affected each class similarly. Since the data analyses were compared across matched subjects with the same teacher, any differences in how these limitations affected each member of a matched pair should be negligible. However, some of the suggestions of teachers and students, as well as observations throughout the project, indicate the need to increase the number of lessons at the representational and abstract levels. This information will help educators prepare future research on the CRA sequence of instruction and research on algebra instruction in general.

Conclusion

This research on the effectiveness of CRA sequence of instruction for algebra learning among students with math difficulties brings continued insight into the effectiveness of hands-on manipulative objects and pictorial representations for complex mathematics. The students in this project who received CRA instruction performed better on posttests and follow-up tests. They committed fewer errors with negative numbers and with transformations of equations before solving for variables. This research contributes to the growing understanding of algebra instruction for students with learning disabilities and those at risk for failure in secondary mathematics. Continued research on helping students with disabilities to understand algebraic concepts will improve students' abilities to think abstractly and may improve graduation rates.

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