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Slower Algebra Students Meet Faster Tools: Solving Algebra Word Problems With Graphing Software

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The article discusses the ways that less successful mathematics students used graphing software with capabilities similar to a basic graphing calculator to solve algebra problems in context. The study is based on interviewing students who learned algebra for 3 years in an environment where software tools were always present. We found differences between the work of these less successful students and the traditional problem-solving patterns of less successful students. These less successful students used the graphing software to obtain a broader view, to confirm conjectures, and to complete difficult operations. However, they delayed using symbolic formalism, and most of their solution attempts focused on numeric and graphic representations. Their process of reaching a solution was found to be relatively long, and the graphing software tool was often not used at all because it did not support symbolic formulation and manipulations.

Key words: Algebra; Computers; Functions; Graphing calculators; Mathematical modeling; Reform in mathematics education; Representations; Secondary 7–9

Studies of alternative approaches to school algebra suggest new aspects of students' understandings and misunderstandings (e.g., Bednarz, Kieran, & Lee, 1996; Bell, 1995; Confrey & Smith, 1995). Studies of current innovations in teaching algebra with technological tools point to a shift from rote memorization to meaningful mathematical action including taking initiatives and making decisions, grasping multiple linked representation, and managing various sources of information to solve new types of problems using technology. Technological tools are not neutral, however, with respect to different conceptions of function and of algebraic symbol representation systems (Yerushalmy, 1999; Yerushalmy & Chazan, 2002). Therefore, fundamental changes in algebra curricula should also address the capabilities of the tools and the ways they are being used by learners. Current research (e.g., Artigue, 2002) endeavors to generate a discourse about these opportunities and to learn how to see them as part of the curriculum.

Algebra reform has taken several approaches, some of which can be categorized as “a functions approach to algebra.” Although there are important differences

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among these approaches, each organizes the algebra curriculum around the concept of function, emphasizes and supports concrete representations, and bases learning on situations that appear realistic and are centered on mathematization in the form of modeling and of abstractions at different levels (Kieran & Yerushalmy, 2004). In such a curriculum, students make conjectures and perform actions with physical tools and representations in ways that were not possible in traditional algebra (e.g., Drijvers, 2000; Lesh & Doerr, 2003; Monk & Nemirovsky, 1994). A large part of the learning is built upon student ideas mediated by tools and activities. It is questionable, however, to what extent this shift in goals and habits works out for all students. The reform discussion in general and specifically about algebra does not focus on low-achieving students. And although algebra for all is an explicit target of current reforms, there is a shortage of studies on how less successful students cope with the new complex environment, including the use of technology (Schoenfeld, 2002).

To examine the learning of less successful algebra students who are intimately involved with technology, we analyzed the work of a sample drawn from the lower 25% of middle school students aged 13–15, studying in Grades 7 to 9, who were taught algebra using a function approach: VisualMath (Yerushalmy, Shternberg, Gafni, & Bohr, 1994/1995). They did not consider themselves to be strong in mathematics and assumed that others perceived them in the same way. They had been tutored in mathematics outside of class, starting in elementary school. They were well regarded by their colleagues and teacher, and a few were encouraged to take courses in high school that were higher than basic-level mathematics. However, they did not dare to do so and continued throughout high school to take the minimum required mathematics. They performed well in other fields and graduated from high school. We followed the students informally in subsequent years and found that they tended not to pursue higher studies in mathematics and the sciences but favored the humanities.¹ To observe the challenges faced by the lower 25%, the analysis focused on the ways in which they used the function graphing tool (software with capabilities similar to a basic graphic calculator that presents two-dimensional graphs and numerical values for any single variable expression) in solving problems in context.

A FUNCTION APPROACH TO ALGEBRA: VIEWS AND TERMS

Function Representations

One of the major attributes of the function as a central idea for learning algebra is its capacity to organize algebra around a small number of concepts. For example, an unknown is not a new concept but a special case of a variable; a symbolic expression is a function rule; equation, identity, and inequality can all be statements asking

¹ Thus, although these students' mathematics achievements were relatively low, the term "low achievers" does not properly describe them.

for relations between the output values that two functions produce for the same input. At the same time, the function as a central idea suggests that work should be focused on the various types of representation systems of the function: words, numbers, graphs, and algebraic symbols (see Figure 1). Solving problems in context requires constant changing of views and mapping between representations.

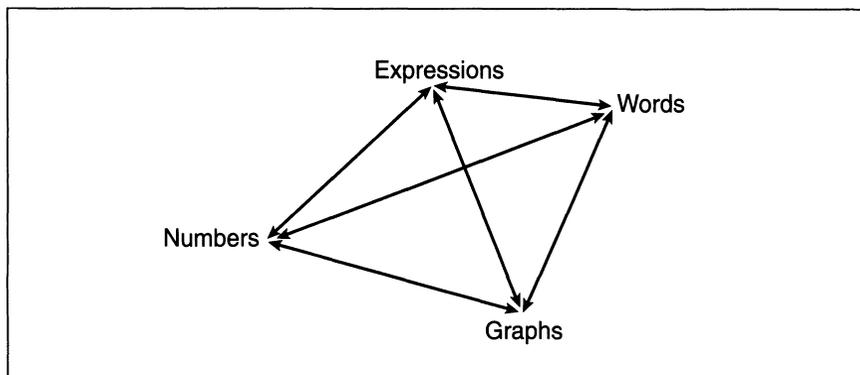


Figure 1. The tetrahedral relations of function representations.

Each representation provides one or more views of the function. I borrow the terms *explicit* and *recursive* (that are generally used to discuss sequences) to describe two such views, each of which provides different information about a function. A recursive view is a process of thinking about patterns and functions by taking a multiplicative or additive view of variations. But to choose an appropriate explicit expression that is computationally efficient, students cannot simply identify the recursive properties of a function. In analyzing tabular information, identifying the input-output pattern would help identify the explicit rule, while the pattern of differences might be described by a recursive rule. Constructing a graph according to an explicit rule consists of point plotting various inputs, whereas a recursive view may invite construction of each point as it relates to the preceding one (these differences are described schematically in Figure 2). Combining both views is what ultimately provides a satisfactory understanding of a function and its properties. Graphing software (as used in this study) requires explicit rules as its input.

Developing competence by solving real-world problems in function-based algebra means learning to move freely along the tetrahedral path described in Figure 1 and between the views described in Figure 2. It means choosing an appropriate representation to describe the situation at hand and identifying the isomorphism between representations. In analyzing issues of competence in a particular domain, Schoenfeld (1986) pointed to several factors that may hamper the development of conceptual knowledge. One factor is the nontrivial interaction between the two types of knowledge of the two representation systems. In solving contextual problems, construc-

Representation \ View	Numbers	Graphs	Expressions
Explicit	$\begin{array}{c c} x & f(x) \\ \hline & \rightarrow \\ & \rightarrow \\ & \rightarrow \end{array}$		$f(x) = ax + b$
Recursive	$\begin{array}{c c} x & f(x) \\ \hline &] \\ &] \\ &] \end{array}$		$f(x + 1) = f(x) + a$

Figure 2. Two views of linear functions in three representations.

tion of a symbolic system for real-world knowledge is a complicated task because “knowing” the actual phenomenon cannot be easily manifested in symbolic language, even when one is familiar with the language. Another factor is the inconsistency or incompleteness in an isomorphism that may not extend to all occurrences of one of the systems. For example, consider the functions $x + 3$ and $4 + x - 1$; from the point of view of actions carried out with the function’s rule, these are two different rules that describe different processes. If, however, one were to plot the output of each of these rules on a Cartesian plane, the two would be indistinguishable. The symbolic representation of function makes its process nature salient, whereas the graph suppresses the process nature of the function and invites encapsulation.

Solving a Problem in Context

The tasks analyzed in this study were problems in context. For example, consider the Parking problem (the ₪ symbol represents the new Israeli shekel):

Two parking lots are located next to each other. One lot charges 2.7 ₪ per hour and a proportional price for fractions of an hour. The second lot charges 2 ₪ per hour and a flat 6 ₪ per entry. Describe the rates of the two parking lots in a way that would allow the customer to choose the best offer.

One algebraic representation of the problem is $2.7x = 6 + 2x$, the solution of which is $x \approx 8.57$ hours. Therefore, a customer who plans to park at least 9 hours should choose the lot that charges for entry but charges a lower hourly rate.

Within the VisualMath function approach to algebra, in which rate of change is a central concept, some or all of the following pieces of knowledge would be used in solving a problem in context (not necessarily in a specific order): identifying two

different processes in which a quantity (price) changes as a function of time (see Figure 3); sampling and computing input-output values directly from the story, organizing them in an ordered table, and computing the sequences of differences in both processes; identifying that both processes are linear because the differences over constant intervals are constant; describing the two processes graphically as two intersecting lines that represent two linear functions (graphs suggesting an approximate relation between the two processes can be sketched on paper or accurate graphs can be drawn on paper using the tabular data or using the graphing tool with function expressions as input). By finding a common output it is possible to arrive at a numeric value that is close to the solution (because the solution of this specific problem is not an integer). An approximated solution can also be read from the graphs (its accuracy depending on the scale of the graphs): the intersection of the lines marks the number of hours for which parking in both lots costs the same. The equation $2.7x = 6 + 2x$ represents the comparison of the two processes. Manipulating the equation to find the solution involves producing equivalent equations, and the simplest equivalent equation (the equation from which it is easiest to read the solution) is $x \approx 8.57$. Solving the equation for x provides the input that in turn provides the common output. No preferred order among the three options (table, graphs, equation) is suggested in the curriculum.

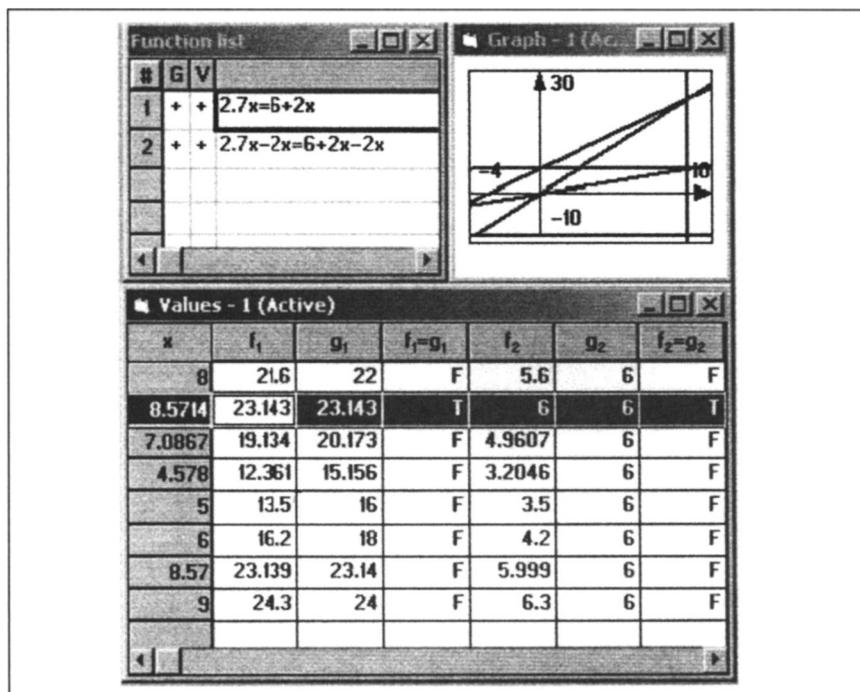


Figure 3. The screen of the solution of the Parking problem with the graphing tool.

The knowledge involved in such a solution consists of mapping between the situation and the function as its mathematical model, describing processes and manipulating objects in numerical and graphical representations, and shifting between recursive and explicit views. Solving problems of this type offers an opportunity to implement the modeling skills acquired earlier, at the presymbolic stage (described in Yerushalmy & Shternberg, 2001) and to use them for thinking about symbolic models and solutions of linear equations and inequalities. In solving algebra problems in context, function graphing software is used most frequently to enter a symbolic expression of a function (in a single variable) and observe a graph over the visible domain and a tabular representation of input-output values in this domain. This requires first an abstraction of the morphism between terms of the real world and formal algebraic symbols to create a symbolic model. The software then provides interactive mapping between the expression and the graph or value table.

Examination of technologically supported school algebra curricula suggests that graphing calculators are frequently used when the curricular focus is on developing understanding of linked expressions, graphs, and tables in order to expect, explain, and conjecture about symbolic manipulations and about qualities of expressions, as well as about the nature of solutions and equations (Heid, 1996). Studies of students working with a function-based approach to algebra, based on relations between quantities, also suggest some important aspects of the role of realistic situations in algebra (e.g., Chazan, 2000; Hershkowitz & Schwarz, 1997; Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000; Yerushalmy & Gilead, 1997). The numeric functionality of graphing technology reduces the need for algebraic manipulations, as one can approach a solution by reading rather than computing the solution. However, understanding the underlying mapping is conceptually complex. This study explores the ways in which graphing software and algebraic manipulations might be integrated in a reform algebra curriculum for less successful students.

LOWER ACHIEVERS AND REFORM CURRICULA

In the traditional teaching approach, low achievers are often seen as only being capable of adopting efficient procedures that secure a correct solution. Their problem-solving processes are usually short, characterized by a rapid decision based on a single method of solution and rapid give-up in case of failure to complete the task (Cardelle-Elawar, 1995). Curriculum efforts to offer students learning situations that would help them access ideas conceptually have often failed, at least in part because students were not involved in meaningful mathematics and thus had difficulty implementing what they had learned (Arcavi, Hadas, & Dreyfus, 1994). Other studies observed a lack of tendency and ability on the part of low achievers to assess the quality of answers appropriately; inability to stay with one task long enough; loss of control when the problem developed into several possible directions; and fragmented knowledge (Arcavi et al., 1994; Chazan, 1996, 2000; Nesher, 1987; Watson, 2002). As a result, teachers usually concentrate on remedial activities—on how to recall and use facts and procedures, focusing on actions. The usual

type of activity recommended for low achievers requires performing simple computational procedures. Students are not taught to face decisions about what to do and how to abstract from actions to processes of several steps but rather learn algorithms in a fragmented way that they often perceive as illogical and difficult to remember over time (Tobias, 1990).

Probably as a result of a history of poor achievement, low achievers are often described as uninterested in participating in class discussion or in obtaining feedback on their ideas. Traditional methods of supporting less successful mathematics students tend to discount the contribution of Standards-based curricula to these students. At the same time, recent studies on Standards-based algebra offer some insight into the work of less successful students. In the intensive work with students and teachers in the low track carried out by Chazan (2000), low-track students, some of whom failed previous algebra courses, were able to develop important aspects of conceptual understanding. Huntley et al. (2000) investigated solutions of contextual problems by lower and higher achievers and suggested that new approaches to algebra could enable unsuccessful students to gain access to problem-solving strategies traditionally unavailable to them. They found that students who were not strong in symbol-manipulation skills could outperform symbolically capable students when the tasks required formulation and interpretation of situations.

STUDY DESIGN CONSIDERATIONS

The Interviewees

The general objective of the project was to study the problem-solving processes developed over 3 years in two algebra classes where various software tools, including graphing tools, were available at all times. Twelve pairs of students, chosen in the first semester of the seventh grade, formed the study sample: three pairs of successful students (upper 25%–U25), three pairs of the less successful students (lower 25%–L25), and three average pairs. The present study analyzed primarily the lower achievers in the sample at a time when they were in eighth and ninth grades, and graphing software was being used routinely. The criteria used to determine the level of achievement of the interviewees included their final mathematics grade in the sixth grade and the impression gained by the interviewer during the first quarter of the seventh grade. This impression was based on student participation in small-group work (in pairs) with the computer and in whole-class discussions, verbal clarity, and cooperation with the teacher and classmates. The students categorized as L25 were those whose final grades in the sixth grade were in the lower 25% and who required additional explanations from the teacher, more demonstration and instruction in computer tasks, and often evidenced dissatisfaction during whole-class discussion.

Students were interviewed five times: in the spring of their seventh grade and in the fall and spring of the eighth and ninth grades. The seventh-grade interviewees had completed a unit of qualitative modeling whose central concept was rate

of change. They were acquainted with descriptions of functions through numeric pattern recognition and had developed a concept of the rate of change of functions using qualitative graphing software tools (further descriptions can be found in Yerushalmy & Shternberg, 2001). This study focuses on data collected during the eighth and ninth grade (interviews 2, 3, and 4), when students were already acquainted with symbolic representation. In the eighth grade, they became familiar with manipulations of relatively simple expressions and of equivalent expressions and solved contextual problems similar to the parking problem described earlier. They did so most often by comparing two or more linear processes, forming an equation, and reaching a solution by reading graphs or tables or by manipulating the equations.

The ninth graders continued to solve contextual problems. They were acquainted with procedures and manipulations of more complex expressions related to a wider variety of types of functions. In this curriculum, expressions were first offered as a tool for describing the generality of tables and graphs of functions encountered earlier. Only later was the symbolic manipulation of expressions and relations (equations and inequalities) practiced. Software tools were always available in class, and students were encouraged to use them at home. The techniques involved in using the graphing tool were practiced routinely in the eighth and ninth grades, including graphing, obtaining a table of values from an expression, analyzing differences of values, reading linked representations of functions, scaling to adjusting for appropriate information, comparing two functions drawn on the same coordinate plane (by evaluating their shapes and values and by plotting a difference function), and reading the values at the intersection of two function graphs. Students learned to manipulate and solve problems both with and without the graphing tool, with emphasis on one or the other depending upon the natural way in which the environment suited the action.

Students usually chose their partners for working in pairs, and frequently remained as a pair for the 3 years. Students were also with the same teacher for all 3 years. Making explicit the students' thoughts, anticipations, and conjectures and their convincing arguments and disappointments was the norm in the class. There were also norms regarding the use of tools. The software was used in different modes: at times, students worked in pairs with a computer on activities that specifically required explorations with the software. At other times, they worked on tasks and would decide whether or not they wanted to use computers. Yet at other times, the teacher used a single computer as a tool for both students and teacher during whole-class discussions.

The Interview Tasks

For the interviews we sought problems that would provide an opportunity to follow the process by which students construct meaning for functions in ways related to algebra and that would provide opportunities for them to think about the meanings of symbols (e.g., as variable/unknown) and of the manipulation of equations

as linked to the problem's story and to parallel representations. All the problems could be solved by an equation in one variable. The problems were new to the students to some extent. Although some problems were of the more familiar break-even type, each one had been structured to include a unique aspect that would make the work interesting and challenging. Table 1 describes the problem itself, the anticipated solution, the structure that characterized each of the problems, and the challenge to students at the time of the interview.

Interview Format and Goals

The structure of the interviews allowed about half an hour for solving each problem. The interviewer, an educational psychologist who was observing the students on a regular basis in class, mostly listened, although she also helped the students move forward when they felt stuck, prompted them to review their work, and asked for their comments. At times, if she found that there was no way for a student to make further progress, she stopped the work. Although the interviews were performed outside the classroom, they followed classroom procedures and norms: Students were interviewed in pairs (the same pairs as in class) and the type of work resembled the problem-solving norms in the classroom, both in environment and discourse. The major goal of the interviews was to observe the processes by which the L25 constructed mathematical meaning while solving traditional word problems with the function graphing software. We did not look for a specific solutions structure in order to identify a type of work but rather attempted to explain student actions as participation in meaningful situations. The interviews were videotaped, transcribed, and translated into English. Clarifications and completions of partial sentences appear within square brackets.

DATA ANALYSIS: USES OF THE GRAPHING SOFTWARE TOOL²

L25 students were frequently hesitant to use the computer, although it was always offered among the other tools (paper, ruler, arithmetic calculator) as part of the interview. These hesitations and efforts to solve the problems without using the graphing software were absent from the work of U25 students, who used the computer at very early stages in the solution process. While interviewing the U25, we often had to ask them to hold back their solution with the computer for a while and talk to us about their initial thoughts.

In one of the interviews with a pair of L25, Vera³ and Mor (then eighth graders), the question of using or not using the tool became an explicit issue. After Vera and Mor solved the Number Game, they talked with the interviewer about their solution process. Their comments about the way in which they participated in tool-mediated

² The data analysis is based on the research of Suheir Nasser described in Nasser, 2000.

³ The names that appear throughout are pseudonyms, which have been coordinated with other papers presenting data from interviews with these students.

Table 1
Tasks Used in Interviews 2, 3, and 4

Characteristics of the Activity	The Problem	Anticipated Solution
The Parking Problem—Second Interview		
<p>A linear break-even problem involving two continuous processes with a solution that is not an integer.</p> <p>Students were familiar with numerical and graphical descriptions of break-even problems but had relatively little practice in solving linear equations.</p>	<p>Two parking lots are located next to each other. One lot charges 2.7 ₪ for an hour and a proportional price for fractions of an hour. The second lot charges 2 ₪ an hour and a flat 6 ₪ per entry (e.g., half an hour of parking costs 1 ₪). Describe the rates of the two parking lots in a way that would allow the customer to choose the best offer.</p>	<p>Problem equation: $2.7x = 6 + 2x$. Can be viewed as a comparison between two linear functions, each representing the parking fee for any amount of time. $x \approx 8.57$</p>
The Number Game Problem—Second Interview		
<p>Involves a quadratic model. Is often given as a puzzle not directly related to algebra and functions.</p> <p>In class, students were learning about functional properties of both linear and nonlinear processes.</p>	<p>You are to participate in a number game with the following rules: Choose an initial number and multiply it by 2. Subtract the result from 10 and multiply the result by the initial number. The winner is the one who obtains the highest number as a result. Can you find the numbers that would enhance your chances of winning? Can you always be a winner? (Note that you can choose any number: an integer, a fraction, a negative number, etc.)</p>	<p>$f(x) = (10 - 2x) \cdot x$</p> <p>The maximum is reached at $x = 2.5$.</p>
The Age Problem—Third Interview		
<p>Unlike other problems, this one is not intended to elicit a comparison between the two growth processes: of the father and his son.</p> <p>Students had practiced in class solving problems by comparing two processes represented by graphs or expressions.</p>	<p>The father is 24.5 years older than his son. In 2 years he will be 1.5 times older than his son. How old is he?</p>	<p>The father's age in 2 years is the unknown, x; the son's age is $x - 24.5$. The equation is $x/(x - 24.5) = 1.5$ or $x = 1.5(x - 24.5)$. If described as a function, the graph describes the relation between two quantities; if described as two linear functions, one describes the aging of the father and the second describes the 1.5 times the aging of the son.</p>

Table 1—continued

Characteristics of the Activity	The Problem	Anticipated Solution
	The Bonus Problem—Fourth Interview	
A linear break-even problem. Requires considering relations between two processes.	An employer wants to give workers a bonus for the holidays. The accountant suggests two methods. Method A: The January payment includes an additional 450€ per person. Method B: The January payment is increased by 1/5 of each employee's monthly salary. Present the cost of both methods to help the owner select a strategy.	$f(x) = x + 450$ and $g(x) = x + x/5$. x represents the range of all possible salaries. If described as two linear graphs, each would represent the range of payments according to one of the methods. The break-even value (2250) can be read from the intersection point, but the full solution should consider inequality relations between the two functions.
Students routinely solved such problems by solving equations with or without the graphing tool.		

algebra learning provide an example of the norms and modes of use that had developed around the graphing software for the L25 sample.

Vera and Mor correctly viewed the Number Game story as the expression $(10 - 2x)x$ and started to substitute numbers for x , obtaining output and organizing it in a table. At that time, they did not have a formal knowledge of the properties of quadratic functions and had limited experience analyzing the geometric and numeric properties of parabolas. Vera suggested computing only even numbers from 1 to 10. She justified this by arguing that values 2 and 3 for x give the same output. Although 2 and 3 are symmetric around the line of symmetry of the parabola ($x = 2.5$), at this stage the students did not yet think about a quadratic model and its properties, so 3 appeared to be redundant. Vera assumed that this would remain true for all odd numbers. Then they realized that for both 0 and 5 they obtained 0, and the strategy changed. Mor argued that the smaller the number the better the result. But Vera returned to the first finding that 2 and 3 gave the same value and computed the result for 2.5. That produced 12.5, which they acknowledged as the highest result they had. Vera wanted to continue calculating the result for 2.75, but Mor wanted to use the computer to see the table and graph and “everything in principle” (a global picture). Vera still believed that the arithmetic calculator was adequate to finish the work, but when she finished the calculations for 2.25 (which surprised them by being the same as the output for 2.75), Mor argued again that the table of values they could obtain using the software would show everything. They turned to the graphing tool, scaled the graph to see the values they had anticipated together with the graph, and declared that “2.5 was the best result and 2.25 and 2.75 were the same as we did.” Thus, their solution was a blend of conjectures, proposals for strategies to be followed, and computations. They were surprised to notice the symmetrical property of the results and were confident in their search and in the results. They used the computer for confirmation, mainly because Mor wanted to have the full picture of the symmetry she

had started to see in her computations and which she had started to describe as a process. They did not mention the shape of the graph and its single maximum, nor did they return to view the properties of the expression. They concentrated on the symmetrical character of the table of values.

Following the interview, MS (the interviewer) asked them to spend some time talking about the method they used to reach a solution. MS was interested mainly in their views about the mathematical meaning of the different uses they made of the paper, pencils, arithmetic calculator, and graphing software.

- 0.1 Vera: Usually when we want to construct something we start with different examples. Like we had said what is the upper number and what is the lower . . . and only after that we checked it.
- 0.2 MS: Different conjectures, you mean?
- 0.3 Vera: Right. . . We conjectured and then we checked.
- 0.4 MS: And am I hearing that you are trying to say that you don't consider this work as valuable? Is that what you said?
- 0.5 Mor: Yes, pretty much.
- 0.6 Vera: I think no matter what, you always need to prove, it is not always what looks logical to you that matters.
- 0.7 MS: And you don't consider your work here as proof?
- 0.8 Mor: Maybe it is. . .
- 0.9 Vera: But if you come to someone without a computer? And if you want to show each value? You would need to describe everything! You need proof.
- 0.10 MS: Let me ask you something: Why did you ask my permission to use the computer? You didn't do that when you borrowed paper and a calculator.
- 0.11 Vera: I do it myself when the tasks ask explicitly to check with a computer—then I check.
- 0.12 MS: But if in this problem you could have done both?
- 0.13 Vera: So I try it first on paper.
- 0.14 MS: And then, do you still use the computer?
- 0.15 Vera: Yes, when it gets hard, when I want to see it. . .
- 0.16 Mor: I use it to check with. But there are tasks where there is no choice—you have to do them on paper—if I need to shorten [simplify] an expression, I must do it on paper.
- 0.17 Vera: Exactly!
- 0.18 Mor: The computer wouldn't do it for me.
- 0.19 Vera: Exactly right!

The first part of the conversation [0.1–0.9] reveals their ambivalent attitude toward their work with the graphing tool. They saw the conjectures as an important but insufficient part of the work. The conjectures had to be proven or explained by a proof and seeing them on the screen did not seem to constitute such proof. Mor hesitated [0.8] about whether the information on the screen might support a different kind of reasoning. Vera, who first argued for insufficient proof, stated that the complexity was a technical obstacle; she needed proof because she had to be able to talk about it when the computer was not available. The second part [0.10–0.19] helps us understand the distinctions upon which their instrumentation

of the tool was based. They applied definite criteria for the modes of use and integration of the computer. Vera would use it for checking and feedback; Mor would use it if she wanted to see the complete picture of a partial image she had already constructed, and when the problem became difficult [0.15]. She also stated when she would definitely not use it: if she had a transformational task that required symbolic manipulations.

The way in which Vera and Mor clarified their strategy for working with the tool illustrates the problem-solving approaches that we identified in the first round of the analysis of solution processes used by the L25. A second round of analysis was then organized to analyze in depth the four types of problem-solving processes that emerged in working with the graphing tool: “When I want to see it” [0.15], “When it gets hard” [0.15], “The computer wouldn’t do it for me” [0.18], and “I try it first on paper” [0.13].

When I Want to See It

For Mor, “seeing it with the computer” (while solving the Number Game problem described earlier) meant that it would be helpful to obtain a global picture (“everything in principle”) after paper and pencil calculations supported by an arithmetic calculator had provided local data that she suspected to be more general. Mor started to notice a process of generating symmetric values, and she wanted confirmation of her view by obtaining a dense table of values that she conjectured about in the course of previous computations.

When Gal and Roni were interviewed about the Number Game problem, they had had experience with graphs of nonlinear expressions and could recognize nonlinear patterns using difference analysis. The Number Game problem describes a procedure in a relatively direct way, but Gal and Roni did not express the procedure in symbols. They followed the procedure given in the story by sampling numerical examples that they had organized in an ordered table of values from -5 to 5 . When they did not recognize a pattern, they decided to expand the table from -5 to 10 (see Figure 4). They conjectured and argued for about 10 minutes about the rate of change (e.g., values drop very fast when one moves from 8 to 9 and to 10). They also tried to identify the source of the symmetries they saw in the table.

When it appeared that they were stuck, the interviewer prompted them as follows:

- 1.1 *MS:* So what would you do with these computations? You created a long table. Would you add more examples and find something? Would you continue to check more values with the calculator to find the answer?
- 1.2 *Gal:* I am not sure it is at all possible to get a correspondence rule since the rate of change is not constant and the increase in the rate of change is also not constant. For example, if the differences would have increased by 2 each time. . . But I don’t think it is [increasing] because 2 and 3 are the same.
- 1.3 *Roni:* [Who seemed absentminded] I see it!
- 1.4 *MS:* What did you see?
- 1.5 *Roni:* It simply looks like a parabola.
- 1.6 *MS:* Yes? So. . .

-5	-100
-4	-87.2
-3	-48
-2	-28
-1	-12
0	0
1	8
2	12
3	12
4	8
5	0
6	-12
7	-28
8	-48
9	-72
10	-100

Figure 4. Organized results of Roni and Gal playing the Number Game.

- 1.7 *Roni:* I am not sure but it comes close to zero and goes up again.
- 1.8 *MS:* Okay, so how would you check it?
- 1.9 *Gal:* In a graph.
- 1.10 *Roni:* No, I see it here in the table of values.
- 1.11 *MS:* So what would you do to check your conjecture?
- 1.12 *Roni:* It is difficult, but we need to have a correspondence rule.
- 1.13 *Gal:* But we can do a graph on paper and then we don't need a rule.
- 1.14 *MS:* Aha, you are talking about a graph? On paper or with the computer, if at all?
- 1.15 *Gal:* Yes, but for the computer it means a rule and it is more complicated. I don't think we must have a rule. The rule helps if you want to evaluate for different numbers and see whether it is true. But it is a problem to construct a rule when this [the differences] is not constant.
- 1.16 *Roni:* There is a sequence, a kind of order here. . . [A few more minutes passed and still they were not making progress. MS suggested that they look again at the story rather than at the table of values.]
- 1.17 *MS:* And with this [points to the text] can you construct a rule?
- 1.18 *Roni:* This is just the givens.
- 1.19 *Gal:* Actually this makes sense.
- 1.20 *MS:* [Deriving it] from the text?
- 1.21 *Gal:* Yes, the *any number* is x , the rule is x times 2, then 10, aha, ten minus $2x$ aha. . . times x .

- 1.22 *Roni*: May we try with the computer? [They typed in a mistaken expression, $(10 - x) \cdot 2 \cdot x$, instead of $(10 - x \cdot 2) \cdot x$, saw a parabola, and started to analyze the graph's properties. But something seemed wrong.]
- 1.23 *Roni*: It doesn't make sense.
- 1.24 *Gal*: If you take a number, see, maybe there's a mistake in the rule. . . . [They needed more time and another prompt to review the story and compare it with the expression they typed in. They found the mistake in the expression.]
- 1.25 *Roni*: There is here only one point that should be [winning]. . . Around 5 [points at the maximum].
- 1.26 *Guy*: Not exactly 10. There is a mistake. In our case it is 5, in here we get zero [in their handmade table; see Figure 4].

Gal and Roni chose an interesting way to approach the problem. They tried to make sense of the story by sampling players' moves, probably because it was the more natural way to proceed with a problem asking about numbers. However, they did not sample to find the answer (as would someone playing a guessing game) but to find the behavior of the model [1.2–1.7]. Using their fine observations about first and second differences [1.2], they quickly started to analyze the table, encapsulating the single entries and describing the properties of the table, such as its nonlinearity. Eventually, Roni suggested it represented a parabola. This observation was expanded by the symmetries that helped them conjecture about an almost correct final model. They could have found the correct answer; when MS asked them to describe how they were going to complete their answer, they were already looking for something else, probably for a convincing argument. The convincing argument could have come either from an accurate graph that they could have plotted with the points they had sampled or by generating an expression and comparing its output with the values they already had.

The interviewer's suggestion, reminding them of the possibility of using the computer, was not favored at first by either student. Roni suggested that the sequence contained the rate of change of a parabola [1.5]. He could have reached this conclusion from the fact that the second differences were constant, but he did not say so. What he did say was that this was apparent in the table, so he did not even need a graph to see it [1.10, 1.16]. Gal wanted a graph. They both realized immediately that to use the graphing tool they would need to devise a rule, which they did not know. They could have plotted points on the screen, but they kept looking for a rule. Although they were well aware of the connection between the situation described in the story and the strength of the symbolic representation, they were unable to exploit this connection. The interviewer's suggestion to go back to the text of the situation came when she realized that another link could help them formulate the rule. At this point, they attempted to retell the story using algebraic symbols and arrived at a mistaken expression [1.17–1.21]. Understanding that they have created an entry that would allow them to use the software, Roni asked to use the computer, and they looked at the graphic (Gal) and numeric (Roni) presentations and compared them with their conjectured values to improve and correct the symbolic model and to make the algebra agree with their preliminary conjectures

[1.23–1.26]. Although they had firm expectations regarding the numeric and the graphic results, they could not debug the expression and reformulated it instead.

The computer screen, with the three linked representations of the story problem, was therefore a way for Roni and Gal to reflect on their conjectures by observing a picture created independently from the results they obtained on paper. The table they first created on paper was a record of actions they had taken, as if they were playing the game. Later, this became an object in itself, as they tried to define its properties in order to find the maximum output value. At the same time, the table and graph on the screen were the result of an algebraic model they formed by rewriting the given story. They were looking for this model to agree with their preliminary conjectures. The initial disagreement between the expected results and the appearance that the maximum of the parabola is reached at the value of 5 signaled a mistake [1.23–1.26]. Once they reformulated the expression, the agreement between the two independent parts of the work formed a strong argument, which in fact proved the conjecture they had developed at first by playing the game.

The work of Mor and Vera, who also developed conjectures and then looked to confirm them with the computer, was of a different sort. Their sampling and table of values were already based upon a correct expression rule, which they then used with the software to obtain additional data in support of their conjecture. In both cases, students started with sampling, anticipated values and the behavior of the sequence, and conjectured about the properties of the sequence or of its imagined graph, relating to the properties of functions as objects (such as symmetry, constant and nonconstant rate of change) without using the software. They then used the computer to reach a more global view, to confirm or improve conjectures regarding the correctness of the algebraic model and more so the behavior of the numerical model.

When It Gets Hard

When Vera used the expression “when it gets hard” [0.15] to describe the circumstances that would make her use the computer, she was referring to the difficulty of tedious computations like those she performed with the arithmetic calculator in her work with Mor on the Parking problem. Immediately after reading the problem, Vera and Mor recognized the structure of the problem as being a comparison of two processes. Using the arithmetic calculator they shifted their observations from a single output to the evaluation of a sequence. Then they concluded that there was no point in continuing the computation with one tool and asked to use the other. The software removed the burden of computations. This shift from the arithmetic calculator to the graphing tool took place when they were already convinced that they would need an infinite number of computations to reach a solution. At no point did they try to obtain the solution by solving the equation $2.7x = 2x + 6$, although they formed and typed expressions in the computer, and (based on their class work) they could have easily solved the equation.

Gal and Roni started their work on the Bonus Problem by formulating the correct equation,

$$x + 450 = x + \frac{x}{5},$$

and then began to manipulate terms in order to solve it. They started by writing all the terms as fractions:

$$\frac{x}{1} + \frac{450}{1} = \frac{x}{1} + \frac{x}{5}.$$

They made an error manipulating the equation, added an x to the left side instead of subtracting it, getting

$$\frac{2x}{1} + \frac{450}{1} = \frac{x}{5},$$

and realized that something was wrong but did not know what it was. After a long and stressful pause, they started over. Gal insisted on having an equation because “if we’ll write an equation we’ll see when, at what income it’s equal and when . . . A is better than B.” Roni, however, wanted a sketch (see Figure 5): “Wait, if it’s $x + 450$, let me make a sketch of $x + 450$. . . If it’s $x + 450$ it looks something like this.”

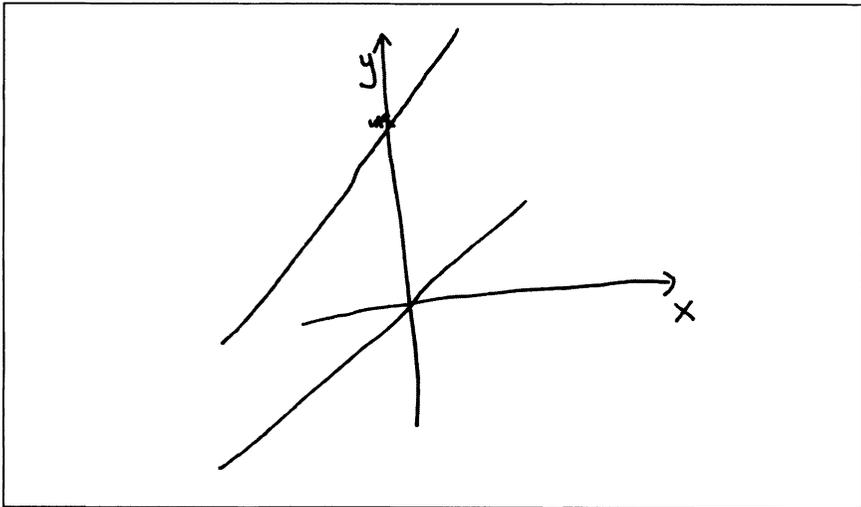


Figure 5. Roni’s sketch of the two methods in the Bonus problem.

Roni’s sketch represented the slopes inaccurately (the lines should intersect in the *first* quadrant). They hesitated about the relations between the lines (Gal: “Wait, it’s x divided by 5 . . . they are parallel. No, but it’s x plus something of x , they are not parallel”) and concluded that they could not be parallel but did not identify any mistake in the sketch. The sketch appropriately served the purpose of demonstrating that the two methods of payment were described by linear but not parallel

functions, and therefore the equation must have a single solution. Roni was certain that use of the tool and analysis of the graphs would lead them to the solution: “If we could use the computer, maybe it would show us right away the two graphs and we could see if they intersect.” Thus, while they failed in performing correct symbolic manipulations, they were confident that they would find the correct answer using the tool. They did not go back to the equation or to reviewing their manipulations but answered the question by explaining what the graphs told them about the advantages of each method of payment, pointing to the graphs and to values on the screen.

Yet another situation of “when it gets hard” occurred when Inbar and Mika worked on the Age problem. They had difficulty expressing the story in a correct algebraic model. They started out by translating the story into an inequality. Inbar suggested $2 + (x + 24.5 \cdot 1.5) > x + 2$, then Mika manipulated the inequality to try to reach a solution (see Figure 6). They suspected that the algebraic inequality might be incorrect only when they ended up with zero x . They then looked for a different way to write the inequality. The interviewer offered to review the inequality with them. During this time, Mika was busy with the techniques of graphing, and Inbar claimed that it was worthless. She kept saying that there was a problem with the inequality sign and that functions could not help. Fifteen minutes later they were still desperately looking for an idea:

- 2.1 *Mika:* Maybe we should try to draw functions with the computer and then we’ll see it?
- 2.2 *Inbar:* How would that help you?
- 2.3 *Mika:* We will see the intersection point.
- 2.4 *Inbar:* And what would the intersection point tell you? How do you know
...
- 2.5 *Mika:* [I’ll know] when they are equal.
- 2.6 *Inbar:* But father and son can’t be the same age!

Inbar was skeptical about the use of the graphing tool as a means toward a solution [2.2]. She argued [2.4–2.6] that looking for an intersecting point might be a bad idea: An intersection usually shows the point of equality of two processes, and the two processes she was envisaging were the ages of the father and of the son, which can never be intersecting graphs. She decided to work some more on the expressions. Mika, who insisted on getting help from the computer, was not involved with Inbar’s work and waited for Inbar to dictate to her the expressions. They took a blank piece of paper and started over. Inbar wrote $2 + (24.5) = x + 2 \cdot 1.5$ (she missed the x on the left side of the parenthesis and the parentheses on the right). She again left the manipulations to Mika and again there were mistakes (see Figure 7).

Inbar was sure that this time they would get a correct answer because she had clear arguments regarding her model and she seemed to settle the complexity of dealing with an inequality that she had struggled with all along. But she kept trying

$$\begin{aligned}
 & 2 + (x + 24.5 \cdot 1.5) > x + 2 \cdot 1.5 \\
 & 2 + x + 24.5 \cdot 1.5 > x + 2 \\
 & \cancel{26.5} + x > \cancel{x} + 1.5 \\
 & \cancel{28} + x > x + 2 \quad | -28 \\
 & \cancel{28} + x - \cancel{28} > x + 2 - \cancel{28} \\
 & \cancel{16} > x - \cancel{26} \quad | -x \\
 & \cancel{16} - x > x - \cancel{26} - x \\
 & 0 > -26
 \end{aligned}$$

Figure 6. Mika and Inbar's first round of attempts to solve the Age problem.

$$\begin{aligned}
 26.5 &= x + 2 \cdot 1.5 \\
 26.5 &= x + 2.5 \\
 24 &= x / : 24 \\
 \frac{24}{24} &= 1 \\
 \frac{x}{24} &= ?
 \end{aligned}$$

Figure 7. The final attempt of Mika and Inbar to solve the Age problem.

to explain the strange result:

- 2.7 *MS*: So what does that mean [$x = 24$]?
 2.8 *Inbar*: When she solved the equation she moved all the numbers to one side and all the x s to the other.
 2.9 *MS*: Ah...
 2.10 *Inbar*: All numbers add up to 24, and there is only one x , and then we do $x/24$. . . We will gain nothing by it. . .

Indeed, the Age problem was more difficult than other problems because it did not invite a description of two processes changing in time as Inbar had said [2.6]. Inbar

made constant efforts to express the story mathematically. She understood what might be gained from graphs of functions and when graphs were and were not informative. Describing the aging of the father and of the son over time would not lead to the required equation, nor would it allow them to obtain a numeric answer from the graphs because the two lines did not intersect. Constructing an equation required seeing the relations between the processes and describing them algebraically. Inbar admitted that she was very weak in manipulations and her knowledge of manipulations was totally instrumental and not tied to the meaning of the model (it seems that she remembered some rules of thumb [2.8–2.10]).

Mika was considered by herself and by Inbar to be the stronger in manipulations, but she obviously made many mistakes. Functions and graphs, whether created with software or on paper, were also approached in a technical manner. She was graphing (on paper at first) because graphing was considered to be a way of finding a solution, but she attached no meaning or conjectures to her graphing. She viewed the graphing tool as an instrument to help her complete these actions. Therefore she insisted on using the computer, probably because she considered it a better grapher than herself and, hoping to reach a solution in this way [2.1–2.3], she ignored Inbar's logical argument about the tool being useless at that point. Apparently Mika did not see strong links (if any) between the solution she might reach by manipulations and the solution she might reach by reading the intersection on the graph.

Vera and Mor, Gal and Roni, and Mika and Inbar all used the graphing tool because things were becoming difficult. Based on their attempts, we identified three modes of approaching the computer for help when things became difficult: (1) The tool is used to replace tedious computations. Mor and Vera used the arithmetic calculator to create a numeric model in two complementary ways (input-output and pattern of differences), but they intended to abandon the completion of the solution because of the difficult technical work involved. (2) For Gal and Roni, the computer acted as a lever to solve the problem by skipping a stage that they failed to complete on their own. They had started by abstracting the situation for themselves into an equation. After the model was graphed on screen, they were able to answer fully and correctly. However, the manipulation problems were merely bypassed rather than supported (they never retraced their process to see whether they could solve the equation after reading the results on the screen). (3) Mika used the computer as a substitute for her unsuccessful actions, but unlike the other students she did not have an explicit suggestion or conjecture regarding the solution. After making many attempts and failing, she chose the computer to explore possibilities that had proven to be helpful on earlier occasions.

The Computer Wouldn't Do It for Me

Unlike in the last episode where Mika was stuck in a long process that she could not complete and used the computer in the hope of obtaining help with the symbolic formation of the equations, other interviewees often explained why they *would not use the computer* even when they recognized that they might be stuck. Vera and Mor used the simplifying of an expression as an example of when they would defi-

nately not use the computer [0.16–0.19]. Mor argued that the computer would not simplify expressions for her.

Gal and Roni deliberately avoided using the computer on the Age problem. They began by discussing whether the problem made sense, and how:

- 3.1 *Gal:* The difference between them does not change; he will always be 24.5 years older, but within 2 years he will also be 1.5 times older. Wait . . . how old is the father?
- 3.2 *Roni:* How old is the son?
- 3.3 *Gal:* Two years more . . . this does not really matter.
- 3.4 *Roni:* Are we missing a given?
- 3.5 *Gal:* No, simply as we do in a table, we substitute for x , which is the son's age, and change it until we get the father to be 1.5 times years old.
- 3.6 *Roni:* Does it make sense? When is the son old enough so that his father is exactly 1.5 times older?

Once they set up these terms and determined the logic of the story, they began plugging in numbers, checking and rejecting extremes. They approached the unknown age of the son as a variable quantity and continued to substitute numbers:

- 3.7 *Gal:* Let's assume the son is 10, so 10 times 1.5 is 15 and the difference is 5—so we should increase it. Let's say it is 100.
- 3.8 *Roni and Gal:* It does not make sense; then the father is 150.
- 3.9 *Roni:* The difference does not allow the father to be so old, so let's say the son is 70, the father 105 and . . . we need a smaller number.
- 3.10 *Gal:* Wait, if he is 50, then 75 and the difference is 25. There should be an arithmetic operation between 24.5 and 1.5, divide or something like that. If he is 45 the father 67.5 the difference 22.5, we are really close, increase it just a bit. He is 47 times 1.5 is 70.5, difference 23.5, so I do x times 1.5 minus x , that's it!

At this point, they were very close to reaching the solution $x = 49$ but instead they moved on to the formulation of the algebraic rule. They were able to state the function's rule, $x \cdot 1.5 - x$, but they were still missing a relation that would express the constraints that appear in the story in full.

- 3.11 *Roni:* $f(x)$ is the father's age and $g(x)$ is the son's.
- 3.12 *Gal:* No, wait a minute. The way I did it x is the son's age. The rule for the father is $1.5x$. From that I need to subtract x .
- 3.13 *MS:* So is this the correct rule?
- 3.14 *Gal:* Something is missing here. . . It is correct but there is something that should be dependent on this difference [$1.5x - x$]. It should be multiplied by something so it will come out that the difference is 24.5.
- 3.15 *MS:* How are you going to do that?
- 3.16 *Gal:* With the computer [they both use it, choosing the $f(x)$ option to write an expression]. I don't know how to write it exactly. . .
- 3.17 *MS:* You were successful with the examples [meaning with the generalization from the examples], weren't you?
- 3.18 *Gal:* I could say it in words, I can get the answer, but I don't know how to write it [using the software].

They did not know *exactly* how to do it because as Gal explained [3.12], he did not see the two functions that Roni suggested [3.11]. For a moment it seemed that plugging the expression into the graphing tool would help them [3.16], but they ended up not using the tool because it introduced a complexity [3.18]. They were able to define a procedure that described the process of how to compute the correct age, but they could not write it as a comparison of functions: $1.5x - x = 24.5$. They viewed the structure of the problem as a comparison of two functions, $f(x) = g(x)$, but apparently had difficulty seeing 24.5 as a constant function [3.14]. They could have typed in only one function and evaluated it for $f(x) - x = 24.5$. This was analogous with their previous work [3.7–3.10], which they probably did not consider to be a robust way of solving the problem but rather a type of exploration that they had already accomplished. Thus, it is conceivable that they did not use the computer because they were looking for a confirmation of another type: They viewed an equation as a way to formally describe and generate an answer that would validate their exploration. They viewed an equation as a comparison of two functions, and they could not find such an equation. To paraphrase Gal [3.18], he knew how to say it, had reason to believe they had found the correct solution, but he could not express it in a way that would cause the graphing tool to support them by supplying the missing formal formulation of their explorations.

For Gal and Roni, the tool could have been helpful had they been able to describe their ideas in a way that agreed with its constraints. For Mor, the graphing tool is not a tool for symbolic manipulations but rather a tool for presentations and representations. Although graphing tools could play a role in such transformational activities (Yerushalmy (1991) and Yerushalmy and Gafni (1992) describe and analyze such roles), it was perceived as a function tool, and its use was solely related to issues of representation and not to the manipulation of symbols.

I Try It First on Paper

Vera and Mor definitely preferred to start their thinking on paper [0.13], which was the general method of work of the L25 in our sample. We observed this tendency using time charts, another method of documenting the interviews. Time analysis offered a clear view of the typical modes in which the students integrated paper and software. Time charts (see Figures 8–12) record length of time with the representations used in operations with and without the tool. Representations are grouped in the following way: the top three rows (Expressions, Equations, and Manipulations) address symbolic representations; the fourth row deals with comparisons in any of the available representations (comparing two expressions, two graphs, sketches or value tables); the next two rows (Accurate Graphs and Sketch) deal with graphing; Arithmetic and Value Table deal with numeric operations; and the last row represents the Solution statement. Time (in minutes) in the interview spent on each mode is marked in black; it is in gray when the graphing tool is being used. Columns with no shading are periods of actions that do not fit into any of the categories.

The first chart (Figure 8) shows the strategies employed by Mor and Vera in the Parking problem. The chart shows extensive work with the value table exploring the vicinity of the solution, then wanting to see it all and checking values and graphs with the software. The second chart (Figure 9) illustrates the attempts of Roni and Gal to solve the Number Game problem, spending a long time trying out numbers and realizing that the differences were not constant. They did not make any further progress until the interviewer prompted them to look again at the story. After they had found an expression for the rule, they started working with the software. The third chart (Figure 10) shows Roni and Gal working on the Bonus problem and shows that, unlike in earlier attempted solutions in previous interviews, here they immediately approached the symbols, the expressions, and the equation.

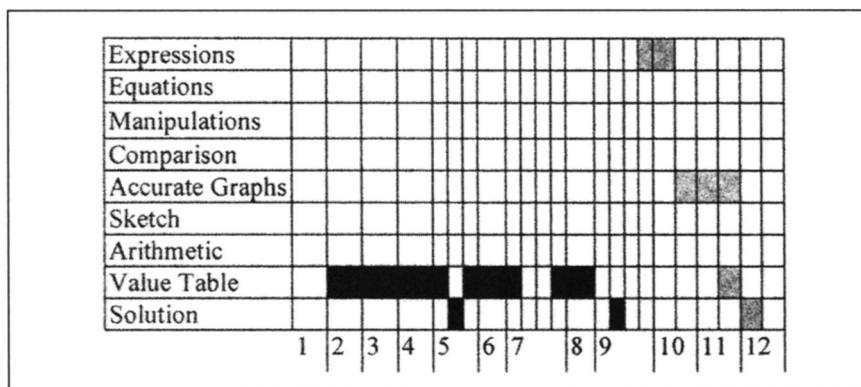


Figure 8. Vera and Mor solving the Parking problem.

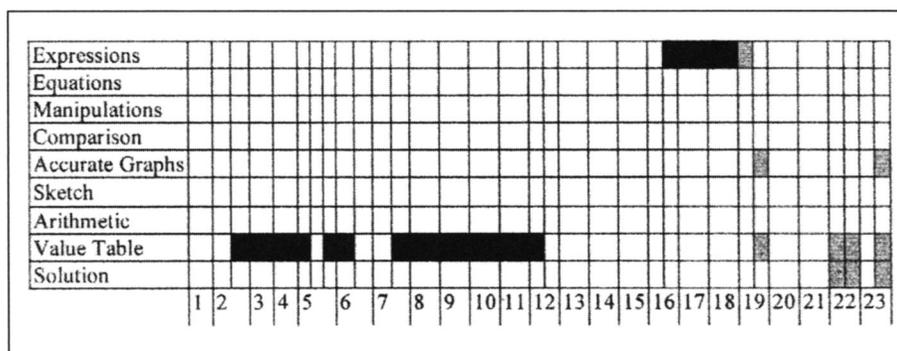


Figure 9. Gal and Roni solving the Number Game problem.

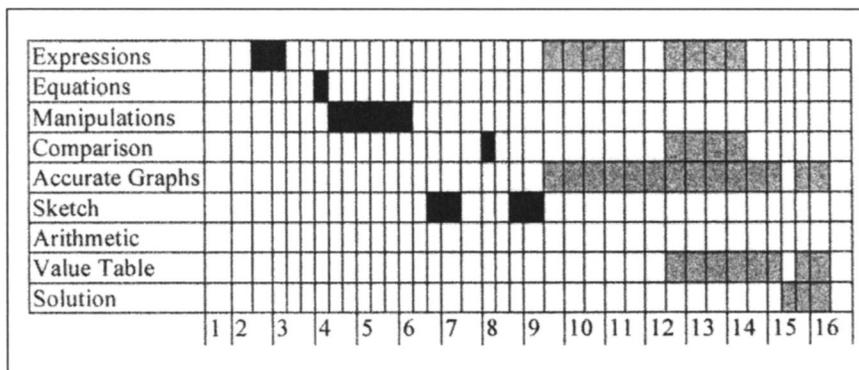


Figure 10. Gal and Roni solving the Bonus problem.

The visual presentation of the solution processes provided by the charts amplified the typicality of these modes of work to the L25. In no instance did the L25 ask for the computer at the beginning of the interview. In contrast, the U25, when they chose to use the software, did so at the early stages of the solution. Although the full analysis of the U25 is not part of this article (see Nasser, 2000), juxtaposing the two groups is instructive. Nick and Leon, a pair of the U25 sample, asked for the tool while solving the Parking problem. They typed in two expressions, $2.7x$ and $2x + 6$, then adjusted the scale of the graph to view the expected intersection:

- 4.1 *Nick*: We can see that for the first [type] more money is added all the time so it must exceed the other one some time.
- 4.2 *Leon*: [Checking simultaneously the table and the graph] Here, after 9 hours, you should use the second option.

In the Number Game problem, Nick and Leon attempted to reach the solution mentally by evaluating the sign of the output of $10x - 2x^2$. Once they realized that the range of positive and negative input could be difficult to evaluate mentally, they asked to use the tool and then typed and scanned the graph and the table:

- 5.1 *Leon*: It gave us exact values; we see a nonconstant rate of change for the graph. It is exact and fast.
- 5.2 *MS*: Which representation exactly is the one you are using?
- 5.3 *Nick*: To be certain, we need all of them, but there are things that we can be sure about without cross checking.

The time charts in Figures 11 and 12 show short processes typical of the U25. In many such cases, interviewees figured out the model and its symbolic presentation, then used the software immediately to complete the solution and confirm their mental plan. In some other interviews, the interviewer, faced with the risk of not having an opportunity to hear much, refused the students' attempt to use software

unless they first described their expectations of what they were going to find. In some cases, they finished the work by solving equations on paper; in others, they both described the expected results and confirmed them with the software and by manipulations. This was almost never the case with the L25. It took longer for the L25 to form or to manipulate a symbolic model because they created it from examples or had to check it first by examples. Unlike the L25 who often delayed the use of the tool because they realized that the computer would not be helpful when they were struggling with the details of the symbolic expressions, conceptual competence with expressions came easy to the U25 and therefore the computer was more useful to them.

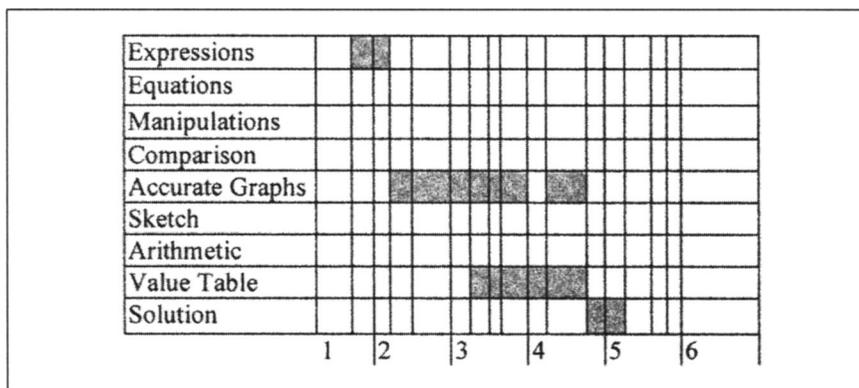


Figure 11. Nick and Leon solving the Parking problem.

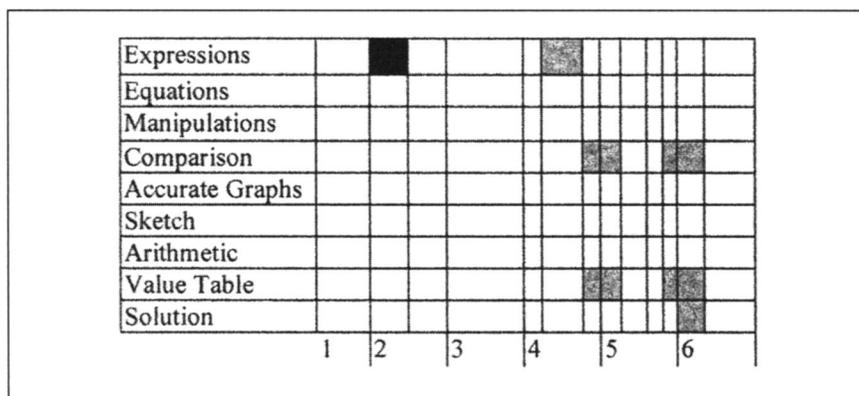


Figure 12. Nick and Leon solving the Number Game problem.

DISCUSSION: CHARACTERIZING THE LESS SUCCESSFUL STUDENTS' (L25) PROBLEM-SOLVING ATTEMPTS

Students learned to appreciate the option of obtaining a better view with less effort using the tool. At the same time they were aware of the constraints, namely that the tool must be driven by symbols, and that functions expressions, equations, or inequalities must be written as a comparison of two functions. Artigue (in Ruthven, 2001) found that for weaker students the difference in the semiotic characteristics of two learning environments, paper and computer algebra systems (CAS), was the most difficult issue to deal with. A similar observation was made by Drijvers (2000), who found that CAS could be perceived by students as idiosyncratic because they require a highly specific syntax that must be obeyed. In the current study, students understood the constraints and did not try to work in ways that would contradict the design of the tool. In that sense, the tool was transparent to the student's algebra and there were no surprises.

Students often started to solve a problem by collecting numeric data or by formulating an equation. In both cases, they would devote a relatively long time for experimentation and initial explorations on paper, without the graphing tool. Only after this experimentation did students use the tool. As they became experienced (by the fourth interview), they would try to skip examples and retell the story symbolically using expressions of functions or equations. When that happened, using the graphing tool to make sense of the problem was an attempt to relate the expression and the graph to the story, as did Gal and Roni working on the Bonus problem (see Figure 5). Not using the tool when it did not make sense to do so for a given story was also an explicit choice, as when Inbar rejected Mika's attempts to describe the two aging processes as two intersecting lines [2.3–2.6].

The tool was part of the students' reasoning and argumentation and was used to reflect on conjectures. In these cases, interviewees did not use the tool without some anticipation. They often generated examples until they felt that the process needed a more global view, and they realized that the tool was a better way of obtaining such a view (e.g., Gal and Roni identifying the properties of the sequence before turning to the computer [1.16–1.23], and Vera wanting to "see it all" [0.15–0.16]). Although a major part of the conversations and actions focused on numeric examples and computations, the discourse could be identified as a conceptual conversation, which according to Thompson and Thompson (1994) is aimed at exemplifying and supporting some underlying arguments rather than merely computing an answer, even when involving calculations.

The tool's and the students' algebra were found to be similar in some aspects and different in others. Students analyzed the given situation and proceeded in the direction they chose along the tetrahedral shown in Figure 1: modeling the story starting with the numeric data, with the graphic model, or directly with the function's expression. The tool did not explicitly support the links between the situation and the models; rather, it supported mapping only among the three representations and only when driven by the symbolic representation. At the same time, the design of

the tool invited students to view or participate in the same fundamental principles of structure that underlie their studies of algebra: functions connected to expressions, equations, and representations. The tool helped students view functions as mathematical objects that exist as screen objects in three linked parallel presentations. An equation is a comparison between two functions, and the solution is viewed as the intersection of two graphs, the x value being valid for both processes having the same output (for example in [2.1–2.6; 3.12–3.16]).

We found a difference between the work patterns of the L25 and the traditional problem-solving patterns of less successful students. In typical algebra instruction, once an efficient algorithm has been taught, there is pressure, especially on less successful students, to use this algorithm rather than idiosyncratic solution methods. Students in this study did learn, adopt, and create for themselves successful mechanisms, using procedures that involved less work with symbols. Despite (or perhaps because of) this difference, we wondered why these students remained the L25 for the 3 years that we followed them. Three complexities likely contributed to students staying in the L25: (1) The L25 understood the meaning of mathematical notions represented by graphs, symbols, and procedures. However, unless they dealt with a problem of a type with which they had relatively long experience, they delayed using symbolic formalism, and most of their solution attempts focused on numeric and graphic representations. (2) They demonstrated mature problem-solving skills: planning, using examples to form a conjecture, looking for a global view, and making choices regarding the use of the tool. This caused their solutions to take a long time—longer than their U25 peers. (3) They used the graphing tool to confirm conjectured equations, to get unstuck, and to correct errors. However, the given graphing tool, which is driven by explicit symbolic description, was less useful in constructing mathematical models and in performing symbolic manipulations. These three complexities are discussed in greater detail in the sections that follow.

Delaying the Use of Symbolic Forms

Compared with the U25, the L25 tended to delay the use of expressions and equations. The symbolic discourse did not come as naturally to them as it did to others. They knew, however, that they needed to make attempts to participate in this discourse in order to use the graphing tool, and sometimes they needed help in doing so. Although they had learned to use symbols and expressions, and they performed symbolic procedures in class, they preferred to model phenomena using tabular and graphic forms. Analysis of rate of change—the pattern of the numeric differences and the linearity or nonlinearity of graphs—was their primary way of describing a process and of conjecturing about a comparison between two processes. Expressions became an integral part of this view only slowly and gradually; it took at least 1 year after they started algebra in the seventh grade for this to become their preferred presentation of the situation (the Bonus problem was given in the ninth grade). The time charts suggest that “expressions first” was the regular choice of the U25 all along and of the L25 in the 3rd year (see Figure 10). At that time (in the ninth grade),

we noted their serious consideration of the meaning of equations, variables, and expressions of functions. Although most tasks did not require using symbols and even less so algebraic manipulations, because at this stage in-class activities included intensive use of symbols and manipulations, students considered a complete solution to include a symbolic formulation, which was often a difficult challenge. One of the concerns expressed by Chazan (2000) in describing his attempts to teach meaningful algebra to all students was the fact that his students did not develop a satisfactory facility with algebraic symbols. A major goal of Chazan's teaching was getting students to appreciate and value symbols. Our L25 interviewees valued symbols, but it was not until about two thirds into the 3-year algebra course that they started to make using symbols their preferred problem-solving choice.

One of the criteria for successful work in school mathematics is reaching a correct solution to a problem. The L25 made manipulation mistakes when they tried to solve problems on paper. Because of their flexibility in alternating among various representations and views of the problem, and because they understood the compatibility of these representations, they found a way out of these errors by circumventing the manipulations with the tool. Although Gal and Roni often reflected on their actions with numbers [3.7–3.10] and expressions [1.23–1.27], they did not retrace their steps to try to solve the problem algebraically without the tool; as a result, they missed the opportunity to tie up loose ends and learn from their mistakes. Although Gal and Roni did not overcome their manipulation mistakes, working with the tool improved their outcome. In the case of Mika and Inbar, manipulation mistakes dominated both their paper solution and their work with the graphing tool and left the problem unsolved. They demonstrated instrumental understanding (Skemp, 1978) of the links between equations and functions. They used rules of thumb that they tried to memorize [2.8–2.10]. Although the question of symbolic fluency remains partially unresolved, it may be that if the only way to approach the problems had been symbolic, our interviewees, as generations of traditional algebra students before them, would have given up.

Work Often Takes Longer Than for Other Students

L25 students worked for a long time to solve the problems. The striking difference in the duration of the work is apparent when comparing the time charts of the L25 (Figures 8–10) with those of the U25 (Figures 11–12). The L25 spent a long time doing things: they took many steps on paper, sometimes repeating them with the computer; they were flexible in using graphs and tables and went back and forth analyzing the situation in different ways; they performed many computations (mentally or with a calculator); and occasionally they tried to solve and manipulate expressions. Sometimes the solution was delayed by the complexity of formulating the symbolic model required as the ultimate input for the graphing tool. At other times, they could have completed the work immediately with the software but chose not to do so (for example, in 0.1–0.15).

This delay may be a type of precaution characteristic of less experienced people. Laborde's (2001) description of instructions given by inexperienced teachers using Cabri resembles this phenomenon. Laborde reported these teachers asking students to perform the task on paper first and only then to use the computer. This was not the norm in the classes our interviewees attended or during the interviews, and students did not seem cautious about using technology. The U25 attempted to model the situation algebraically with or without the computer and were able to do it either way. The L25 tried to make abstractions by hand, then attempted to understand the abstraction using the graphing tool. It seems that the L25 would benefit from being granted additional time, both for finding solutions and for learning and grasping procedures and representations. As mentioned previously, this is not a new observation in the literature about low achievers in general. Their willingness to be involved in long and compound problem-solving processes, however, contradicts the frequent finding about less successful mathematics students being unwilling to spend a long time on activities.

Lacking Support in Symbolization and Manipulation

A central use of the tool in this study was for elaborating conjectures initially developed by experiments on paper. Although the tool supports numeric input, it was not used in this way, probably because its most obvious use is that of entering expressions and equations. Conjectures had to be formed symbolically as an expression in order to follow and trace behavior previously identified as a pattern on paper. Undoubtedly, the tool was useful in obtaining a global view, confirming conjectures, and bypassing manipulations. It was not helpful, however, in formalizing the pattern (as in the Age problem of Gal and Roni) or in supporting manipulations (as in Inbar and Mika's case).

Tools could be used to support further mapping between the problem's story and the algebraic symbols. For example, spreadsheets and graphic calculators can accommodate the use of numeric examples and provide an approximate symbolic model for selected numeric data represented as points in a plane. Instruction could then focus on the nontrivial task of evaluating and appreciating the resulting symbolic models (for an example of intervention using graphic calculators, see Hershkowitz & Kieran, 2001). Another way of addressing the problem of providing meaning to algebraic models is to use technology that encourages investigations of phenomena in experiments that link physical bodily action to analytic symbols. Activities that support lines to become motion (Kaput, Noss, & Hoyles 2002; Schnepf & Nemirovsky, 2001) and motion to become mathematical models (Shternberg & Yerushalmy, 2003) are examples of uses of tools that can help the L25 develop a feel for symbolic expressions.

Symbol manipulations were obviously difficult for the L25. Graphs could be used to indicate false operations during expression manipulation. A correct manipulation should leave the graph unchanged, and mistakes like those made by Mika ($x + 2 \cdot 1.5$ was simplified to $x + 2.5$ in Figure 7) could be easily identified. But in the

case of complex mistakes (as in the mistake by Gal and Roni discussed on page 372), the change in the intersection x value of the two graphs would signal a mistake but would not necessarily help in identifying its source. For that they would have to analyze the difference between the graphs and identify the reasons for the change of the intersection values. Graphs that suppress the process nature of the function's expression and invite encapsulation can help locate erroneous result but require further attention and maybe proficiency to be helpful tools in the orderly performance of manipulations. Although it is important to organize activities that encourage explicit links between manipulations and solving problems in an environment of functions, additional help can be sought from symbolic manipulators frequently embedded in CAS. We should explore the possibility of integrating software with symbolic manipulation capabilities in the work of beginning algebra students, especially for those who need it to become fluent in algebra, even at the price of giving up mastery of manipulations. Little CAS research deals with beginners in algebra or with the L25—two groups that require future attention. In general, it is important to continue analyzing the epistemological aspects that underlie the design of various tools, to study the ways in which algebra beginners use the tools, and to explore connections between these aspects and the mastering of algebraic symbols.

Final Note

The participants in this study were chosen at the beginning of their seventh grade based on their performance, which was then slower than that of other students and required more help from the teacher, and based on the observation that they were not immediately able to meet the teacher's demands in class. All this remained unchanged over the next 2 1/2 years. However, the students learned to appreciate good mathematics and mathematical structures and to understand some important patterns of mathematical thinking. In this respect, they came closer to high achievers; they were doing well, but they were doing it differently. They needed no enrichment in cognitive skills, as their cognitive capabilities were already impressive. They lacked neither the motivation to explore nor the willingness to spend a long time on tasks. However, they were not doing well enough to have all available future paths open to them. We should learn to identify those components of meaningful learning environments for algebra that allow everybody to demonstrate their strengths.

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