Carla, the left coset, and the normal group

Excerpts from *Learning and Understanding in Abstract Algebra*

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The case of Carla was developed as part of exploratory study that sought to describe students' understandings of abstract algebra in the context of an abstract algebra course. Using Tall and Vinner's (1981) notion of a concept image, which is the entire cognitive structure associated with a concept, the study identified prominent characteristics and components of students' concept images in group theory, up to and including the concept of quotient group.

The context for the study was an abstract algebra class for mathematics majors, covering many of the standard topics from group theory, but in which lectures were replaced by collaborative and individual work on problem sets designed to promote connections between students' prior understandings and the content of the course. Analysis and the ensuing results were based largely on interview data with five students from the class.

The following case is intended to illustrate two aspects of Carla's concept images of homomorphisms, cosets, and quotient groups, and related concepts. First, by this time in the course, Carla had developed considerable fluency with many of the concepts in the course. Second, Carla attached names to the concepts in unconventional ways.

<u>The Left Coset</u>

The common core of the third interviews consisted of determining whether a particular function was a homomorphism and finding the cosets of the kernel of that homomorphism. A *homomorphism* is a function f from a group G to a group G', with operations * and *', such that for all a and b in G, $f(a*b) = f(a)*'f(b)$. The idea is that the function preserves the group operation (and hence some of the group structure) in the sense that it doesn't matter whether the operation occurs in G and the result is sent through the function or, alternatively, the elements are sent individually through the function and their images are combined under the operation in G'. In this class, homomorphism was introduced as a generalization of the formal version of isomorphism.

A typical task involving the concept of homomorphism is verifying that a given function is a homomorphism. As part of interview 3, I chose a function f from U_8 to Z_4, given by $f(1) = 0, f(3) = 0, f(5) = 2$, and $f(7) = 2$, where U_8 is {1, 3, 5, 7} under multiplication modulo 8 and Z_4 is {0, 1, 2, 3} under addition modulo 4. When the function is given formulaically, the verification is an exercise in symbol manipulation, but without a formula here, students needed to verify that $f(a*b) = f(a)*'f(b)$ for all 16 pairs of elements a, b from U_8.

To determine whether f was a homomorphism, Carla first listed the elements in the groups and then explained her plan:

33 Carla: So first I am going to try an example where I am going to pick two elements that map to the same thing, and then if that isn't a counterexample, I'm going to try two elements that map to different things.

34 Carla: So first one is going to be … I want to show that f of … Let's see U_8 is a group under… [pause] … I am trying to think if it is a group under addition or multiplication. But it must be multiplication, because if it was addition then zero would be in there.

35 Brad: Why?

36 Carla: Because zero is the identity for addition. So it must be under multiplication.

Although Carla was momentarily unsure of the operation in U_8, she was able to determine the operation quickly by reasoning from the group axioms, demonstrating some fluency with the groups involved. She then verified the homomorphism property for two specific cases, but rather than have her verify all the cases, I wanted Carla to find the cosets of the kernel of the homomorphism.

The *kernel* of a homomorphism is the set of elements from the domain that map to the identity in the codomain. If the kernel is called K, computing a particular coset aK requires multiplying a particular value a from the group by each element in K. Computing all the cosets involves completing such calculations for all elements a in the group. Carla correctly identified the kernel of the homomorphism as the set $\{1, 3\}$. I moved on to cosets:

62 Brad: Okay, Let's call that set K and what I want to do is investigate these sets aK.

63 Carla: Okay. Left cosets. Alright. So to do aK we see what happens … Okay. The set $\{1,3\}$ is always going to be on the right, and we want to work with every single element that is in… Let's see…. Every … $\{1,3\}$ is a subset, in this case it is a subset, of U_8.

66 Carla: So you are going to take out every element of U_8, I think it is. The definition is … I think you take every element of U_8 and you multiply, in this case you multiply because that's the operation. [Okay.] Is that right that it's U_8? I'm not sure … We just said this today, but … [mumbles]

67 Brad: Well show me what it is you are going to do here, and then maybe you can answer your own question.

68 Carla: Well I am going to multiply 1, 3, 5, and 7 each individually by the, by K. So $1\times\{1,3\}$ gives me $\{1,3\}$. $3\times\{1,3\}$ gives me $\{1,3\}$.

For Carla, the notation aK evoked the term coset and seems to have supported her reasoning: a is an element that varies through U_8 and K is a set. The individual elements of K required little of her attention, as evidenced by the fact that she didn't say or write down intermediate calculations and quickly considered $\{3, 1\}$ to be the same set as $\{1, 3\}$. The lower-level process of iterating through the elements of K was essentially automatic. She computed the other cosets similarly and quickly (see Figure 1) and concluded "So we have two elements in the left coset" (line 70).

Figure 1. Carla's cosets of {1, 3}

$\{1, 3\} = \ker(f) = K$

aK

$1\times\{1, 3\} = \{1, 3\}$

$3\times\{1, 3\} = \{1, 3\}$

$5\times\{1, 3\} = \{5, 7\}$

$7\times\{1, 3\} = \{5, 7\}$

From the ease with which Carla performed these calculations, it seems that she had a firm understanding of the concept of coset, but her statement "we have two elements in the left coset" was unexpected. I asked her to explain:

73 Brad: Two elements in the left coset. What do you mean?

74 Carla: Because {1,3} and {5,7} are each of, are the two elements that result from *aK*.

75 Brad: Are the two… They are elements …

76 Carla: Well, they are sets, but if you look at them as one… If you rename {1,3} to *a* and {5,7} to *b* then there are two elements in *aK*.

77 Brad: Oh, okay. So you are saying on the one hand we can talk about the set of cosets, does that makes sense? [Yup.] So how many cosets are there?

78 Carla: Two.

79 Brad: Two. But now does *aK* refer to the set of cosets, does it refer to all of them, or does it refer to one specific one?

80 Carla: *aK* tells you how you find the cosets. So that was none of your choices. What were your choices?

81 Brad: Does *aK* refer to all of the cosets, or just one of them?

82 Carla: *aK* is the general formula that gives you all of the cosets. Then you choose specific *a*'s for whatever *K* you are given and you will find all the specific cosets.

Thus, Carla had an efficient and reliable procedure for computing cosets, and the procedure was supported by the notation *aK*. Nevertheless, she did not maintain a clear distinction between a single value of *a* and the set of all of such values, and thus did not distinguish between the particular coset *aK* and the set of all cosets of *K*. Instead, *a* varied as part of a procedure specified by the formula *aK*, which would give all the cosets. Psychologically, it seems to be an easy step to then imagine that *aK* is all such cosets, without needing to distinguish between a non-specific, particular one and the collection of all of them.

Analysis: The Left Coset

Carla was happy to let a vary through all the elements of G and construct the collection of cosets aK that result. What she didn't see, however, was a need to distinguish notationally between the particular coset aK and the collection of all such cosets. By calling the cosets elements, it seems that Carla had little difficulty seeing cosets as objects, suggesting that she had encapsulated the process of coset formation. Furthermore, her language "the left coset" for the collection of cosets suggests that she had further encapsulated that collection as an object, a set of sets, a point of view which is helpful in order to see the set of cosets as itself a group under the appropriate coset arithmetic.

Figure 2 shows schematically the objects and processes involved in coset computation and delineates the two levels of processes. Carla's language suggests that her thinking was in the transitional process between two objects: the particular coset $aK = \{ak \mid k \in K\}$ and the set of all cosets of $K = \{aK \mid a \in G\}$. In the midst of the process, a is varying, so aK denotes neither a particular coset nor all of them. For Carla, aK denoted and specified the process. Thus, my question about the distinction (line 81) was not relevant to her.

Figure 2. Objects and processes in coset computation

A particular a in G The subgroup K	$\longrightarrow$ Process: By letting k vary through K, calculate all the products ak. Collect them into a set.	The coset aK	$\longrightarrow$ Process: By letting a vary through G, calculate all the cosets aK. Collect them into a set.	The set of all cosets of K

Being embedded in the process allowed Carla some flexibility in her thinking. On the one hand she could consider a particular coset by stopping the process for a moment. On the other hand she could consider all the cosets by completing or imagining she had completed the process. From within the process she could broaden her viewpoint slightly and see both a particular coset and the set of all of them as two aspects of the concept of "coset." Thus, "*aK* tells you how to find the cosets," and together there are "two elements in the left coset." Carla maintained this dual role of the term coset through the fourth interview, and even maintained her process orientation, saying, for example, "The left coset *gives* the set of (23) and (132)" (Carla 4, line 239, emphasis added) rather than the left coset *is* that set.

<u>The Normal Group</u>

Given a collection of cosets, a follow-up question is whether those cosets can form a group under an operation related to the operation of the underlying group. In this case, because the cosets {1, 3} and {5, 7} came from U_8, in which the operation multiplication modulo 8, the product of two cosets involves multiplying, in the appropriate order, all possible pairs of elements, one from each coset. Continuing in Carla's third interview, I asked her whether she could use these cosets to form a group.

> 85 Carla: Well let's see … We will create a table with {1,3} and {5,7}, of course, on the top and on the right. Actually, we talked about this in class. If we know that the right coset is equivalent to the left coset then it does create a group. I think it's called a normal group and these will … I think we will have a group because I think that the right coset equals the left coset cause it really doesn't matter if you multiply on the right or on the left.

Based on results from class, Carla was convinced that the result would be a group, because, as she correctly observed, the left and right cosets were the same. Carla was essentially correct, except for her use of the words coset and normal. The correct statement is that when the left cosets of a subgroup are the same as its right cosets, the subgroup is said to be *normal* and the cosets will form a group under the set operation described above. The resulting group is called a *quotient group* or a *factor group*. I have already discussed Carla's use of the word coset to denote a set of cosets. As for her use of the word normal, it seems she had lost track of the subgroup that gave rise to the cosets and was applying the word normal to the quotient group rather than to the subgroup. Furthermore, it is worth noting that as sets of cosets, Carla's left coset and normal group were the same, suggesting that the set of cosets was a very salient object for Carla.

Carla completed the coset calculations easily and quickly, taking advantage of the commutativity of the underlying group. See Figure 3 for the resulting table.

Figure 3. Carla's group table for {1, 3} and {5, 7}

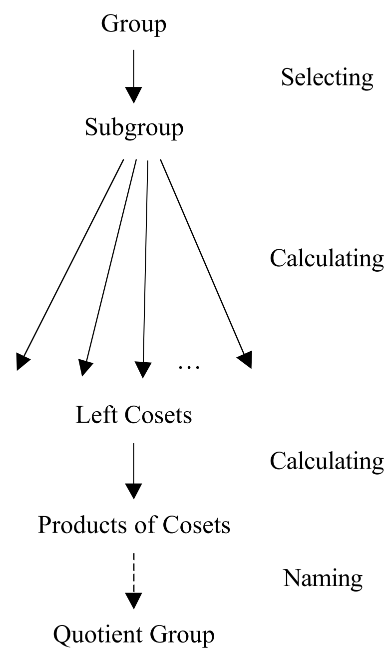
×	{1, 3}	{5, 7}
{1, 3}	{1, 3}	{5, 7}
{5, 7}	{5, 7}	{1, 3}

As the interview continued, Carla noticed that the group table was isomorphic to a familiar group. Soon thereafter, I explored Carla's understanding of some of the proofs from the second midterm exam. Because these portions of the interview don't have direct bearing on issues of language use, they will not be discussed here.

<u>Analysis: The Normal Group</u>

To analyze this episode further, it will help to step back from the particular objects and processes and consider them more generally. The above episode may be described as a instantiation of the general coset activity represented in Figure 4. Because the coset activity can proceed with a subgroup that is not necessarily the kernel of a homomorphism, the subgroup is denoted by H in the following discussion to portray that generality. Given a group G and a subgroup H, the left cosets of H are calculated and collected. The cosets are then considered elements in a new structure, a set of cosets. Coset products are computed, and results may be organized in an operation table. If the set of cosets constitute a group under this operation of coset multiplication, then the set of cosets is called a quotient group. The construction depends on both the group and the subgroup. Thus, the resulting group is called the quotient of H in G, denoted G/H, and read "G modulo H."

Figure 4. Coset activity



Of course, the set of (left) cosets do not always constitute a group, and the arrow is dotted in Figure 4 to indicate this potentiality. It turns out that the key issue is one of closure: whether the product of two cosets is again a coset. When this criterion is satisfied, all the other axioms follow: in G/H, the identity element is the coset H; the inverse of the coset aH is the coset $a^{-1}H$; and associativity follows from the associativity of the operation in G. Carla's observation, that the left and right cosets were the same, turns out to be both a necessary and sufficient condition to guarantee that the product of two cosets will be a coset.

What are the cognitive requirements in this activity? Assuming that the group G and subgroup H have both been determined, there are several crucial steps. First, computing a left coset requires choosing a specific element $a \in G$ and calculating aH, which is determined by multiplying a, on the left, by each of the elements in H. Second, this process is repeated for each of the elements $a \in G$, yielding a collection of cosets. (Either of these processes may be infinite, and such situations carry the additional requirement of being able to describe the results without actually completing the processes.) Next, the products of the cosets must be calculated, but this also involves a subsidiary process. So third, given two cosets, computing their product involves multiplying, in the appropriate order, all possible pairs of elements, one from each coset. And fourth, this process is repeated for all pairs of cosets. Thus, there are four relevant processes in two pairs. In each pair, one process operates at the level of elements, and the other operates at the level of sets. Carrying out such processes requires some flexibility: first conceiving of a set or an element as fixed and letting other sets or elements vary, and then letting the fixed set or element vary to carry out the higher processes.

Carla had participated in an activity depicted structurally in Figure 4 and had created written records similar to Figure 1 and Figure 3. How does one make sense of such an activity? What does one take from it? Because the difficulty seems to be in the language, I am suggesting that, for the analysis of the activity, we separate the concepts and procedures from the names. Making productive sense of the activity requires distinguishing among the various components in Figure 4 and then assigning names appropriately. How might one attach words such as coset, quotient, and normal to aspects or portions of these events?

Looking back again at Carla's case so far, it seems that she admirably negotiated the various kinds of processes involved in the calculations and seems to have had a good sense about the kinds of entities (e.g., sets or elements) she was dealing with. Where she had trouble was in attaching the names to the objects and processes.

<u>Cosets in Z_{12}</u>

The fourth interview concentrated on examples and non-examples of quotient groups, which required computing cosets and paying attention to whether the left and right cosets were equal. When I mentioned to Carla, as a preface to the fourth interview, that we would be discussing quotient groups, she showed discomfort:

6 Carla: I have got to figure out what they are. Quotient group… This is one of the things I haven't… I've needed to study it, but I haven't done it yet. [Okay; pause] What is the other name for a quotient group? That we gave? It wasn't a normal group right, that's different.

Thus, although Carla had computed a quotient group in the previous interview, she had not yet attached the name to the idea. I suggested we start with an example and return to the term later.

To get started, I asked her to find the subgroup generated by 3 in the group Z_{12}.

15 Carla: There would be 3, and 9 would be in it because 3 squared is 9. And 0 would be in it because … Well, actually, Z_{12} is a group under addition. So it's not … I can't really think of it as 3 squared … So 9 is in it, but not because it's 3 squared. 9 is in it because it's 3 cubed when you are adding. So, in other words three 3s.

Carla was momentarily confused about the operation in Z_{12}, but quickly resolved that issue and soon determined that the subgroup generated by 3 was $\{0, 3, 6, 9\}$. I asked her to find the cosets of that subgroup.

22 Carla: Okay. First we will do left cosets. Probably the left coset and the right coset will be the same, but we'll start with the left coset. So, do you want me just to use the kernel as the …

23 Brad: Do it whatever way you think is best.

24 Carla: Well, I'll start with the kernel at least because I am most comfortable with that. The kernel is 0 because 0 goes to 0. 0 is the identity.

25 Brad: You said 0 goes to 0. What does 1 go to?

26 Carla: No. 1 is not in there.

27 Brad: Oh. Okay. So what does 3 go to?

28 Carla: I don't think … We haven't really … Well, we know 0 goes to 0 because the identity always goes to the identity in two groups. But we haven't really defined what the others … [inaudible] The other, you know, function that we are dealing with … [inaudible] I am doing it with 0 and that's it… because I know I can do it with the kernel because the kernel is… Oh, what a sec … We do cosets … We have to have a subgroup … And we have to perform an operation of the members in Z_{12} on the left, for the left coset. So you have members of Z_{12} * with the subgroup. And I already know what the subgroup is, so it's 0, 3, 6, 9. Took me a while but …

Carla momentarily thought that she needed a kernel to talk about cosets, but reasoned that only a subgroup was necessary. It seems from line 22 that Carla was still using the term "left coset" for the set of cosets, just as she had in the third interview.

Carla then called the subgroup S and listed the elements of Z_{12}. In describing how she knew the operation was addition mod 12, she talked at first about a group G', suggesting she had not completely abandoned the idea that there was a homomorphism involved. Then she decided:

35 Carla: Let's just keep it simple and call S is a subgroup of G. So S is 0, 3, 6, 9, as I said before. And G is Z_{12}. And I am going to tell the truth here: The reason I chose addition mod 12 is because that's the only real operation we have right here. [Laughs.]

Carla began the coset calculations without hesitation. Her language evolved in the course of doing the calculations, becoming abbreviated: "If you are doing the operation of 0 on 0, 3, 6, 9 …" (line 37), "If you do 1 star (which is adding mod 12) with 0, 3, 6, 9 …" (line 41), "2 with S" (line 41). Soon she stopped doing all the calculations and instead reasoned about the results, based on pattern:

43 Carla: So we can see that there is a pattern here. 0 and 3 have the same left coset. And so 6 will and 9 will, because they are all multiples of three, because I can see a pattern of 3 here. So 1, the left coset, containing 1 is 1, 4, 7, 10. And that is the same as the left coset containing 4. So 1, 4, 7, and 10 are going to have the same left coset. So then it will be 2, 5, 8, and 11 that will have the same left coset.

The results of her calculations is shown in Figure 5. At first she had not recorded an operation in some places, such as between 2 and *S*, but when I asked her about the notation, she inserted stars and said that it was addition mod 12.

Figure 5. Carla's cosets calculations for {0, 3, 6, 9}

$S = \{0, 3, 6, 9\}$

$* = +_{12}$	lt coset		
	0*{0, 3, 6, 9}	=	{0, 3, 6, 9}
	1*{0, 3, 6, 9}	=	{1, 4, 7, 10}
	2*S		2, 5, 8, 11
	3*S		0, 3, 6, 9
	4*S		1, 4, 7, 10

Carla went on to compute right cosets. She first listed the calculations she intended to make, and then stopped after computing *S**0.

55 Carla: And right away I see that the right coset is going to the same thing as the left coset because we are adding mod 12 and it's commutative. So it doesn't matter which side your single element is that changes.

57 Carla: So we have a normal group because … I think it's a normal group that says that the left coset equals the right coset.

58 Brad: So what is the normal group here? When we have the left cosets and the right cosets being the same we have this thing called normal, but I want to know precisely what is normal here.

59 Carla: So the normal group is a set of sets. So one of the smaller sets will be 0, 3, 6, 9. [Okay] Another of the sets will be 1, 4, 7, 10. [Okay] Another one will be 2, 5, 8, and 11. So our normal group consists of 3 sets.

Two points are to be made here. First, she did this work with little interaction from me, and she noticed quickly that right coset calculations would yield the same result as the left coset calculations she just completed. Second, Carla's usage of the terms coset and normal group was consistent with her usage in the third interview. For Carla, the construction of the normal group required that the left coset equal the right coset. When I asked in what sense it was a group, she began by stating that the identity would be 0, 3, 6, 9. To explain this, she decided to construct a table (see Figure 6).

Figure 6. Carla's normal group with {0, 3, 6, 9}

	{0, 3, 6, 9}	{1, 4, 7, 10}	{2, 5, 8, 11}
{0, 3, 6, 9}	{0, 3, 6, 9}	{1, 4, 7, 10}	{2, 5, 8, 11}
{1, 4, 7, 10}	{1, 4, 7, 10}	{2, 5, 8, 11}	{0, 3, 6, 9}
{2, 5, 8, 11}	{2, 5, 8, 11}	{0, 3, 6, 9}	{1, 4, 7, 10}

While filling in the table, Carla at first performed the set addition by listing aloud all the pairs to be added. As she continued, she filled in the table according to what she believed it should look like. She used abbreviated procedures, partly to check her expectations, and she described much of her thinking aloud:

79 Carla: Alright. So I know that this is a normal group so I … because I know it's a group, I know it's going to create a group table. So I can … I have a pretty good idea what this table is going to look like. It's going to have 1, 4, 7, 10 in the first column second row and it's going to have 2, 5, 8, 11 in the position below that because 0, 3, 6, 9 is the identity so it needs to just reflect what row it's in because I am working on the first column.

81 Carla: Now I think that the middle spot in the table is going to give me 0,3,6,9. So I am going to … Well, but I can see that it's not going to do that. So then I know that it's going to be 2, 5, 8, 11. The reason that I knew it wasn't going to do that is because I saw 1+1 is 2 and that's not in it. So that cancels that right there.

85 Carla: Yes, [you should always get another coset in these calculations] with a normal group. And I know that with my 3 by 3 group tables one of the options is to have all of the identities down the diagonal, so that's why I automatically thought that might be the identity in the middle position. But then when I checked it I realized it wasn't. But it still can be … I can still have a group table.

88 Carla: Since this is a group, I know that every element only appears once in each row or column, so I see that 0, 3, 6, 9 is left, for the second row. And then I can just check that 2+1 is 3, 5+1 is 6, 8+1 is 9, 11+1 is 12, which is 0 mod 12.

92 Carla: Even if you just see one of the elements you know which coset it is going to belong to because the cosets don't overlap.

98 Carla: It has to contain 3 and that's the only coset that contains three. So then I have one spot left so I know that has to be 1,4,7,10 because both the third column and the third row lack that set.

Carla's approach, based on patterns and facts, was reasonably efficient and included some redundancy, which she used to catch errors. First, she knew every pair of cosets would produce another coset (line 85). Second, she knew the result would be a group, so she had expectations about the patterns in the table (line 79), and she used the fact that in group tables every element appears exactly once in each row and column (lines 88, 98). Third, she knew the cosets didn't overlap (line 92), so she used representative calculations to determine which coset should appear in a cell and to check errors. She believed, for example, that, as a 3×3 group table, it would have the identity along the diagonal (lines 81, 85), which is impossible, although both 2×2 and 4×4 tables can have the identity along the diagonal. This error of memory did not cause much trouble, however, because on the basis of a representative calculation she quickly realized that that was not correct.

<u>Exploring the Language</u>

Carla had previously stated that the result would be a normal group, but I wanted to get some clarity on how she was using the phrase. When I asked her what the resulting table was and what it had to do with the word normal, she responded that it was a group table that had "everything to do with the word normal" (line 102). She continued:

105 Carla: Well, the thing that it has to do with is that this table is a demonstration of the group of those three sets—that they are a group.

109 Carla: The normal part of it just says that you got it because you had left and right cosets that were equal to each other.

111 Carla: That's how you know … That's how it differentiates from just any old group.

So the normal group was a group, as demonstrated by the table, and it was normal because there were left and right cosets that were equal to each other. These associations of the words normal and group, it should be pointed out, are essentially correct. In particular, when the left and right cosets are equal to each other, the word normal applies, but it is supposed to point backward (see Figure 4) to the subgroup that led to the particular set of cosets not forward to the quotient group.

I next explored whether Carla could make any connection to the subgroup:

112 Brad: But these were left and right cosets of what?

113 Carla: Of *S*.

114 Brad: Of *S*. So does the word normal have anything to do with *S*?

115 Carla: I am not sure what you mean. Normal group, like just as far as words in the English language, doesn't really have any meaning to me, that's just what it's called.

116 Brad: Okay, what what is called? That's what I really want to get at.

117 Carla: Oh. The normal group is the group of cosets where the left and right coset of *S* are equal.

I take this last statement to be Carla's definition of normal group. She recognized that it was the subgroup that led to the cosets but did not see any reason to attach the normal to the subgroup. Furthermore, she stated moments later that if the left and right cosets of *S* had not been equal, she would not have gotten a group.

<u>Trying to Refine the Language</u>

At this point in the interview, it was clear that Carla had the right ideas, but was using the words in unconventional ways. I then explored what Carla would do with the conventional language:

121 Brad: Does the phrase normal subgroup, would that mean anything?

122 Carla: My guess would be that a normal subgroup would be a subgroup of a normal group.

126 Carla: If that's true, then a normal subgroup of this normal group that we are talking about could be this set 0, 3, 6, 9 because that's in the kernel … That is … Well, actually I am mixing things here. I know it could be 0, 3, 6, 9 because that's just the identity, so it's obviously closed and has it's own inverse and …

In response to my suggestion, Carla held on to her notion of normal group and applied her concept of subgroup to that. She proposed {0, 3, 6, 9} as a one-element subgroup of her normal group because it was the identity of that group. Carla went on to legitimize this group by noting that there is no "rule against" a one-element group and that the group axioms were satisfied. I then decided to be more explicit about the term quotient group:

164 Brad: Now what if I were to tell you that the thing you have actually created here, this table with an operation table—I mean a group table for these three cosets—that's a quotient group.

165 Carla: Oh. Maybe the quotient group is… Um… My only guess would be that the … The quotient … I don't know if we ever defined it, actually. But my guess would be that the quotient group would be… Well in this case the quotient group equals the normal group… Do you see what I'm saying? I mean it's the same …

166 Brad: What if I were to tell you that the thing that you have been calling the normal group is the thing that *is* called the quotient group?

167 Carla: [Laughs.] Oh, okay. What is a normal group then?

Carla was willing to accept the term quotient group, but was reluctant to let go of her usage of normal group. Despite my direct statement, she at first was still trying to figure out what a quotient group was (line 165) and only gradually came to decide that it was the same as her normal group, though perhaps only in this case. She had not yet detached the term normal group from her previous meaning. My next statement (line 166), however, seems to have caused Carla to consider a different meaning for normal group.

As the interview continued, Carla said, "I guess I have a problem with names" (line 171). I suggested that the idea was already there and the issue was sticking a name on it, and then encouraged Carla to connect the word normal to the subgroup:

178 Brad: But in order to talk about cosets you need to have what?

179 Carla: A subgroup.

180 Brad: Right, so this word normal tells us something about the subgroup that led to those cosets. Now let's see if *that* can make any sense.

181 Carla: Maybe it's when the subgroup is the kernel? I don't know. Because that's the only thing that I can think of that we've been talking about that kind of sets the subgroup apart. Like that's the only more specific characteristic of a subgroup that I can think of right now that we have talked about. The other subgroups were just subgroups, but if you have a subgroup that contains only the elements that map to the identity then that's more specific.

182 Brad: Okay. But in order to have…

183 Carla: We don't even have a homomorphism here. So, there goes that theory.

Even when I told her directly that the word normal applied to the subgroup (line 180), Carla was not able to make the appropriate gluing. The suggestion she made about the kernel, it should be mentioned, is correct in principle, in that the kernel of any homomorphism is a normal subgroup and any normal subgroup is kernel of a homomorphism. This is a pretty sophisticated view, however, and one that might have required some major results, such as the first isomorphism theorem, which were not yet available.

Cosets in D_3

I next turned to a the group D_3, which has both normal and non-normal subgroups. To begin, I asked Carla how she would write down the elements of D_3. She acknowledged the possibility of representing them as rotating and flipping triangles, but chose to use permutation representations instead, which she also called the "123 way." She wrote down the six elements quickly. I asked her for the subgroup generated by (12), and she immediately responded that it would be just (1) and (12):

202 Carla: Because any subgroup has to have the identity, so that's (1). That's why (1) is in there. And then if you … And (12) obviously has to be in there because it's generated by it… So if you operated (1) on (12) you just get (12). So it's in there so we are all set. And (12) operated with (12) just gives you (1). Another way you can think of it is that the order of (12) is 2. So if you call alpha (12) then the subgroup is only going contain alpha to the 0 and alpha to the 1.

Carla carried out 6 coset calculations (see Figure 7), making one small error that was quickly corrected. She saw that she got only three cosets as a result, and suggested that the right cosets would be different because here "the order that you do that in does matter" (line 222). When computing the right cosets, her calculations were guided by expectations that grew from noticing which elements were paired with other elements in the cosets, an approach that implicitly took advantage of the fact that cosets don't overlap.

Figure 7. Carla's cosets of {(1), (12)} in D_3

S = {(1), (12)}

Lt Cosets			Rt Cosets		
(1)*{(1), (12)}	=	{(1), (12)}	S*(1)	=	S
(12)*S	=	S	S*(12)	=	S
(23)*S	=	{(23), (132)}	S*(23)	=	{(23), (123)}
(13)*S	=	{(13), (123)}	S*(13)	=	{(13), (132)}
(123)*S	=	{(123), (13)}	S*(123)	=	{(123), (23)}
(132)*S	=	{(132), (23)}	S*(132)	=	{(132), (13)}

Upon completing the calculations, Carla noted that the left cosets were not equal to the right cosets and claimed that with these cosets, attempting a group table would be a mess. I asked her to try to make a group table with just the left cosets. Before computing the product of {(1),(12)} and {(23),(123)} she predicted the result would be a "four element thing" (line 246). When her prediction turned out to be correct, she noted, "It's not closed" (line 256), in the sense that the product of the two cosets was not another coset. (See Figure 8 for the partial table.)

Figure 8. Carla's table for the cosets of {(1), (12)} in D_3

	(1),(12)	(23),(132)	(13),(123)
(1)(12)	(1)(12)	(23)(132)(123)(13)	
(23)(132)			
(13)(123)			

Note: Carla included no braces for the sets and omitted commas below the top line.

I next asked Carla to consider the subgroup generated by (123). She responded immediately:

263 Carla: The subgroup generated by (123) is (1), (123), and (132). The reason that I know that's 'cause I remember (123) and (132) are inverse of each other, from working with them before.

She seems to have known by recall that (132) and (123) are "inverse of each other," and yet moments later she didn't remember what (123) squared was. She called the subgroup *S* again and computed the left cosets (123)**S* and (132)**S* quickly (see Figure 9), reasoning from the fact that *S* is closed. For the other cosets, her calculations were guided by expectations based, for example, on the fact that cosets don't overlap. By the time she computed the right cosets, her procedure had become quite abbreviated.

Figure 9. Carla's cosets of {(1), (123), (132)} in D_3

$S = \{(1), (123), (132)\}$

Lt Cosets			Rt. Cosets		
$(1)*S$	=	S	$S*(1)$	=	S
$(123)*S$	=	S	$S*(123)$	=	S
$(132)*S$	=	S	$S*(132)$	=	S
$(12)*S$	=	{(12),(23),(13)}	$S*(12)$	=	{(12),(13),(23)}
$(13)*S$	=	{(13),(12),(23)}	$S*(13)$	=	{(13),(23),(12)}
$(23)*S$	=	{(23),(12),(13)}	$S*(23)$	=	" "

After completing the calculations, Carla noted that the left and right cosets were the same. She then reflected on the fact that, while order mattered in D_3, the order didn't seem to matter in these coset calculations:

303 Carla: Right, but we had the right combination of things so that … Where the order mattered … Because the elements of the sets can be in any order … So where the order mattered it covered up for it because you had the right combination of elements.

314 Carla: Well we just see that the subgroups in case of the subgroup generated by (12), we didn't have the right combination of elements … or "right" as in to make the cosets equal. The right combination of elements so that they were equal. And in this case we did have the right combination of elements.

318 Carla: Well one thing that might have helped us is that this subgroup has more elements in it. So it might cut down on the error.

Rather than just ascribing the differing results to the original subgroups, Carla began looking inside the subgroups to determine what might have caused the results.

<u>Refining the Language Again</u>

Equipped with these additional examples, I pursued the language again:

320 Brad: Do you suppose we could use the word "normal" to help us out here?

321 Carla: [Laughs.] I don't know … Well which one is normal? Can you tell me that? I don't, I have no idea what normal is. I thought I knew about it. I have no idea.

322 Brad: Well you were saying before that you have a situation kind of called normal when you have left and right cosets being the same, right? You said something like that, didn't you? So where are left and right cosets here?

323 Carla: Well, left and right cosets are the same here. Maybe it has something to do with their inverses. Because this one (123) and (132) were inverses of each other, whereas in this one each element is its own inverse. That probably didn't answer your question. I was kind of half listening. What was it?

Carla recognized that she wasn't sure about the use of the word normal, and she recognized that the two subgroups were somehow different, in that one led to left and right cosets that were the same and the other didn't. Rather than merely using (and not using) the label normal to distinguish among them, Carla was looking for something deeper: some differing characteristics of the subgroup that would explain the differences in the cosets. As a result, she didn't hear my question, so I repeated it:

324 Brad: Before you said that normal had something… You used the word normal as having something to do with when the left cosets and the right cosets are the same.

325 Carla: Yeah. But I wasn't right on that cause that was quotient group.

326 Brad: Ah, well now, but what was quotient group?

327 Carla: Quotient group is one by, the set of cosets where the left and right cosets are equal. I guess.

This last statement is almost identical to Carla's previous definition of normal group (line 117), suggesting that Carla had substituted the new name for the old idea. As for the differences between the subgroups, she was still trying to find a reason:

339 Carla: Well, yeah, but the only thing that I see … the only difference that I really see between, at least right now, between them, is the thing about the inverses. I'm not seeing any other… unless it is just normal because it's generated … [inaudible] doesn't make sense. I don't know. It doesn't … If a quotient group is the set that you get when, from the cosets, if the left and the right cosets are equal, then maybe the normal group is the group that you use to get the left and right cosets in that case.

340 Brad: What do you mean?

341 Carla: Is the normal group the group that is always going to be your *S*, when you find your

left and right cosets in the way we do the quotient group?

342 Brad: Oh, so you mean the subgroup we started with?

343 Carla: Yeah.

351 Carla: So I would say that this subgroup is a normal group. If, well if my guess is right. This subgroup is a normal group and that one isn't. [Pointing first to the subgroup {(1),(12)} and then to {(1),(123),(132)}.]

It seems that at this point in the interview she had at last made the desired connection between the word normal and the subgroup that determined whether the left and right cosets were the same. The only remaining difficulty, it seemed, was that she was calling it a normal group rather than subgroup.

352 Brad: What about saying "normal subgroup?"

353 Carla: [Laughs.] Well, that kind of would be doing left and right cosets of left and right cosets, I would think. You know. Go another step into it.

354 Brad: What do you mean?

355 Carla: Well, I would think a subgroup is usually a smaller group. So I would think that a normal subgroup, you have to get that from a quotient group. And that it would have to give you a different quotient group.

Carla couldn't accept the new term because the term normal was still close to the quotient group, maintaining some of her previous meaning. Yet a few moments later, she reconsidered:

363 Carla: Right. So, well, then maybe we were all … we only talked about normal subgroups and not normal groups. Is that what you're saying? Because I'm fine with calling it a subgroup. I really don't care. Because if you don't really talk about normal groups and you only call them normal subgroups you're just focusing on the fact that they are subgroups.

While thinking about the terms used in class, Carla relaxed the tie between normal and its old meaning and was willing to consider alternatives. She realized that she may have been remembering the wrong term, and if the correct term was normal subgroup then perhaps it made sense to apply it to the subgroup that led to the cosets. I closed the interview by checking on how she labeled the various objects and the connections among them.

364 Brad: So, which is the subgroup here? Which is the normal subgroup?

365 Carla: S.

366 Brad: And it's a subgroup of what?

367 Carla: D_3.

368 Brad: And it's normal because…

369 Carla: It is what you use to get left and right cosets which are equal.

370 Brad: And this subgroup here, generated by (12)…

371 Carla: It's not a normal subgroup.

372 Brad: Okay. But it is a subgroup of …

373 Carla: D_3.

374 Brad: D_3 still. So here we have one subgroup of D_3 which the left and right cosets were different. So we say its…

375 Carla: Not a quotient group. And S is not a normal subgroup.

376 Brad: Well, you don't get a quotient group.

377 Carla: Right. This set of these is not a quotient group.

384 Brad: All the ideas were there. It's just a matter of…

385 Carla: [Laughs.] Yeah. Defining them.

386 Brad: You know, getting the right words connected to the right ideas.

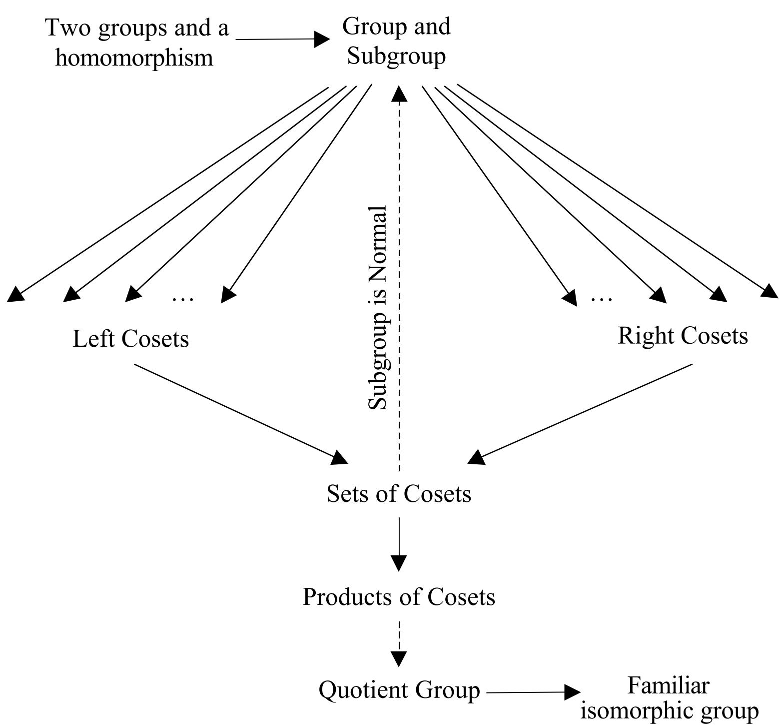
387 Carla: Right.

From Carla's correct usage of the term normal subgroup in the above exchange, it is clear that some learning had occurred. Because this exchange required only short answers from Carla, however, it is not clear to what extent she had changed her meaning of normal. On the final exam, she did use the word normal correctly, writing, for example "H is not a normal subgroup of G because lt. cosets $\neq$ rt. cosets" in response to the question, "Is H a normal subgroup of G? Explain" (problem 9d). It would be interesting to know how Carla would have responded to a more open-ended question.

Analysis: Quotient Group Activity

The schematic diagram in Figure 10 is an expanded version of Figure 4, showing a few more of the processes and objects involved in activities related to quotient groups. The core of the activity is depicted vertically along the center of the diagram. Given a group and a subgroup, compute the left and right cosets. Compare the set of left cosets with the set of right cosets. If they are the same, then designate the original subgroup as a normal subgroup. Calculate the various products of left or right cosets. If the sets of left and right cosets were the same, identify the set of cosets (with their products) as a quotient group.

Figure 10. Quotient group activity



Note: Arrows indicate processes. The dotted arrows denote designation processes that are legitimate only if the left and right cosets are the same (the result of an implicit comparison process on the sets of right and left cosets).

This core activity may be enlarged in two ways. First, the group and the subgroup may be identified as the domain and kernel, respectively, of a homomorphism between two groups. This is similar to the treatment in Fraleigh (1989), in that the concept of quotient group is introduced as the group of cosets of the kernel of a homomorphism. This was similar to the way that quotient groups were introduced in this class. Cosets and homomorphisms were introduced independently, although at about the same time (see Problems, April 22, Appendix C). Second, once the quotient group has been calculated, it is useful for considerations of structure to determine a known (familiar) group with which the quotient group is isomorphic.

These extensions are useful in the analysis because for Carla these extensions were sometimes natural parts of the activity. In particular, in one task she at first wanted the subgroup to be the kernel, demonstrating a connection between the core activity and the concepts of kernel and homomorphism (Carla 4, line 28). And after completing the group table for {1, 3} and {5, 7}, Carla went on to show, without any prompting from me, that the group was isomorphic to a familiar group with 2 elements.

The extensions are also useful for mathematical reasons because the entire diagram can be tied together with the first isomorphism theorem, which says that, given a group homomorphism, the quotient of the domain and the kernel of the homomorphism is isomorphic to its image. This is illustrated in the isomorphism that Carla noticed between the quotient group {{1, 3}, {5, 7}} and the subgroup {0, 2} in Z_4.

What is important in the diagram in Figure 10 is not the names of the processes and objects but the structure—the relationships among the various components. Just as learning group theory requires abstracting from the particular names of the elements and the operations, thinking about the learning of group theory requires abstracting from particular names of the objects and processes. This conclusion alone is not very surprising, as it is obvious that different languages use different terms for concepts. The above case, however, suggests something more: learning group theory requires not only attaching names but also carving the activity into concepts.

Building on the analysis above, it seems that for Carla, the entire center of Figure 10, from just below "Subgroup" to "Sets of Cosets," was essentially one concept: coset. Although within that concept she could distinguish between left and right cosets, she did not distinguish vertically between the processes and objects. Regarding her use of the word normal, she correctly associated the word with the comparison of the cosets, but rather than reaching backward in the activity to attach the label to the subgroup, she reached forward and attached it to the quotient group. This is not surprising when one realizes how distant the subgroup is when one is comparing the sets of right and left cosets. Furthermore, once the name normal was attached to the quotient group, it was very difficult for Carla to make any connection between it and the subgroup.

Questions

1. What does the case of Carla say about the relationships among mathematical names, definitions, symbols, concepts, and the objects they are meant to denote and describe?
2. What are the implications for teaching?
3. To what extent does the notion of a "concept image" help explain Carla's thinking?
4. What kinds of additional theoretical constructs might help?
5. What are the implications for theory and research?

References

Fraleigh, J. B. (1989). *A first course in abstract algebra* (4th ed.). Reading, MA: Addison-Wesley.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.