Learning and Understanding in Abstract Algebra

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The pages that follow contain offshoots and excerpts from my dissertation, *Learning and Understanding in Abstract Algebra*.

- The Operation Table as Metaphor in Learning Abstract Algebra (Paper to be presented at PME-NA this coming weekend)
- Carla and the Left Coset: The specific and the general in mathematical language, notation, and thinking (Proposal for a presentation at the joint mathematics meetings)
- Wendy and $N + N$ (Excerpt from the dissertation)

Most of the findings and results of the dissertation fell under two themes: balancing precision and managing abstraction. These three pieces will allow us to talk about each of these themes and the relationship between them.

THE OPERATION TABLE AS METAPHOR IN LEARNING ABSTRACT ALGEBRA

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This paper presents an analysis of the ways that one particular student used operation tables to support her reasoning about groups, subgroups, binary operations, and their properties. Based on interviews in the context of an undergraduate course in abstract algebra, the student's reasoning processes were external and based in the operation table, so that in large measure the operation table *was* the group for this student and, to a lesser extent, for other students as well. This result provides insight into the role of multiple representations in supporting the emergence of abstract mathematical objects in students' thinking.

Introduction

Among the growing research literature in undergraduate mathematics education, there is comparatively little attention to the learning of abstract algebra. This paper presents selected results from a dissertation (Findell, 2001) on learning and understanding in abstract algebra. To complement to the extant literature describing action, process, and object conceptions of topics in elementary group theory (see, e.g., Dubinsky, Dautermann, Leron, & Zazkis, 1994; Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Brown, DeVries, Dubinsky, & Thomas, 1997) and related literature describing difficulties with particular concepts in group theory (Hannah, 2000; Hazzan & Leron, 1996; Leron, Hazzan, & Zazkis, 1995; Zazkis, Dubinsky, & Dautermann, 1996; Zazkis & Dubinsky, 1996), the study sought to identify prominent features and components of students' concept images (Tall & Vinner, 1981) in the context of an undergraduate course in abstract algebra.

Analysis of interviews and written work from five students provided insight into their concept images of topics in elementary group theory, revealing ways they understood the concepts. In attempting to describe and explain the data, well-known theoretical constructs seemed insufficient. Thus, analysis became a process of theory generation, and much of the conceptual and analytic framework emerged during the analysis through use of the constant comparative method (Cobb & Bauersfeld, 1995; Glaser, 1992; Glaser & Strauss, 1967). Because many salient episodes in the data involved seemingly idiosyncratic uses of language and notation, the analysis was essentially semiotic, using the students' linguistic, notational, and representational distinctions to infer the students' conceptual understandings and the distinctions they were and were not making among concepts. The perspective on semiotics was based in the work of Peirce (1955) as well as recent contributions by Sfard (2000) and others (e.g., Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Pimm, 1995). The framework also took advantage of the distinction between process and object conceptions (e.g., Dubinsky, 1991; Sfard & Linchevski, 1994) as well as various perspectives on abstraction and generalization (Dienes, 1961; Frorer, Hazzan, & Manes, 1997; Hazzan, 1999; Wilensky, 1991) and the metaphorical nature of mathematical thinking (Lakoff & Núñez, 1997, 2000). As the conceptual and analytical framework emerged, the data were synthesized under two themes: making distinctions and managing abstraction. This paper discusses the latter theme, focusing in particular on the ways that students used operation tables to manage abstraction.

Analysis

This research report concentrates on one student, Wendy, whose reasoning was particularly tied to operation tables. All interview excerpts are taken from the first of four interviews with Wendy. The guiding question for this interview was, "Is Z_3 a subgroup of Z_6?" (see Hazzan & Leron, 1996). Before presenting data and analysis, I discuss Wendy's use of language, which bears on the interpretation of the data presented.

Wendy often misused words, saying one word while meaning another, such as multiplication for addition, identity for inverse, or associativity for commutativity. Sometimes she corrected herself immediately, other times she corrected herself when I asked for clarification, and occasionally her linguistic missteps suggested deeper conceptual difficulties. In any case, such slips-of-the-tongue were shown to indicate strong connections among the concepts that the words represented (see Findell, 2001) . Because slips-of-the-tongue are not a focus of this paper, I use brackets in the transcripts to indicate the word that she likely meant.

What is the operation in Z_6?

As Wendy began the interview, she was uncertain of the meaning of Z_6 but surmised that it was "integers mod 6" (line 11). She wanted to create "a total table" (line 13) and then realized that she would need to determine the operation. She discussed both addition and multiplication as possibilities and began constructing a table for multiplication in Z_6 (see Figure 1).

Wendy: Okay. Well, Z_6 is not going to be, when I start with my chart, and I do the first row, 0 times any element is going to equal 0, so if you look at that…. Actually, okay. Let me just…. It's not going to have…. You have to…. I'll just finish it [the row]. Okay, now it has to hold four properties to be a group. Let's write these down. It has to have an identity, an inverse, it has to be closed, and it has to be associative, which we're going to leave for last. [Laughs] (line 20)

Wendy: If you look at this row [the "1 row"], you multiply…. If I…. I am calling 1 the identity. If you multiply 1 by every element, you get the element back, get the original element back. So, like 1 multiplied by this row gives you the same row back. (line 26)

Wendy's inclination to finish rows and to write down names of the group axioms suggests that she preferred to have visual supports for her reasoning. She reasoned about both the action of the 0 and 1 on individual elements and the action of the identity on an entire row of the table. As she constructed the next row in the table, she considered the inverse property.

Wendy: So if you look at the second row [the "2 row"], there is no number when you multiply…. If you take *m* equalling 2, if you take a number equalling 2, when you multiply, there is nothing to multiply by 2 to get—in mod 6, cause it has to be an element, to be closed, you can only work with the elements within mod 6. And I have tried every element, 0, 1 … 0 through 5, multiplied by 2 to see if I can get the identity, 1, and I can't get it. So therefore, Z_6 is not a group under multiplication. (line 34)

Figure 1. Wendy's tables for addition and multiplication in Z_6

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3				
4	0	4				
5	0	5				

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Thus, Wendy had developed a procedure that could be used in verifying the inverse property. To find the inverse of any element in a set, she multiplied that element by every element of the set to see whether the result was ever the identity. Although the operation table may be considered as merely providing an organizational scheme for the necessary calculations, Wendy's procedure was based "in the table" in the sense that she preferred the whole row to be present. In fact, she used this procedure to reconsider the "0 row":

Wendy: Actually, up here, in multiplication, I didn't even have to look at the second row [the "2" row] because if you look at 0 there is nothing you can multiply by 0 to get the identity element back, 1, because 0 times every element is going to equal 0. (line 38)

Having abandoned multiplication modulo 6 as the operation in Z_6, Wendy began constructing a table for addition in Z_6 (see Figure 1) and quickly stated, "I can see by filling out the first table [row] that the identity" is zero (line 47), suggesting that her recognition of the identity was tied more strongly to visual aspects of the table than to mental reasoning about the operation. She went on to consider the other group axioms.

Wendy: So next I am going to check the inverse property. And 0 has an inverse so 0 + 1, or…. Excuse me. Since 0 is the identity we have to check that when you add 0 to 0 you get the identity 0. So 0 is the inverse element for itself. And then 1. When you multiply, when you add 1 and 5 it equals 6, but that equals 0 (mod 6) cause 6 is divisible by 6. That's pretty obvious, but…. So 1 has a inverse. 2 has an inverse because 2 + 4 = 6, which equals 0. 3 + 3 has an in-…. equals 0 (mod 6). 4 + 2 = 0 (mod 6). And 5 + 1 = 0 (mod 6). So each element has an inverse. So you know that Z_6 is a group under addition.

Wendy: And then it's closed. You can see that there are no elements other than 0 through 5, looking at the chart, because we have all possible combinations on elements in Z_6. So it is closed also.

Wendy: And associative. You can see, because the chart has symmetry, that the group will be, is associative. This is how I look at it, anyway, because if you look at 2 × 5 you are going to get 1 and if you look at 2 + 5 you get 1. (lines 50-52)

Wendy correctly verified the inverse property by using the table to find the inverse of each element. She supported this process by making a check mark alongside each row of the table as she identified the corresponding inverse. In verifying the closure and associative properties, Wendy explicitly referred to the table to support her reasoning, but she made several errors in her attempt to verify associativity. First she stated that she was comparing $2 + 5$ and 2×5 when, based on her statement about the symmetry in the table, she probably was comparing $2 + 5$ and $5 + 2$. A more significant error was that she was describing commutativity but calling it associativity. Finally, the penultimate sentence implies that she thought that Z_6 is a subgroup of Z. During the interview, I pursued and she quickly corrected the first two errors.

Observations

Although the students had been working with the groups Z_n, all of the students who were interviewed expressed some uncertainty about the operation in Z_6, and most of these students used operation tables to support their reasoning. To verify the group axioms, Wendy had created procedures based in the operation table. Her reasoning was largely external, however, in that it seemed to require the presence of the table. Other students developed similar procedures, but none were as tied to the table as Wendy.

The above excerpts provide some explanation for the fact that Wendy sometimes confused the words and syntax while discussing the group axioms and related properties. The identity and inverse properties were closely related in Wendy's thinking partially because finding an inverse for an element involves looking for the identity in a particular row or column of the table. Similarly, when using an operation table, commutativity is more salient than associativity, leading to misstatements by Wendy as well as other students. Noting the symmetry in an operation table, they claimed that the operation was associative when the symmetry indicated instead that the operation was commutative. Thus, although an operation table supports students' reasoning, it may also contribute to conceptual and linguistic difficulties.

Is Z_3 a subgroup of Z_6?

After determining the operation in Z_6, Wendy was able to take on the question that had been posed at the beginning of the interview.

> Wendy: Now is Z_3 a subgroup of Z_6? Now, we have to check that Z_3 is going to be a group because it has to have all of the elements [axioms] of a group, which means it has to have identity and inverse; it has to be closed. So I am going to start checking Z_3. Z_3 would consist of 0, 1, and 2 under addition. But Z_3, the table is going to be different. See I am going to have to explore right now whether or not…. When you say Z_3 is a subgroup of Z_6, whether it means you are taking Z_3 out of Z_6, or if you are just looking at Z_6 [Z_3] and seeing whether it's a group. See when you say something is a subgroup of something else [pause] I am not quite sure what way to look at it. Like how it exactly, like how Z_3 ties into Z_6, like to be a subgroup of Z_6. What, that…. Like I know how to check whether or not Z_3 itself is a group and whether Z_6 is a group, but to check whether Z_6, Z_3 is a subgroup of Z_6, I don't know exactly what to look at. (line 76)

Wendy had a sense that the operation table for Z_3 would be different, depending upon whether it was constructed on its own or taken out of the operation table she had just constructed for Z_6. Consistent with the hypothesis that Wendy's reasoning was highly dependent on *looking* at an operation table, her statement "I don't know exactly what to look at" suggests she didn't know what *table* to look at.

The literature on the learning of group theory indicates that although students think of a group as a set, they are not always sufficiently aware of the operation (Dubinsky et al., 1994). For Wendy, however, the operation remained prominent because of her reliance on the operation table, which suggests a much stronger observation than has been made thus far. Rather than saying the operation in Z_3 is different, Wendy said, "But Z_3, the table is going to be different," implying that the table was not merely supporting her reasoning but rather was substituting for the group in her thinking. The phrase "taking Z_3 out of Z_6" implies again that, for Wendy, Z_6 was not merely a list of elements that appeared on the edges of the table but was in fact the table. This observation is further supported by the following excerpt in which Wendy referred not to the group Z_6 but again to the table:

> Wendy: Because if you use the elements of Z_3, which is 0, 1, and 2—are the elements of Z_3. But if you look at them in terms of Z_6, like if you just look at this section of the table Z_6 [dotted square in Figure 2], this isn't going to be a group…. Because it is not closed…. Because 4 isn't an element of Z_3. (lines 78-82)

Thus, through her reliance on the table, Wendy had correctly identified the central issue behind the interview question: whether the addition was to take place based on the operation in Z_3 or in Z_6. Nonetheless, she was not ready to come to a conclusion.

> Wendy: See, it doesn't make sense. Like, I started over here to do, to look at whether or not Z_3 was a group itself, but that didn't make sense to me … [t]o look independently to see whether Z_3 was a group under addition. (lines 86-88)

Wendy was bothered by a contradiction that was implicit in the interview question. Z_3 and Z_6 are each groups if they are considered separately, but the concept of subgroup prohibits such separate consideration. Her thinking coalesced later in the interview:

Wendy: I think we have to look at it Z_3 as like part of the set of Z_6, which, like subgroup, like as a group in Z_6. So if you look at … which is why I kind of choose the elements Z_3 out of the Z_6 table. (line 111)

Wendy was clearly thinking of Z_6 as more than a set and of Z_3 as more than a subset. She was choosing entries out of the Z_6 table that corresponded to the restriction of the binary operation to the subset Z_3. On the conviction that this was the appropriate method for thinking about a subgroup, Wendy decided that Z_3 is not a subgroup of Z_6 because the subset was not closed under the operation.

Figure 2. Wendy's table for Z_6, annotated version

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Note: circles added to clarify transcript

Are there subgroups of Z_6?

The fact that Wendy had answered the interview question was apparently of little concern, for she immediately began looking for subsets of Z_6 that could be subgroups. In particular, she considered the set {3, 4} and saw from the table that 4 wouldn't have an inverse. She generalized the question "Is Z_3 a subgroup of Z_6?" to consider whether Z_n might be a subgroup of Z_6 for other n, but saw that the closure property wouldn't be satisfied.

I asked explicitly whether she could find a subset that was a subgroup.

Wendy: See Z_6, it's hard to take a subset because you have to make sure you include the identity element in the set that you pick. So let's, just for instance, I'm going to take this. Because if I am looking in the fact that you have to have an identity element. Here, if you look at 1, 2, and 3 they each have and 3, 4…. You can't do that. 'Cause now it's not closed, really. You can't take 3, 4, 5 and 1, 2, 3. It wouldn't work. (line 144)

Wendy saw that she needed to include the identity, but she was simultaneously considering "blocks" in the operation table, and she saw that this would not work. After I suggested that she consider nonadjacent elements, such as 3 and 5, I noticed that she was covering up the 4 in the table. She explained:

Wendy: It distracts me. Technically you are only looking at the 0, 2 … 2, 4 [circled in Figure 2]. Right? Because, in other words, you can make that table…. You can't look at the other elements. You can't look at the whole row, 3 and 5. You know what I mean? Because you can only look at the addition of those two. (line 150)

Wendy indicated how she was restricting her view of the table, listing precisely those entries inside the table (0, 2; 2, 4) that are relevant to whether {3, 5} is a subgroup. Furthermore, she justified this view by noting that "you can make that a table." From this view, she noticed that 5 does not have an inverse in {3, 5} because "when you add 3 or 5 to 5, you can't get 0" (line 152).

Wendy next decided to begin with the set {1, 5} to see whether it is a subgroup of Z_6 because "if you took those two separately, it upholds the inverse property" (line 162). Realizing that she also needed an identity element, she then considered the three-element subset {0, 1, 5}. She chose to "move this over" (line 166) to create a new uncluttered table (see Figure 3).

Figure 3. Wendy's tables for {0, 1, 5} and {0, 2, 4}

+	0	1	5
0	0	1	5
1	1	2	0
5	5	0	4

+	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

After completing the table for {0, 1, 5} and verifying the identity and inverse properties, Wendy noticed, with some frustration, that "It's not closed. [Laughs.] Oh, no. It's got 2 and 4 in it. This is just getting really difficult" (line 173).

With the failure of the closure property, Wendy focused on closure and realized that any subgroup that contains 1 must also contain 2, then 3, 4, and 5. I asked her what would happen if she started with something else.

Wendy: If you start with 2 you are going to need 0. You are always going to need 0, 'cause, like you said. Okay. So, things are getting kind of messy. I need a new piece of paper. If you start with 2, you're going to need 4.

Wendy: And when you're doing 4, you need 0. Well…. Ooh.

Wendy: You need 0, anyway. You need 4 though. [inaudible] So, 2 … 'cause 2 and 4 is going to equal 0. Uh oh.

Wendy: It works! You don't…. It's closed. It's got an id-, everything has an identity element … 0 is the identity element for all, each element. Well, they have to have the same identity element, but…. And it's got an inverse. (lines 196-202)

Wendy had found a subgroup of Z_6, but she was able to conclude that it was a subgroup only after she had completed both the operation table and her table-based verifications of the group axioms. For the first time in the interview, she considered several axioms in quick succession, although her explanations were somewhat muddled. Her language suggests that she was surprised—an impression that she confirmed when I asked whether there are any other subgroups of Z_6: "I don't know; I'd have to play and try. I just found one; I didn't think that we could find one, but I just found one" (lines 212).

Observations

Wendy's reasoning about groups and subgroups was largely external, often requiring that relevant portions of the table be present before her eyes without extraneous information interfering with her perception. When considering whether {3, 5} was a subgroup, she covered up the 4, and when building a subgroup with {1, 5}, she created a new table separate from the Z_6 table. In large measure, the operation table *was* the group for Wendy, although later in the interview she began to separate the group from the table, as evidenced by a suggestion that the table she called $4Z_3$ was a rearrangement of the table for {0, 2, 4}, which she called $2Z_3$. The operation table both supported and limited Wendy's ability to reason about groups and subgroups. On the one hand, the table helped her see quickly the central issue about considering Z_3 to be a subgroup of Z_6. On the other hand, her reliance on the table made it difficult for her to find subgroups.

A symptom of the external, table-based nature of Wendy's reasoning was that she often considered only one group axiom at a time when reasoning about groups and subgroups. Toward the end of the interview, however, she had developed more fluency and was able to move more quickly among the axioms. Considering the axioms multiply and flexibly might be described as a matter of proficiency and fluency with the group axioms and with the particular examples, which may be a result of gradual internalization of some of the external processes that were based in the table.

In a later interview, Wendy continued to struggle with the distinction between a group and its representation in a particular operation table. She recognized isomorphisms among various groups of order 4 and had a strong sense that there are only two such groups (up to isomorphism), but her methods remained largely external, relying upon the appearance of the operation tables and upon processes of renaming and reordering elements in those tables.

Results

Operation tables served to mediate abstraction for the students in this study in that they worked with a concrete representation to gain access to abstract objects and their properties. A group's operation table makes the group more concrete by making aspects of its form directly visible. Furthermore, by squinting one's eyes or coloring the operation tables by elements or by cosets, the abstract group—of which the particular table is an instantiation—can almost become visible. Abstracting the essence of a group from an instantiation seems a quintessential example of an activity that requires reflective abstraction—abstraction based on action (i.e., operations) alone. With the help of the operation table, however, perhaps only empirical abstraction is required. Thus, the table becomes both a tool for reasoning and an object of reflection. Under Wilensky's (1991) view of abstractness as a measure of one's familiarity with a situation, the table serves to increase one's familiarity, thereby making the abstract more concrete.

Operation tables served a metaphorical role for many students in that the tables supported their reasoning and helped them think of groups as objects. For some of the students in the study, the table *was* the group rather than a representation—a metonymic substitution of the concrete for the abstract. Their reasoning seemed to be largely external, in the sense that it was based *in* the table and in procedures that required that the operation table be present rather than in reflection on the binary operation. Most of the group axioms, for example, were verified through such table-based procedures. The cancellation laws (e.g., $ab = ac$ implies $b = c$) became embodied in the requirement that each element appears exactly once in any row or column.

The table served also to heighten the students' sense of anticipation about the way the calculations should turn out, similar to Boero's (1993) observation about the role of anticipation in algebraic manipulation. One student, for example, expected the coset {5, 7} to be its own inverse. Many students came to expect certain patterns in their operation tables and likened those patterns to cycles, which caused some potentially problematic connections with the cycle representations of permutations.

The prominence of the operation table helped explain the fact that students demonstrated strong connections between distinct concepts. For example, students sometimes confused the identity and inverse properties in part because finding the inverse of an element requires looking for the identity element in a particular row or column. Students confused commutativity and associativity in part because when an operation table is present, commutativity is more salient than associativity. Commutativity is quite visible through symmetry in the operation table, whereas associativity is more difficult to see and much more difficult to verify when the group is given by an operation table.

The operation table as metaphor has other limitations as well. First, it becomes cumbersome for large groups, and extending the metaphor to infinite groups requires some sophisticated patterning abilities since it is not possible to write out the whole group table. Second, the students expected subgroups to occupy a corner of the table, probably because of reliance on Groups-Are-Sets and Set-Are-Containers metaphors. Third, writing down a group table requires one to choose an ordering of the elements, which sometimes made it difficult for the students to recognize isomorphisms and to think of the order as nonessential. Nonetheless, through experiences in renaming and reordering operation tables, the students began to separate the table from the group—the signifier from the signified—and thus began to develop concepts of abstract groups.

Discussion

The results of this study suggest that the operation table can play a useful metaphorical role in students' thinking about group theory because of the conceptual support that the metaphor can provide. This paper discusses three implications. First, the results of this study provide insight into and support for Peirce's (1955) semiotics, in which a sign becomes a sign when it is interpreted by someone as a representation of something else. In the case of Wendy, at first each operation table *was* a group. She began to develop understanding of groups as abstract objects only through considerable conceptual struggle, supporting Sfard's (2000) contention that "The transition from signifier-as-object-in-itself to signifier-as-a-representation-of-another-object is a quantum leap in a subject's consciousness" (p. 79).

At the same time, the results of this study call into question the developmental relationship between process and object conceptions. Through use of the operation table, some of the students appeared to have strong object conceptions of groups with relatively weak process conceptions. One possible explanation is that the tables were not objects but pseudo-objects for these students (Sfard & Linchevski, 1994; Zandieh, 2000). The processes were largely external in the sense that the students used procedures that depended upon the presence of the operation table. Another possible explanation is that through the use of the operation table, the developmental trajectory was reversed, with process conceptions emerging slowly from object conceptions.

Finally, the results of this study provide additional support for the use of multiple representations as a way of building understanding of abstract mathematical concepts and objects. Thompson (1994) observes:

> Our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one representational activity to another, keeping the current situation somehow intact. Put another way, the core concept of *function* is not "represented" by any of what are commonly called the multiple representations of function, but instead by our making connections among representational activities. (p. 39)

In this study, the students similarly developed abstract conceptions of objects and concepts in group theory by translating among what are taken to be equivalent representations of the same object.

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Carla and the left coset: The specific and the general in mathematical language, notation, and thinking

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Abstract

This paper discusses seemingly idiosyncratic ways that students make distinctions among mathematical concepts. The findings are taken from the author's dissertation on student learning and understanding in an undergraduate abstract algebra class. Analysis of interviews and written work from five students revealed that the students often made nonstandard linguistic and notational distinctions. For example, Carla used the term *coset* to describe not only individual cosets of a subgroup but also the set of all cosets. This kind of understanding was characterized as being immersed in the process of generating all of the cosets of a subgroup. Similar characterizations described and explained instances of the phenomena of failing to make standard distinctions between a set and its elements, between quantifiers such as "any" and "all," and between the specific and the general and their notation in mathematical arguments. Implications for teaching and for the process/object distinction are discussed.

Detail (from Findell, 2001)

Given the function f from U_8 to Z_4, given by $f(1) = 0$, $f(3) = 0$, $f(5) = 2$, and $f(7) = 2$, Carla had identified the kernel of the homomorphism as the set $\{1, 3\}$. When I suggested that she call that set K and that she investigate the sets aK, she responded, "Left cosets. All right. So to do aK, we see what happens …" (line 63). She struggled momentarily to recall the definition of coset and reconstruct the process for computing them, but then completed the calculations quickly, yielding $\{1, 3\}$ and $\{5, 7\}$, and concluded, "So we have two elements in the left coset" (line 70). Her nonstandard syntax suggested unusual thinking. She explained, "$\{1, 3\}$ and $\{5, 7\}$ are each of, are the two elements that result from aK" (line 74). I ask her whether aK refers to all of the cosets or one specific one.

82 Carla: aK is the general formula that gives you all of the cosets. Then you choose specific *a*s for whatever K you are given, and you will find all the specific cosets.

Thus, Carla had an efficient and reliable procedure for computing cosets, and the procedure was supported by the notation aK. By calling the cosets *elements*, she seems to have had little difficulty seeing cosets as objects, suggesting that she had encapsulated the process of coset formation. Furthermore, her language "the left coset" for the collection of cosets suggests that she had further encapsulated that collection as an object, a set of sets, a point of view that is helpful in order to see the set of cosets as itself a group under the appropriate coset arithmetic.

Nevertheless, Carla did not see the relevance of the distinction I was trying to make, and thus did not distinguish between the particular coset aK and the set of all cosets of K. Instead, a varied as part of a procedure specified by the formula aK, which gave all the cosets. Carla's language suggests that her thinking was in the transitional process between two objects: the particular coset $aK = \{ak \mid k \in K\}$ and the set of all cosets of K, which might be written $\{aK \mid a \in G\}$. In the midst of the process, a is varying, so aK denotes neither a particular coset nor all of them. For Carla, aK denoted and specified the process.

Being immersed in the process allowed Carla some flexibility in her thinking. On the one hand, she could consider a particular coset by stopping the process for a moment. On the other hand, she could consider all the cosets by completing or imagining she had completed the process. From within the process she could broaden her viewpoint slightly and see both a particular coset and the set of all of them as two aspects of the concept of "coset." Thus, "aK tells you how to find the cosets" (line 80) and together there are "two elements in the left coset" (line 70). Carla maintained this dual role of the term *coset* through the fourth interview, and even maintained her process orientation, saying, for example, "The left coset *gives* the set of (23) and (132)" (line 239, emphasis added) rather than the left coset *is* that set. Other students in the study used the term *coset* in similar ways.

The paper generalizes the characterization of this phenomenon to include similar difficulties that the students had distinguishing, for example, between a particular multiple of 4 and the set of all multiples of 4. Together, these findings provide explanations for the common phenomenon of failing to make notational and linguistic distinctions between a set and its elements. The findings also provide nuance in the relationships between process and object conceptions of mathematical ideas.

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Finally, Carla herself showed a marked contrast between her work with particular groups, such as U_8 and Z_{12} and her symbolic reasoning about cosets generally, where the group remained unspecified. In particular, she seemed to forget that aK and bK were cosets and instead treated the letters like numeric variables from high school algebra. This point is demonstrated in the subsection entitled "Carla and $aK = bK$."

Wendy and *N* + *N*

This subsection explores a failure to distinguish between a set and an element and provides insight into the kind of learning that occurs as students begin to make such a distinction. During her fourth interview, Wendy described a function from Z to Z_4 where x goes to x (mod 4)—the same function that Carla had described at the beginning of her third interview. Then Wendy listed the cosets of the kernel of the mapping:

50 Wendy: Well, we can take the generator group now, and we can find cosets a lot more easy because if you take … $4x$ … every element … every multiple of 4 is going to get mapped to 0 in x mod 4 … in Z_4 … so $4x$ is going to equal the kernel. So the cosets are going to be $4x$. Okay. $4x + 1$, $4x + 2$, and $4x + 3$, because $4x + 4$ is just going to be [the same as $4x$].

52 Brad: Okay, and what's x?

53 Wendy: x is going to be an integer.

54 Brad: Okay, now is x…? In these 4 cosets, is x a specific integer?

55 Wendy: No.

56 Brad: What do you mean?

57 Wendy: It can be any…. Like any integer you put in here will give you…. Any integer you put in for x will give you 0 … will give you 0 mod 4.

58 Brad: So like you could put in 2 for x.

59 Wendy: Yeah, you could put in anything for any of them, and this [$4x$] is going to equal 0. This [$4x + 1$] is going to equal 1, this [$4x + 2$] is going to equal 2, and this [$4x + 1$] is going to equal 3.

Like Carla, who had used $4n$, Wendy was using $4x$ to denote both a particular multiple of 4 and the set of all of them. My language "a specific integer" did not help her make the distinction I was trying to make. Of course, x was not a *specific* integer. The question is whether she was imagining that it was a *particular* integer or the set of all of them. Yet, perhaps even this distinction would not have helped, for she was focusing on the images of these integers in the codomain, Z_4. And for that purpose, $4x + 1$ was going to map to 1 whether it was a particular value or any set of values.

68 Brad: Okay. Now…. But now you are talking about x as being … you can put anything in there for x. Any integer, right? But now is the coset then any integer? Or is it all of them. Or is it one specific one? Or…? Do you understand what I am asking?

69 Wendy: No, not really. Is the coset one specific…. Like should we name this something? [She writes N.]

71 Wendy: Yeah. If I call it N, it's always, it's going to be congruent to … any integer N that maps to the identity.

73 Wendy: So, it's not going to be a specific, it's going to be any multiple of 4.

75 Wendy: So it is not specific. You know what I mean? Like, we are going to call this coset…. If we call this coset N, N is infinite.

76 Brad: Oh. Is it a set then? Or is it a specific number?

77 Wendy: It's a set.

The letter N denoted what she had previously called $4x$ and was "any integer that maps to the identity," which was not specific but "any multiple of 4." Simultaneously, N was a set. Thus, Wendy was not distinguishing between any multiple of 4 and the set of all of them.

When I mentioned again that the set was infinite, she said, "That is why I was calling it $4x$" (line 81). Because she could not list all the elements, I asked her to show the pattern of N. She began with positive multiples of 4 and then included 0. She included negative multiples of 4 only after I asked explicitly whether there would be any negatives in the set.

88 Brad: So then, when you write this thing $4x$, you mean this set of all the things together, taken as a whole. Or do you mean individual specific …?

89 Wendy: Uh huh. Taken as a whole. Take as a whole.

So just like N, $4x$ was simultaneously any multiple of 4 and the set of all of them. From the fact that Wendy used both N and $4x$, a glance at her work might have suggested that she was using the standard convention of using capital letters for sets and lowercase letters for individual elements. Wendy was making no such distinction, however. The two notations were identical in meaning.

I asked what could be done with the cosets. She reaffirmed that "$N = 4x$, where x is all integers" (line 98) and renamed the other cosets to $N + 1$, $N + 2$, and $N + 3$. She struggled for a moment over whether to add or multiply the cosets but then decided to add them because "integers are only a group under addition" (line 112). Then she tried to calculate $N + N$ and $N + (N + 1)$.

115 Wendy: All right. So, you are going to have, if you have $4N$…. $4x$ plus, because $N = 4x$. It's going to equal $8x$. All right? And $8x$ is congruent to $4x$, which is congruent to 0 mod 4. So this is congruent to N.

116 Wendy: So N is…. You can tell here that N is the identity element. So, you know that … [inaudible] when you add N plus, and $N + 1$ you are going to get $2N + 1$ and that's going to give you…. That's the same thing, that is congruent to $N + 1$. Now I have to figure out why. [Writes $2N + 1 \equiv N + 1$]

It seems that Wendy knew that $N + N = N$ and that $N + (N + 1) = N + 1$. Her reasoning was flawed, however, relying on algebraic procedures that work for symbols that stand for numbers but not symbols that stand for sets. Her symbol manipulation was guided more by her expectations about the results than about the meaning of the symbols.

I asked her what $N + N$ meant.

120 Wendy: You are adding the same set together, so it is going to be the same set. You know like…. If you add two of the same sets together you are just going to get all elements … the same elements in the set. Like if you add 1, 2, 3 … the set of {1, 2, 3} and the set of {1, 2, 3} you are still going to have the set {1, 2, 3}. Your elements aren't going to change any?

Wendy saw $N + N$ as a sum of sets. When I asked her how to compute {1, 2, 3} + {1, 2, 3}, she did not remember at first how to do it but eventually decided that the sum would be {2, 3, 4, 5, 6}. She was not happy with the result, however, because it did not fit her expectations.

134 Wendy: But if you add any multiple of 4 and any multiple of 4, you're going to get another multiple of 4.

136 Wendy: So in this case it's different. So I know that it looks like. I was like, oh well that disproves what we are saying here, but it doesn't because …

137 Brad: So then is it right to say $4x + 4x = 8x$? Is it right to even call it this thing $2N$?

138 Wendy: Well, $4x$…. You can say $4x + 4x$. I don't know about this [Crosses out $2N + 1 \equiv N + 1$]. But if you say $4x + 1 + 4x$, you are going to get $8x + 1$, which is going to be congruent to…. This is still a multiple of 4, so it is still going to be equal to—congruent to; I don't want to say equal to—a multiple of 4, plus 1, which is what $N + 1$ is.

Wendy clearly had some thinking that was not reflected in the symbols. Furthermore, she had not understood what I was implying by my question about whether $4x + 4x = 8x$ was an appropriate calculation to verify that $N + N = N$. I suggested that $8x$ looked more like multiples of 8, not multiples of 4.

140 Wendy: That's true, it doesn't include every multiple of 4. But we're talking about mod 4. And if you are talking about mod 4, we got this because it is $4x + 4x$. All right? So if you have a multiple of 4 here and a multiple of 4 here, and then you add one, it is still going to be…. If you add two multiples…. Like, if you add…. 8's a multiple of 4, and 8's a multiple of 4, and that equals 16. 16 is still a multiple of 4.

141 Brad: Do the two multiples of 4 have to be the same in this way you are writing $4x + 4x$?

142 Wendy: No. 8 + 12, okay? That equals 20 and that is still a multiple of 4, so I don't think so.

143 Brad: Okay, so…. But now does the way you have written it, $4x + 4x$, does that handle both of these cases? When they are the same and when they are different?

144 Wendy: Uh huh.

Thus, Wendy's work with the symbols depended upon her thinking. Furthermore, she did not see a need to distinguish notationally between the two multiples of 4. I pursued this directly in her notation.

145 Brad: So by $4x$ you just mean …

146 Wendy: A multiple of 4.

147 Brad: *A* multiple of 4. And by this $4x$ you mean *a* multiple of 4.

148 Wendy: $4y$.

150 Wendy: This is going to be $4xy$. So it's still going to be a multiple of 4 and then you add 1.

151 Brad: $4xy$ do you mean? Or …

152 Wendy: Plus y. So, all right. This explains it better. Do you see why this explains it better?

It did not take much intervention to get Wendy to distinguish between the two multiples of 4, but even after she had made the distinction, her symbolic moves were problematic, suggesting that there were not yet strong connections between her thinking and the symbols. From this point on, however, her symbolic reasoning improved dramatically:

154 Wendy: If you have a multiple of 4 and a multiple of, another multiple of 4, but where they're not the same multiple of 4, and then…. But this multiple…. This is…. Okay, this is N. Like if this is your multiple of 4, you add a multiple of 4, which we are calling N, and you are adding it to $N + 1$, you have another multiple of 4, + 1. Okay? But because it is not necessarily the same … like this isn't necessarily 8, and this isn't necessarily 9. They are not necessarily consecutive numbers. You have to have different values for x and y. Okay? So, we can, because they have a common factor, we can pull it out. [Writes $4(x + y) + 1$.]

156 Wendy: Okay, which is the same thing…. So this is still going to be 4 times…. This is still going to be an integer. An integer plus an integer is going to be an integer. So I am going have to call $x + y = z$ so $4z + 1$, so this is still going to be $N + 1$. That's how I can explain this.

Wendy's reasoning and symbolic representation seemed sound at this point. I then attempted to learn how she had previously been thinking about the symbols.

158 Brad: Okay. But now here, when you are saying this $4x + 4y$, are you imagining that this is one specific x, for now, and this is one specific y, for now?

161 Wendy: Yeah, but it would work for any x and y.

162 Brad: Okay. But are you imagining for a minute that they are fixed?

165 Wendy: It helps me think of it better, but it doesn't necessarily have to be. Because no matter what x or y you put … any integer you put in there, it will work. So, and these are all the integers. x and y are all the integers for … are just, are all integers. So in that case and since it works for all integers, you can look at it as the whole set. But yes, you were right, I was looking [inaudible].

167 Wendy: … But it doesn't make a difference, because…. It did help me clarify at first, but because if you show for a specific case it does work, and then if you take a step back and say it does, it works for every case … like x, no matter what value for integer x, or y, it will still work.

Thus, Wendy agreed that it helped to think of x and y as fixed, but any integer will work, "and since it works for all integers, you can look at it as the whole set" (line 165). Her thinking could move smoothly from a fixed value, to any value, to all values, and finally to the set of all values without ever a need for a clear distinction. Nonetheless, at this point, she had begun to articulate a distinction in her thinking: first showing that it works for a specific case and then stepping back to see that it works in general.

I asked her to reflect on possible differences in meaning between $4x$ and N, and she asserted that "this set and this set are going to be the same" (line 171).

173 Wendy: I was calling x and y different, but they are not. Because x is going to be all integers. In this set, y are all the integers. Like this is $4z$, and this is $4z + 1$. This is the set $4z + 1$ … all integers …

174 Brad: By z there you mean …

175 Wendy: All integers in z.

Thus, the distinction in Wendy's thinking was not yet reflected in her interpretation of the notation. Wendy's written work through this point included only a single lower case z, but her statement "All integers in z" seems to suggest that she was thinking about the set of integers.

176 Brad: Oh, do you mean the big Z that means all integers?

177 Wendy: By calling it x, I think it is like making me think towards … by saying x … like we usually use that for a specific value. But…. So if you write Z … if you write $4Z + 4Z + 1$ … $4Z$ is…. Go ahead. [She writes capital Zs.]

178 Brad: This Z … is that the same as this z?

179 Wendy: Uh huh.

181 Wendy: The set Z.

Thus, just as Wendy was not distinguishing between a single value of x and the set of all of them, she had not been distinguishing between a single value of z and the set of all of them. After I asked whether she meant the set of integers, however, she changed her notation from what appears to be a lowercase to the capital that is typically used to denote the set of integers (i.e., $\mathbb{Z}$).

183 Wendy: All right, so if we think of that away from x, a specific case…. I think we, when we have x, I tend to think of it … and I think that is what you are trying to get to up there …

185 Wendy: … So if we call it $4Z + 4Z + 1$, $4Z + 4Z$ is going to still equal $4Z$, + 1. See I think that makes it more clear.

186 Brad: Although, now how do you explain to someone why $4Z + 4Z$ is just $4Z$?

187 Wendy: Because it's…. Because of the fact that it's infinite, you're taking every ele-, every multiple of 4…. Well, you can look at that specific example. If you take an x in, and a y, a $4x$ from $4Z$ and you add it to $4y + 1$, well x and y weren't equal. 4…. You can show what I was showing up there that x and y, because they are integers, because you took them from the set of integers, x and y will be an integer, so 4…. This is still going to be an integer.

It seems Wendy was still somewhat uncomfortable about x being a particular value. When I asked her to explain the sum $4Z + 4Z$, she started talking about the whole set but then resorted to using x and y to illustrate a particular case. I asked whether she was beginning to see a distinction between $4x$ and $4Z$.

190 Wendy: Uh huh. Yeah. I am a lot more so than I was when I was up here.

191 Brad: Is that helpful to make that distinction?

192 Wendy: Yeah. I don't like calling it x now because it does, here looks like a more specific value, except it is not. But I think in a sense I was thinking of it, even though I didn't think I was.

193 Brad: You were thinking of it in which way before?

194 Wendy: As a more specific value. Although I knew that I should keep in mind that it was a set, but I think I was still thinking of it too specifically. I think I was right to show … to explain it using a specific example, but I think it was important to go back, to begin and to end, showing that it was the whole set.

195 Brad: So before when you were talking about this $4x + 1$, were you trying to imagine both ways at the same time?

196 Wendy: Yes. I was definitely trying to…. I definitely knew that it works for all values of

x and y, but I wasn't really thinking of it as a whole, as like a Z, like integers.

Wendy was seeing a conflict in her earlier work: thinking of x as a specific value while also keeping in mind that it was the whole set. She stated that she had been thinking "too specifically" and had not really been thinking of x as the "whole set." Her language again suggests an intermediate position between a particular element and the set of all of them, because "it works for all values of x and y."

She began to focus on the notation:

198 Wendy: For some reason just visually, if you look at this $4x + 2$ [points on paper] and $4Z + 2$, visually, if you look at the two, it's easier to see one being a set and.… It's easier to see $4Z + 2$ being a set over $4x + 2$.

200 Wendy: $4x + 2$ looks more like a value than it does a set.

202 Wendy: But I think it is clearer to write, you know, like Z as, like an upper case Z, like writing the notation as a set for the integers. So $4Z + 2$ is going to equal, obviously be a set, and it is not going to be a value.

203 Brad: So it is useful, then, you think now, to distinguish between those things that are sets and those things which are sort of generic values.

204 Wendy: Like I use N, an uppercase N here. I think it makes more sense to use like the.… I think.… Do we use uppercase values for sets, rather than lower case?

So by the end of the interview, Wendy saw that the standard convention of using uppercase letters for sets could be useful in making a conceptual distinction. It seems she had much earlier developed a sense that uppercase letters were usually sets, though perhaps she had never before been in a situation where she felt a need to make a clear distinction between a set and an element.

Wendy seems to have learned to distinguish between a set and an element. Because this interview took place after the final exam, I can make no claims about the stability or durability of this learning. Instead, I would like to point out what seems to have been required. First, Wendy did not make any distinction between a particular value and the set of all values until she saw that her equation $4x + 4x = 8x$ did not support the idea that the sum of any two multiples of four would be a multiple of four. Yet even after she began to make the distinction verbally, she still did not make it notationally. And even after she made the distinction notationally, she had to revisit her intermediate conceptual position of something that "works for all values of x." Then, by reflecting on the notation, she was able to see how the notational distinction could support her emerging conceptual distinction. Furthermore, it seems that I provided cognitive support by asking Wendy whether she was imagining x as fixed, suggesting that this metaphor was not available to Wendy at the beginning of the interview.

Robert and What Varies

Of the key participants, Robert had the most trouble negotiating the processes involved in computing the collection of cosets and, in particular, in keeping track of what kinds of entities he was dealing with and where they were situated. During his third interview, he was working with the same homomorphism as Carla above and had just determined its kernel. I asked him to find the cosets of the kernel.

140 Robert: Oh boy. Cosets … equals the set of all *ah*s such that h is in H. [Writes $ah = \{ah \mid h \in H\}$.]

141 Brad: And what's H here?

142 Robert: Yeah, that's what I was wondering. Good question. I am not so good with these cosets yet.

143 Brad: Well, is the kernel a subgroup here?

144 Robert: Yeah, it's a subgroup of U_8.

146 Robert: Isn't that one of the things we proved on the take-home exam? That if f is a group homomorphism then the kernel of f is a subgroup of G.

147 Brad: Okay. So what would be the cosets of the kernel?

148 Robert: See, I am not really sure, like you say, what this H is. These are…. Okay. H would be 1 and 3 because those are the things that are in the kernel. So it would be the set $\{1, 3\}$, and each little h would be 1, 3. So then the question is the a. Which side of