Abstract

Recent research has demonstrated that teachers need not just content knowledge that many educated adults have, but also knowledge specialized for teaching particular topics. The present study contributes to research on knowledge that teachers use in practice by examining fraction multiplication instruction in two 6th-grade classrooms that used the Connected Mathematics Project materials. The materials use lengths and rectangular areas to represent fractional quantities. The central question was the following: What knowledge did the teachers use and to what extent did they adapt as they responded to their students’ thinking? To explain where each teacher did, and did not, adapt to her students’ explanations and drawn representations, I examine the unit structures and knowledge of multiplication that each teacher evidenced and the purposes for which they used drawn representations. The results highlight the importance for teachers of reasoning with three levels of units, and with flexibility supported by the distributive property, when responding to students’ representations of fractional quantities.
Fraction Multiplication and Adaptive Representation of Unit Structures

Research on teacher knowledge has expanded from studies of teachers’ subject-matter knowledge of various content areas to the organization of teachers’ knowledge for teaching particular content to students (e.g., Ball, 1991; Ball, Lubienski, & Mewborn, 2001; Borko & Putnam, 1996; Davis & Simmt, 2006; Ma, 1999; Sherin, 2002; Shulman, 1986). This expansion follows a generation of research that did not find clear connections between teacher knowledge and student achievement and reflects increasing awareness that teachers need not just content knowledge that many educated adults have, but also knowledge specialized for teaching particular topics (Ball et al., 2001). In light of this realization, current discussions of teacher knowledge are often framed in terms of subject matter, pedagogical, and pedagogical content knowledge (e.g., Borko & Putnam, 1996). When introducing the notion of pedagogical content knowledge, Shulman emphasized knowledge of students’ thinking about particular topics, typical difficulties that students have, and representations that make mathematical ideas accessible to students. Subsequent research has further developed the notion of pedagogical content knowledge by examining connections among various understandings that support such knowledge. Examples include knowledge packages (Ma, 1999) and content knowledge complexes (Sherin, 2002).

In a related line of research, Ball and colleagues (Ball & Bass, 2000; Ball et al. 2001) have used the term mathematical knowledge for teaching to emphasize knowledge that teachers might use when solving problems that arise in practice—for instance, using curricular materials judiciously, choosing and using representations, skillfully interpreting and responding to students’ work, and designing assessments. In the most visible line of current research on mathematical knowledge for teaching, Hill,
Schilling, and Ball (2004) reported a multiple choice instrument that reliably measures elementary teachers’ knowledge of number concepts; operations; and patterns, functions, and algebra. Hill, Rowan, and Ball (2005) demonstrated through a large-scale study that first- and third-grade teachers’ knowledge of these topics, as measured by the same instrument, had an effect size comparable to SES on student gain scores on the Terra Nova. The gains occurred over a one-year period in urban and suburban schools serving higher poverty populations. These results established a positive correlation between teacher knowledge and student achievement, a link that has proved elusive in previous research.

In the present study, I examine fraction multiplication instruction in two sixth-grade classrooms that used a combination of units from *Connected Mathematics* (CMP; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002) and draft revised units that have since been published as part of *Connected Mathematics 2* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). These units ask teachers and students to use various manipulatives and drawings to reason about problem situations in which fractions are embedded. My central research question was the following: What mathematical knowledge did the teachers use and to what extent did they adapt as they responded to their students’ thinking? When using the word adapt, I mean adjustments that teachers made in their thinking about the content as a consequence of interacting with students.

The study contributes in three ways to research on mathematical knowledge that teachers use in practice. First, past research has not examined teachers’ knowledge of fraction multiplication closely. Second, I used results on students’ reasoning about fractional quantities to gain insight into teachers’ reasoning and places where teachers did, and did not, adapt in response to their students’ thinking. Third, the analysis led to a theoretical frame for describing mathematical knowledge for teaching in this domain.
that foregrounds teachers’ unit structures and knowledge of multiplication and
teachers’ purposes for using drawn representations. I use the phrase drawn
representations to refer to persons using drawings to reason about or convey
information they perceive in situations distinct from those inscriptions. I focus on the
use of drawn representations in teaching not only because discussions of pedagogical
content knowledge and mathematical knowledge for teaching refer to representations,
but also because reform-oriented curricula in the United States often place new
demands on teachers and students to interpret and reason with a variety of
representations.

Background

This section demonstrates that little research on teacher knowledge has focused
directly on fraction multiplication and summarizes past results on unit structures that
informed the theoretical frame presented in the following section.

**Teachers’ Knowledge of Fraction Multiplication**

Research on teachers’ knowledge has examined fraction division and decimal
multiplication more closely than fraction multiplication. Research on fraction division
has reported that teachers can confuse situations that call for dividing by a fraction with
ones that call for dividing by a whole number or multiplying by a fraction (Armstrong
& Bezuk, 1995; Ball, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992;
Ma, 1999). Ball asked 10 preservice elementary and 9 preservice secondary teachers to
generate situations that would illustrate $1 \frac{3}{4} \div 1/2$. Seventeen could compute the
correct answer but only five were able to generate an appropriate word problem or
situation. Five others generated situations that would illustrate $1 \frac{3}{4} \div 2$, one generated

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a situation that would illustrate 1 3/4 x 2, and eight were unable to generate any situation, correct or incorrect. Ma reported similar results for U.S. teachers after including the same task in her investigation of 23 U.S. and 72 Chinese inservice teachers’ understandings of core mathematics topics taught in elementary grades. Borko et al. (1992) reported a case in which a teacher candidate, Ms. Daniels, was asked by a student to explain the invert and multiply rule for fraction division. The class had just computed the answer to 3/4 ÷ 1/2. Ms. Daniels generated a situation and area representation that illustrated 3/4 x 1/2, realized that her example showed fraction multiplication instead of division, and was stumped.

A second set of studies about teacher knowledge of rational numbers has built upon the notion that people have intuitive models for arithmetic operations (Fischbein, Deri, Nello, & Marino, 1985) and that the model for multiplication is repeated addition. The repeated addition model implies that the multiplier or operator can only be a whole number and supports the notion that multiplication always makes larger. Several studies have extended Fischbein et al.’s (1985) results on fifth-, seventh-, and ninth-grade students to elementary school teachers (Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Harel & Behr, 1995; Post, Harel, Behr, & Lesh, 1991; Tirosh & Graeber, 1989). In these studies, teachers have had a harder time solving word problems in which the multiplier was a decimal less than one. For instance, Graeber and colleagues (Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989) administered to 129 preservice elementary teachers a written test consisting of 26 word problems adapted from those used by Fischbein et al. (1985). They found that a higher percentage of the preservice teachers solved multiplication word problems correctly when the multiplier was a whole number but often used division in problems that should have used a decimal less than one as the multiplier.
A few further examples in the literature suggest that teachers may also have a hard time using drawings to explain the product of two rational numbers. Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993) reported that during an interview the same Ms. Daniels began, but could not complete, an explanation for .7 x 2.35 using a rectangular region. Armstrong and Bezuk (1995) gave inservice middle school teachers a word problem for which computing 1/3 of 3/4 would be appropriate. The teachers recognized that the situation called for fraction multiplication but had a hard time explaining their thinking, drawing diagrams to match algorithmic solutions, and understanding the appropriate unit or whole for the problem. Finally, Ball et al. (2001) present a classroom vignette in which a teacher struggled to explain products of decimals using dimensions and areas of rectangles.

Unit Structures

Conceptual units of various types have played a central role in research on children’s understandings of whole and rational numbers. In work closely related to the present study, Steffe’s (1988, 1994) analyses of emerging multiplication schemes have relied on the notion of composite units. A child who has formed composite units understands the number five simultaneously as one group of five and as five individual units. In such cases, a child coordinates two levels of units. Through interiorization of composite units, a child can produce the number five as five groups of a second composite unit. Steffe (1994) referred to the distribution of one composite unit across the elements of a second composite unit as a units coordination operation. A child with such an operation can assimilate a display of 20 blocks as five composite units, each of which contains a second composite unit composed of four individual units. In such cases, a child coordinates not just two, but three levels of units.
Two lines of fractions research have examined in detail the formation and reformation of nested levels of units. In the first line, Steffe and Olive (Olive 1999; Olive & Steffe, 2001; Steffe 2001, 2003, 2004) have examined how elementary students modified their knowledge of whole-number counting and multiplication to construct knowledge of fractions when engaged in tasks involving lengths and areas. The students in Steffe and Olive’s reports solved tasks by coordinating two and three levels of units and by using disembedding, iterating, and partitioning operations. Differences in students’ ability to coordinate levels of units and contexts in which they engaged their existing disembedding, iterating, and partitioning operations led to significant differences in the fraction schemes and operations they constructed. The most successful students constructed two new operations, recursive partitioning and splitting, that were fundamental to the construction of schemes for commensurate fractions (e.g., 1/3 is equivalent to 5/15), improper fractions, fraction composition (e.g., finding 1/3 of 1/4), and addition of unit fractions.

I describe recursive partitioning to illustrate the role three levels of units can play in reasoning about fractions and because it will be important to one of the case studies reported below. Steffe (2003, 2004) has defined recursive partitioning to be taking a partition of a partition in the service of a non-partitioning goal. For instance, to understand the result of taking 1/3 of 1/4, students might first partition a unit length into four pieces and then partition the first of those pieces into three further pieces. Determining the size of the resulting piece is a non-partitioning goal, and students could accomplish this in more than one way. Students might simply iterate and count to see that 12 copies of the smallest piece fit in the original unit. This solution requires decomposing an initial unit into a unit of units (one unit containing 12 twelfths). Alternatively, students might recursively partition by subdividing each of the
remaining fourths into three pieces. In contrast to the first solution, recursive
partitioning involves decomposing an initial unit into a unit of units of units structure
(one unit containing 4 fourths, each of which contains 3 twelfths). The first solution is
based on two levels of units, the second on three levels of units that could be described
as twelfths units within fourths units within a one-whole unit.

In the second line of research, members of the Rational Number Project have
focused on five rational number subconstructs—part-whole, quotient, ratio number,
operator, and measure (e.g., Behr, Harel, Post & Lesh, 1992). Although members of the
Rational Number Project have drawn from research on conceptual units (e.g., Steffe,
1988) and on the mathematics of quantity (Schwartz, 1988) to gain insight into each of
the five subconstructs, the sources for their analyses are different than Steffe’s. Steffe
reasoning, whereas members of the Rational Number Project have based theirs on their
own understandings of the domain. Of particular relevance to the present study is the
operator construct, in which a fraction is thought of as a function applied to a number,
object, or set.

Behr and his colleagues (Behr, Harel, Post & Lesh, 1993) decomposed the
operator construct into three main subconstructs—duplicator and partition-reducer,
stretcher and shrinker, and multiplier and divisor. They used the problem 3/4 of 8 to
contrast the duplicator and partition-reducer subconstruct with the stretcher and
shrinker subconstruct. In the duplicator and partition-reducer subconstruct, the
numerator specifies how many copies to make of the operand set. In the present
example, three sets of eight objects are created. The denominator is understood in terms
of partitive division, creating four sets of six objects, and one of the resulting parts gives
the answer. Behr et al. also explained that one could partition first and then duplicate.

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In either case, both the numerator and the denominator transform the number of composite units but not the size of those units. In contrast, the stretcher and shrinker construct acts on the size of composite units and is based on quotitive division. In this case the numerator stretches by exchanging each of the eight singletons by a unit of three objects. The result is rearranged into groups of four using quotitive division, and the denominator shrinks by exchanging the six groups of four for six groups of one. Behr et al. also explained that one could shrink first and then stretch (see Behr et al., 1993, for further details). The stretcher and shrinker construct contains an implicit version of the distributive property because the numerator and denominator both operate on all of the subunits.

In a subsequent study, Behr, Khoury, Harel, Post, and Lesh (1997) interviewed 30 preservice elementary teachers on the operator subconstruct. This is the only other study of which I am aware that analyzes unit structures to which teachers attend when reasoning about fraction multiplication. The teachers were given 8 bundles of 4 sticks and asked to complete three tasks. The first task was simply to show a pile that has $\frac{3}{4}$ as many sticks. For the second and third tasks, the teachers were asked to imagine that each bundle of sticks represented boards that one carpenter used on a job in one day. The second task stipulated that only $\frac{3}{4}$ of the carpenters came to work one day, and the third stipulated that all eight carpenters came to work but only worked $\frac{3}{4}$ of a day. In both cases the teachers were asked to arrange the boards for the carpenters. The researchers sought to (a) characterize the strategies the preservice teachers used as duplicator and partition reducer, stretcher and shrinker, or other and (b) determine the conceptual units that the teachers formed and transformed during their solutions. They reported that even though the intent of the third problem was to elicit stretcher and shrinker strategies, only 12 of the documented strategies were of this type. Nineteen of
the remaining strategies were consistent with the duplicator and partition reducer construct. In these cases, teachers tended not to offer a stretcher and shrinker strategy even when the interviewers attempted to elicit such strategies. Apparently these teachers found it hard to distribute fractions as operators across their conceptual units, which in turn suggested constraints on the flexibility with which they formed and transformed conceptual units when finding a fraction of a whole number.

The present study contributes to the bodies of literature summarized above by examining knowledge of fraction multiplication that two teachers used while teaching; by highlighting the flexibility with three-level unit structures, supported by the distributive property, necessary for making adaptations when engaging students’ thinking; and by connecting the teachers’ unit structures to the purposes for which they used drawn representations.

Theoretical Frame

Analyses of the two case study teachers presented below informed the theoretical frame that organizes knowledge for teaching fraction multiplication with drawn representations into two categories. The first category is unit structures and knowledge of multiplication; the second is uses for drawn representations. I will use these two categories to demonstrate that the teachers’ unit structures and knowledge of multiplication shaped how they used drawn representations and the contexts in which they did, or did not, engage their students’ thinking and adapt in response. Before proceeding, I make two quick points. First, I did not begin the analyses paying particular attention to unit structures. Rather, levels of units emerged as central to instances where the two teachers seemed more and less able to adapt in response to
their students’ thinking. Second, these categories do not span all knowledge that a teacher might use during the course of fraction multiplication lessons.

*Unit Structures and Knowledge of Multiplication*

The common parts-of-a-whole entry point into fractions emphasizes two levels of units. The extension of fractions from parts of wholes to parts of parts creates opportunities to establish three-level unit structures when relating parts of parts back to the original whole. Because the first solution to 1/3 of 1/4 discussed above illustrates that reasoning about parts of parts is not necessarily the same as reasoning with three levels of units, I will distinguish between the two throughout. To reason with three levels of units one must relate all three levels at once, not just two of the three levels at a time. Furthermore, three-level structures are necessary but not sufficient for adapting in response to the range of ways that students might assemble such structures: Attention to the distributive property is also required.

In addition to unit structures, both teachers evidenced several other ideas related to multiplication. Examples included whole-number factor-product combinations, multiplication is the same as repeated addition, a fraction times a number is the same as a fraction of the number, and products of dimensions give rectangular areas. Data from the case studies will include instances in which teachers and students attended to part of a whole when thinking about a fraction times a number and to part of a part when thinking about a fraction of a number. Thus, I will not use the words "of" and "times" interchangeably and will be careful to use the same word that the teachers, students, books, and interviewers used in any given instance.
Uses for Drawn Representations

Both teachers used lengths and rectangular areas to represent fractions, and both said that drawn representations showed students “why.” Comparing the two teachers’ enactments to my understanding of the CMP materials led to four different purposes for which a teacher might use drawn representations to solve fraction arithmetic problems. Any of the four could be interpreted as showing “why.”

The first use for drawn representations is simply to illustrate solutions also arrived at using an alternate method, such as a numeric computation. The second use is to infer a computation method by using drawn representations to determine solutions for various problems and then looking for numerical patterns (for fraction multiplication the pattern might be products of numerators are numerators of products and products of denominators are denominators of products). Each case study teacher emphasized one of these two purposes.

The third use for drawn representations is to deduce a computation procedure from the represented structure of quantities. In the case of fraction multiplication, a teacher might focus on his or her own understandings of the whole, parts of the whole, and parts of parts of the whole to determine a general computation procedure. For instance, a teacher might partition a unit square horizontally to represent one fraction and vertically to represent the second, creating a smaller array representing the part-of-the-part nested inside a larger array representing the whole. The number of pieces in the smaller array is determined by the product of numerators and that in the larger array by the product of denominators. Therefore a general computation procedure can be based on products of numerators and products of denominators.
The fourth use is to adapt how one represents structures of quantities in response to students’ thinking by (a) inferring their understandings of the whole, parts of the whole, and parts of parts of the whole and (b) attending to the variety of ways that they might begin to assemble three-level unit structures as evidenced by their explanations and drawings. This fourth case would require the ability to perceive and produce three-level unit structures in a variety of ways, to understand the role of the distributive property in comparing alternative approaches, and to understand opportunities for determining general numeric methods afforded by different approaches. Neither teacher used drawn representations to deduce or adapt.

Connected Mathematics and Fraction Multiplication

The CMP materials shaped, but did not determine, the perspectives on fraction multiplication and the purposes for drawn representations that the teachers used during their lessons. The first unit, *Bits and Pieces I*, introduces various interpretations and representations of fractions, decimals, and percents. The main interpretations include parts of a whole, measures of quantities, and indicated division. I underscore that at least the parts of a whole interpretation emphasizes two-level unit structures. The representations include fraction strips, number lines, and rectangular areas. Fraction strips are strips of paper subdivided by folding. They use lengths to represent fractions and can be identified with the interval from zero to one on the number line.

The second unit, *Bits and Pieces II*, develops fraction arithmetic by building on the interpretations and representations introduced in *Bits and Pieces I* but does not discuss three-level structures explicitly. Each case study teacher taught from a slightly different draft of the revised *Bits and Pieces II* unit that has since been published (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). At different points, the overarching mathematical
goals stated in the teacher’s editions appear consistent with each of the second, third, and fourth uses for drawn representations discussed above. The goals include developing ways to model sums, differences, products, and quotients using fraction strips, number lines, and rectangular areas; looking for and generalizing patterns in numbers (the second purpose); and developing algorithms (possibly the third or fourth purpose). The introduction to the teacher’s edition states that *Bits and Pieces II* does not teach a preferred algorithm. Rather, students are to develop solution strategies for problems in which fractions and operations on fractions are imbedded. Teachers are to help develop students’ strategies into general algorithms (the fourth purpose).

Pierce Middle School, Ms. Archer, and Ms. Reese

The present case study was conducted as part of the Coordinating Students’ and Teachers’ Algebraic Reasoning (CoSTAR) project. The project conducts coordinated research on teaching and learning in grades six through eight at Pierce Middle School. At the beginning of the study, the district had replaced traditional instructional materials focused on skill in computation with the standards-based CMP (Lappan et al., 2002) materials. Pierce Middle School is in a rural community outside of a large southern city. At the time of the study, the school had approximately 700 students who were racially and economically diverse: 35% were African American, 63% were White, and 42% qualified for free lunch.

The first teacher, Ms. Archer, \(^1\) was in her first year as a full time teacher, but she was not new to teaching. She reported several previous non-teaching careers as well as teaching in the district as a long-term substitute, mostly in high school classrooms. For one assignment, she had taught the CMP seventh-grade *Stretching and Shrinking* unit in Pierce Middle School. Ms. Archer had a bachelor’s degree in mathematics and, at the
time of the study, was enrolled in a 2-year alternative certification program for teachers who have been hired by a district and have a bachelor’s degree in an appropriate field. The program assumes that teachers have the requisite content knowledge, emphasizes other aspects of teacher preparation such as teaching methods, and requires that certification candidates have an on-site mentor. Ms. Archer reported that her own experiences as a student had emphasized drill and repetition, and her collegiate mathematics appeared rusty when she reported having to “brush up” on functions for an exam of content required for licensure.

The second teacher, Ms. Reese, was confident in her teaching. Prior to the study, she had taught algebra to seventh-, eighth-, and ninth-grade students for approximately ten years. Ms. Reese reported that her high school classes had focused on “traditional mathematics” and algorithms, had rarely used manipulatives, and had included drawn pictures only occasionally to introduce a topic or when there was confusion. Ms. Reese was in her second year of CMP implementation, but at the time of the study was teaching Bits and Pieces II for the first time. Author (Date) described Ms. Reese in greater detail.

CMP is an ambitious program that embeds mathematical ideas in problem situations, and teachers often need significant support when first using the materials in their classrooms. Prior to the present case studies, teachers at Pierce Middle School had limited professional development opportunities to support their transition to reform-oriented materials. The district hired a consultant to help each grade level select and prioritize units for the first year. Ms. Reese participated in this initial professional development, but Ms. Archer began teaching in Pierce Middle School afterwards. Both teachers participated in monthly after school professional development sessions provided by the CoSTAR project but had to rely primarily on their existing

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mathematical understandings when implementing the CMP materials. One aspect that was particularly new for both teachers was solving fraction arithmetic problems using lengths and areas as representations, which required them to grapple with unit structures and drawn instantiations of the distributive property. Both teachers seemed earnest about using the CMP materials successfully with their students and combined aspects of more traditional and more reform-oriented instruction.

Data and Methods

Several members of the CoSTAR project were involved in data collection. We used two cameras to videotape one teacher’s instruction every day during the same class period. One researcher set the first camera in the back of the classroom and recorded the entire class, adjusting the levels of ceiling and wireless microphones to hear all students during whole-class discussion and to zero in on conversations between the teacher and individual students during group work. A second researcher used the second camera to record written work, staying at the back of the classroom to record the whiteboard during whole-class discussion and shadowing the teacher to record work that she discussed with students at their desks. The choice to dedicate the second camera to written work reflects the project’s emphasis on the role inscriptions play in problem solving. Later the same day, we combined the video and audio from the two cameras using an audiovisual mixer to create a restored view (Hall, 2000) and used the result to identify instances where the teacher and her students discussed drawn representations. From these instances, we selected examples for use in interviews where the teacher and students apparently struggled to understand one another.

I interviewed pairs of students from the same classes in which we gathered lesson videos (four pairs in Ms. Archer’s classroom and three in Ms. Reese’s). Pairs were
selected with the aide of each teacher to represent as best as possible a cross section of achievement, a balance between boys and girls, and the racial composition of the class. Each pair of students was interviewed once a week for approximately 50 minutes starting the second week of the study. The interviewer used problem-solving tasks and lesson video excerpts as prompts in semistructured interviews (Bernard, 1994, Chapter 10). The tasks resembled those at the center lesson excerpts. Students worked the tasks using pencil and paper and watched the lesson excerpts on a laptop computer. One camera recorded students, and one recorded written work and viewed lesson excerpts. Project members first created a restored view that captured much of what the students said, wrote, and watched and then located excerpts for use in teacher interviews that revealed aspects of students’ mathematical thinking and difficulties not evidenced during lessons.

Finally, project members not listed as authors conducted weekly, hour-long interviews with the teachers to gain access to their interpretations of the CMP activities, their understandings of their students, and the pedagogical decisions they made. The interviewers used a combination of lesson and student interview video excerpts selected in consultation with me. These consultations insured that many of the same lesson excerpts were used in student and teacher interviews. These interviews occurred after a completed cycle of student interviews and usually within a week of the original lessons. When playing the lesson excerpts, the interviewers asked the teachers to discuss the mathematical content and to reconstruct their thinking. When playing the student interview excerpts, interviewers asked the teachers to respond to students’ problem solving and interpretations of lesson excerpts. One camera recorded the teacher, and one recorded viewed lesson clips. We created a restored view that captured much of what the teacher and interviewer said and watched.

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We collected data on Ms. Reese first and only during her enactment of *Bits and Pieces II*, which began mid-March and continued to mid-May in 2003. Roughly, the first month concentrated on fraction addition and subtraction and the second month on fraction multiplication. We followed up on the 11 multiplication lessons in five student interviews and two teacher interviews. In analyzing the data, questions emerged about how partitioning had been discussed in the preceding *Bits and Pieces I* unit. Thus, the next year we collected two rounds of data in Ms. Archer’s classroom. The first round occurred in late January through February and consisted of 13 lessons from *Bits and Pieces I* in which Ms. Archer and her students used linear representations to determine products of fractions and whole numbers. We followed up on those lessons in eight student interviews and four teacher interviews. The second round occurred in late April through May and consisted of 17 lessons from *Bits and Pieces II*. Only the last two focused entirely on products of fractions using area representations, but portions of earlier lessons touched on products of fractions using number lines. We followed up on fraction multiplication instruction in 11 student interviews and 3 teacher interviews.\(^2\) Thus, the analyses below are based on significant corpora of data.

Once data collection was complete, I analyzed the lesson and interview videos using methods similar to those described by Cobb and Whitenack (1996) and Schoenfeld, Smith, and Arcavi (1993) for analyzing longitudinal sets of video recordings. (We transcribed only the interviews.) In particular, I analyzed verbal references, hand gestures, and added inscriptions (e.g., line segments and shading) for evidence of understandings that the teachers and their students engaged. I compared the knowledge teachers evidenced during lessons and interviews and was more confident in my attributions when teachers evidenced similar understandings in both

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the lesson and interview contexts. This was particularly important given the delay of up to a week between initial lesson taping and the following teacher interview.

I also sought to understand apparent discrepancies between knowledge evidenced in lessons and interviews. I considered the possibility that teachers evidenced different knowledge in the two contexts and the possibility that my initial attributions needed refinement. I generated multiple hypotheses and sought to discriminate among them by going back and forth between the interview and lesson videos until my descriptions of teachers’ knowledge became stable and appeared to account for understandings evidenced in both contexts. I also asked project members for their interpretations of video excerpts that were hard to interpret.

Results on Ms. Archer

The two main results on Ms. Archer will feed into those on Ms. Reese:

(1) Ms. Archer was sufficiently flexible with two levels of units to adapt in response to a student’s alternative strategy but did not evidence three-level unit structures in situations where one might expect her to produce them if she could.

(2) Ms. Archer used two-level unit structures and other understandings that she associated with multiplication to reason about parts of parts and used drawn representations to illustrate already computed solutions to fraction multiplication problems, the first use of representations discussed above.

Products of Fractions and Whole Numbers

Ms. Archer introduced products of fractions and whole numbers during *Bits and Pieces I*, and her enactment provided data for result (1).
Opportunities to Attend to Three-Level Unit Structures

*Bits and Pieces I* opens with a set of problems about fund raisers and introduces thermometers as representations for recording progress toward fund-raising goals. Students are asked first to make fraction strips by folding 8 1/2 inch strips of paper into halves, thirds, fourths, up to twelfths. For prime numbers, there are few alternatives to estimating directly the size of one piece and folding to check, a strategy that emphasizes two levels of units. Composite numbers afford opportunities to attend to three-level unit structures. For instance, Ms. Archer asked a student who had made thirds but was struggling to make ninths, “So if you fold it in thirds, and I know that three times three is nine, so what would I have to divide the thirds into?” The second day, Damien demonstrated to the class how he used factors to make a twelfths strip. He explained that half of 12 is 6 and half of 6 is 3 as he folded his strip in half, in half again to make fourths, and then in thirds to make twelfths. If Ms. Archer attended to three levels of units at such moments, her concluding remarks for the activity emphasized that the strips were going to be used as measuring devices and that students should remember to find equal sized parts of a whole (two levels of units). Ms. Archer emphasized the same two points when discussing these lessons in interviews, and her enactment appeared to align with the task describe in the materials.

Using Two Levels of Units to Adapt in Response to Damien’s Reasoning

The fund raising thermometers are also 8 1/2 inches, allowing students to use their folded strips to measure thermometers directly. The thermometers show progress toward raising a specified number of dollars. Ms. Archer’s students began determining fractions of whole numbers to find how many dollars had been raised at different points in several fundraisers. Over the course of two and a half days, students worked
on 11 such problems in groups and presented their solutions to the whole class. The vast majority of solutions relied on determining the dollar value of one equal part of the whole number (interpreting fractions as parts of a whole) and then using multiplication or repeated addition to determine the final answer. As one example, Angie determined 3/5 of $300 by finding that 1/5 was $60 and then calculating 3 x $60 = $180. Although one could think of three-level unit structures in these situations—for instance by thinking of 300 divided into five groups, each of which consisted of 60 ones—comments made by Ms. Archer and her students made reference to just two levels of units.

Ms. Archer’s reasoning with two levels of units appeared sufficiently flexible to engage an alternative strategy that Damien proposed for determining 11/12 of $240. He found that one of 12 equal parts would be $20 and then subtracted that from $240. Ms. Archer first observed this while working with Damien at his desk and commented, “Oh. That is a good idea. Hold that thought so you all might share that with the class.” During subsequent whole-class discussion, Ms. Archer called on Damien to present his alternative solution. As Damien presented, however, a debate emerged because some students had got $216 as the final answer using tenths strips. Ms. Archer ran out of time and did not resume the discussion the next day.

These data are significant because they suggest that Ms. Archer adapted in response to students’ alternative solutions in some contexts. Further data presented in a subsequent section will demonstrate that Ms. Archer had trouble responding to and even rejected students’ alternative strategies, strategies that from my perspective were reasonable but required flexible three-level structures to understand.
Struggling to Respond to Students

The final episode from *Bits and Pieces I* that I discuss occurred two weeks after those presented above and suggested that if Ms. Archer could produce three-level structures, she did not do so readily during instruction. (I use the word *produce* to emphasize assembling three-level structures while reasoning in a given context as opposed to perceiving all three levels at once from the beginning.) Students were comparing 2/3 to 3/4 to determine whether 2/3 was closer to 1/2 or to 1, and some said that 2/3 and 3/4 were the same distance from 1 because both were one piece away from making a whole. In an apparent attempt to direct students’ attention toward the size of each fraction, Ms. Archer drew two squares and emphasized that both were the same size. She then subdivided the squares and shaded “three of four equal parts” and “two of the three.” When she asked which fraction was bigger, some students continued to say that the fractions were the same while others said that 3/4 was larger. Ms. Archer switched to numeric computations, determined equivalent fractions, and asked if 8/12 or 9/12 was larger. Figure 1 shows the white board as it appeared at the end of her explanation. Note that in writing 3 x 3/4, Ms. Archer asked how many times 4 went into 12 and then multiplied 3 times 3 to get 9. Her discussion of 4 x 2/3 was similar.

Figure 1 About Here

That Ms. Archer did not partition further to generate 8/12 and 9/12 provided preliminary evidence that she struggled to use drawings when responding to students’ thinking because she did not produce three-level unit structures. Further evidence came from a subsequent interview during which Ms. Archer reviewed an interview excerpt
with Emily and Angie.³ The students reviewed the lesson excerpt summarized above, and the interviewer asked if, starting with 3/4, Ms. Archer could have made thirds. Angie explained that Ms. Archer “could not make thirds exactly” but could make twelfths. Ms. Archer commented, “She’s right. I could have, but I wasn’t thinking about that at the time.” Furthermore, when discussing how to divide fourths into twelfths, Ms. Archer and the students created either eighths or sixteenths. Ms. Archer’s comments during the lesson and her interview suggested that she attended to the whole divided into fourths (two levels of units) and the same whole divided into thirds (two levels of units) but not to a third level of units, such as twelfths, that could simultaneously repartition thirds and fourths.

*Products of Fractions Using Number Lines*

Ms. Archer used lengths to introduce products of proper fractions. Although the tasks afforded opportunities to produce three levels of units, her speech and gestures did not index such structures explicitly. She apparently based her reasoning about parts of parts instead on successive applications of two-level unit structures. Moreover, Ms. Archer’s emphasis on two-level structures apparently impeded her ability once more to respond to students. Finally, Ms. Archer used number line solutions to illustrate already computed answers, the first use of representations discussed above. These data contributed to results (1) and (2).

*Using Two Levels of Units to Reason About Parts of Parts*

*Bits and Pieces II* intends for teachers and students to work with rectangular areas before number lines when studying products of proper fractions, but Ms. Archer began with an example that used number lines. The problem was to find one fifth times two
thirds and, as Ms. Archer explained in a subsequent interview, the completed number line representation came from the teacher’s edition (see Figure 2). The problem was not situated in any particular context.

Ms. Archer began by reminding her students that a fraction of a number means a fraction times the number. She pointed out that the interval from 0 to 1 was divided into three parts. Next, she covered all but the first third, and students commented that it was divided into five parts. Ms. Archer then reminded students that the problem asked about two thirds:

So we have one fifth here (pointing to the shaded fifth of the first third)
and we have one fifth here (pointing to the shaded fifth of the second third.) But this one fifth (pointing to the shaded fifth of the first third)
really is the same as what if we look across the whole number line?

Emily explained that the answer was one fifteenth because “in the whole number line there’s five equal parts in each section and you add them all together.” Ms. Archer agreed, pointed again to the shaded fifth of the first third, and said this would “really would be one of fifteen equal pieces.” Emily’s comment suggested attention to three levels of units, but Ms. Archer’s focused explicitly on only two levels at a time as she juxtaposed interpreting each shaded portion first as one fifth and then as one fifteenth.

Jeff may not have understood the difference between one fifth of a whole and one fifth of a third because he stated, “If you wanted one fifth, it would change to, it would be like three fifteenths.” Jeff’s tone implied confusion about discussing one shaded piece both as one fifth and as one fifteenth. Ms. Archer did not address Jeff’s
confusion directly. Instead, she read a sentence on her overhead that also appears in the teacher’s edition: “Each one fifth of a third is one fifteenth so that two parts marked would be two fifteenths.” Jeff still seemed confused when he stated once more that one fifth would be three fifteenths. Ms. Archer may have thought Jeff was thinking of one fifth of three thirds because her response emphasized that this problem required looking at only the first two thirds. She then confirmed the answer multiplying numerators together and denominators together. Ms. Archer concluded her demonstration by stating that “the number line shows why” and moved on to the next task in the students’ books that introduced fraction addition and subtraction.

Why Ms. Archer Inserted the Number Line Example

We used this lesson segment in the next interview with Ms. Archer. She said that she had not known how to use number lines to multiply fractions, thought the example was interesting, and explained her students’ existing knowledge that she had considered when deciding to insert the demonstration:

I like to give them a taste of what we’re going to be doing and to show them how there are other ways that things can be done. And since we had been using the number line, and I want to show them that you could also use the number line to, to multiply fractions. I mean they knew where one third was, they knew about one third. They knew how to divide things into parts and, and I, they knew that one part would be one fifth of that.

Ms. Archer’s comment about giving students a “taste” helps explain why she inserted the number line example and why she did not continue the lesson by having students work further similar problems. Her comments also made clear that, from her point of view, students should be able to understand the example because they knew about the
location of fractions on number lines and about dividing things into parts, an apparent reference to the parts-of-a-whole interpretation and two levels of units.  

Difficulty Responding to Jeff

Further interview data suggested that if Ms. Archer could produce three levels of units, her reasoning was not sufficiently flexible to engage Jeff’s thinking. First, she still had trouble responding to his comments that 1/5 was the same as 3/15. After reviewing the lesson excerpt, she commented:

Maybe I should’ve said sixth fifteenths to let him know we talking about three fifteenths plus three fifteenths. Yeah. And that would’ve helped him to understand that now I’m listening to it….He’s right. That’s three fifteenths, but we’re concerned about two thirds. So therefore we have [to do] three fifteenths plus three fifteenths, which is six fifteenths. And when you reduce it, it still gonna give you two fifteenths.

One possible explanation for Ms. Archer’s comments is that she did not focus on the distinction between one fifth of a whole and one fifth of a third and, in a momentary lapse, assumed that 6/15 could be reduced to 2/15. A second possible explanation is that she meant three fifteenths of one third plus three fifteenths of one third makes six fifteenths of one third. If that were the case, however, her final comment switched implicitly from fifteenths of one third to fifteenths of the whole. Neither explanation would be consistent with an explicit three-level structure in which five fifteenths make up each of three thirds of a whole.

Second, Ms. Archer viewed an interview excerpt in which Jeff and Emily reviewed her solution to 1/5 times 2/3 on the number line (see Figure 2). When asked why Ms. Archer shaded the way she did, Jeff proposed arranging the two shaded pieces
side by side in the first third. He explained, “You’re really thinking of two thirds, I mean, two fifths of a third.” Understanding that Jeff’s proposed diagram for $\frac{2}{5}$ of $\frac{1}{3}$ could also be interpreted as $\frac{1}{5}$ of $\frac{2}{3}$ would require sufficient flexibility with three-level unit structures to see a length of $\frac{2}{3}$ divided into ten equal pieces, the first two of which were shaded. Recognizing that Jeff’s proposed diagram and Ms. Archer’s could both represent $\frac{1}{5}$ of $\frac{2}{3}$ would require additional flexibility supported by a visual understanding of the distributive property formally symbolized here as:

$$\frac{1}{5} \times \frac{2}{3} = \frac{1}{5} \times \frac{1}{3} + \frac{1}{5} \times \frac{1}{3}.$$ 

At an earlier moment during the same interview, Ms. Archer’s reported not thinking about the distributive property during the original lesson. Now she rejected Jeff’s proposal:

This is one third by itself (pointing to the first third) and this is another third by itself (pointing to the second third). And the concept is, was then one fifth times two of three parts. So, you gotta have the two of three parts. You can’t just put both of them down in there, then you won’t get what you’re supposed to have.

Ms. Archer’s insistence that each third be separate surfaced again when the interviewer asked her to use the number line to solve $\frac{3}{5}$ times $\frac{2}{3}$. Ms. Archer first calculated $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$ and $\frac{3}{15} + \frac{3}{15} = \frac{6}{15}$, correctly, and then shaded three fifths of the first third and three fifths of the second third. At one point, she emphasized “you gotta do three sections of two separate things.” Thus, in both examples, Ms. Archer appeared to use the parts-of-a-whole interpretation twice in succession (once to establish thirds and once to establish parts of one third) and then applied repeated addition to the result.

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Ms. Archer’s Unit Structures and Uses for Drawn Representations

Ms. Archer’s comments made explicit the purpose she saw in using the number line. When asked about the value of showing students both the algorithm and the diagram, she said, “I think the diagram showed to me, it showed them why one fifth times two third is two fifteenths, you know, and I just wanted to show them why.” In the next exchange she went on to explain:

[The number line] is just another way of doing things, you know, which I’ve often told them there’s more than one way to do math problems. You know, if it, that might help some of the students to understand things better. They may wanna do it that way.

These comments were consistent with using drawn representations to illustrate solutions also arrived at through alternative numeric methods, the first use for representations discussed above. That Ms. Archer computed answers before using lengths to represent parts of parts, combined with her difficulties producing three levels of units, suggested that she would have had trouble with the remaining uses for drawn representations discussed above. Data presented in the next section will provide further evidence that her difficulties with three-level unit structures precluded further uses of representations.

Products of Fractions Using Area Representations

Ms. Archer introduced area representations when she returned to fraction multiplication ten days later. Initially, she blurred the distinction between part of a part and part of a whole when interpreting problems as finding one fraction times another. These data suggested that Ms. Archer continued to use just two levels of units to reason about fraction multiplication, result (2). As Ms. Archer introduced tasks that
emphasized finding a fraction of a fraction, however, she made the distinction between part of a part and part of a whole more explicit and became somewhat more adaptive in response to her students’ representations of parts of parts. Thus, she appeared to be learning with her students.

*Blurring Part of a Part and Part of a Whole*

The CMP materials introduce rectangular areas as representations of fraction multiplication with the example 1/2 of 2/3. Ms. Archer began her introduction with just a few minutes left in the class period. She presented the problem as 1/2 times 2/3 and had students tell her to multiply the numerators together and the denominators together, resulting in 2/6. On one transparency she partitioned a unit square vertically into thirds and on a separate transparency she partitioned a second unit square into halves. She then laid one unit square over the other as shown in Figure 3a, reproducing the drawing in the teacher’s edition. She asked students how many parts were in the square and told students to “look at the area that has the most lines in it.”

Figure 3 About Here

Ms. Archer returned to the same example the next day, reproduced the representation of two thirds from the previous lesson, and asked the class how to show two thirds times one half. Several of the students’ attempts suggested that they viewed both two thirds and one half as being of one whole. The first student drew a second square, shaded one half, and got stuck (Figure 3b). Theo then repartitioned the square Ms. Archer had drawn into sixths and shaded one more piece, recreating Figure 3a. He then started to express two thirds and one half in terms of sixths. Ms. Archer cautioned
Theo not to confuse this problem with addition and subtraction. Angie stated that the answer could not be two sixths because five of six pieces were shaded. Ms. Archer said that the picture was correct and that she wanted to know “why” it showed that "two thirds times one half [was] two sixths.” Finally, Ms. Archer accepted Emily’s explanation that the two thirds and one half “mixed” in two of the six squares. All of these explanations relied on the parts-of-a-whole interpretation of fractions and none attended explicitly to parts of parts.

To this point, Ms. Archer and her students had discussed one fraction times another, but a few moments later she had students work on problems in the book that used squares to represent brownie pans and that asked students to "show part of the part in the brownie pan." Ms. Archer and her students worked on these problems for two days. At one point, Ms. Archer represented 1/2 of 2/3 of a brownie pan (see Figure 4) in two ways and, in so doing, evidenced a more flexible perspective on part of a part than she had during her number line lesson, where she had insisted on showing 1/5 of each third. The first solution was based on cross partitioning but showed more clearly that the 1/2 was just of the 2/3, not the whole; the second solution double shaded one third. Ms. Archer used hand gestures to show that the two sixths from the first method could be rearranged to coincide with one of the two thirds in the second method. For the balance of the lesson, students used rectangular areas to determine 3/4 of 1/2. Classroom data collection ended here because the school year was about to end, and Ms. Archer moved on to fraction division.
Adaptive Representation of Parts of Parts

In conjunction with another member of the CoSTAR project, I planned a final interview with Ms. Archer to examine the extent to which she accepted as correct the variety of ways students represented parts of parts and what she meant when she told students that drawings show “why.” The prompts included reproductions of student class work on 3/4 of 1/2 shown in Figure 5 and excerpts from the final round of student interviews. Ms. Archer’s responses suggested strongly that when examining students’ work she focused primarily on whether the picture showed part of a part and did not attend as explicitly to opportunities for deducing numeric algorithms. I emphasize that the understandings Ms. Archer evidenced during her final interview may have been shaped by her experiences helping students with the part-of-a-part language. Thus, I did not use these to infer understandings she may have used when first introducing fraction multiplication with brownie pans.

Figure 5 About Here

The interviewer asked Ms. Archer to review the set of examples in Figure 5 and to indicate which ones students could use “to understand why the answer to three-fourths of one half is three eightths.” Ms. Archer indicated the first or second. She stated that the first method (Figure 5a) was what she had taught, but that the second (Figure 5b) was also possible. These data suggested that Ms. Archer now accepted as correct a range of representations of parts of parts, but did not focus on the cross partitioning aspect. Initially, however, she rejected Theo’s solution (Figure 5c):
That’s not right. Wait a minute. Four, eight, twelve, sixteen. That’s not right. Well, hold up. Three, sixteen. Two, four, six, eight, sixteen. Four, eight, wait, four, eight, twelve, sixteen. Two, four, six. Six, sixteen; three, eight. Okay. All right. I see what he did.

After a few exchanges about the first two examples, Ms. Archer considered Theo’s work once more, “One, two, three, four, five, six of sixteen and that’s three eighths. Yeah.” These data are inconsistent with the ability to flexibly produce three levels units when responding to students’ work.

For the balance of the interview, the interviewer replayed excerpts and showed written work from the final round of student interviews. A primary goal for the interviews was to find out how students might extend Ms. Archer’s demonstrated solution to other examples. Ms. Archer’s response to data from the student interviews provided further evidence that she attended more flexibly to parts of parts but did not attend to opportunities for deducing numeric procedures from partitioning activities, the third use of representations discussed above.

During one interview, Damien and Theo solved $\frac{2}{5}$ of $\frac{3}{4}$ using brownie pans. Figure 6a shows how Damien first divided the square into four parts. When he said that he was not sure how to “divide into five,” Theo suggested drawing vertical lines but subdivided the fourths into only four pieces. Damien built on Theo’s suggestion, subdivided each fourth into five parts (see Figure 6b), and got stuck once more. Finally, Theo suggested taking two parts from each of the three shaded fourths (see Figure 6c) and explained that the answer would be $\frac{6}{20}$ because there are “four pieces divided up into five.” The students’ drawings and explanations evidenced their production of three levels of units and distributive reasoning.

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Ms. Archer watched Damien subdivide the fourths into five parts (Figure 6b) and at this point commented, “He’s just gotta shade in two of each of those fifths, which would give him, like, six twentieths, I think.” Although Ms. Archer also evidenced distributive reasoning when anticipating Theo’s suggestion (Figure 6c), she still had trouble describing parts of parts correctly. A few moments later she stated, “That is six twentieths of three fourths. Yeah. That’s right.” At another point she affirmed Damien and Theo’s approach: “We’re talking about parts of parts regardless of where the parts are. Right? As long as it’s a part of that part.” She concluded, “It shows me that they do have the concept in a round-a-bout way.”

For the next example, the interviewer simply showed Ms. Archer a reproduction of Emily’s work on the same problem. Emily partitioned the brownie pan vertically into four parts and shaded three. She then repartitioned the three shaded fourths vertically into five parts and double shaded two of the resulting parts. Although Emily’s construction showed part of a part correctly, her resulting drawing made it impossible to relate the double shaded region to the whole. When asked if Emily’s approach was useful for determining “why it works,” Ms. Archer thought for 15 seconds before commenting:

Yeah. ‘Cause all she’s done is shown, what is that? One, two, I think, it’s gonna be, what, three fifths? No, no, no, no. What is it? [Ms. Archer reduced 6/20 to 3/10 in the margin] …. Let’s see. How could she get three tenths out of this? One, two. Well, yeah, I guess it could work because if you count, it should be ten across when you get through counting and then she would’ve done, this probably would be three if she put another
thing here. You know what I’m saying? And, that would be the same concept.

Why Ms. Archer reduced 6/20 when thinking about Emily’s work remained unclear. Her comments focused on whether Emily might have arrived at the correct answer, not on whether from her partitioning Emily might deduce a computation procedure.

To summarize, Ms. Archer’s response to Damien’s method for determining 11/12 of $240 made clear that she valued students’ alternative solutions and could adapt in a case where just two levels of units were required. Further data on her enactments of *Bits and Pieces I* and *Bits and Pieces II* provided strong evidence that she could not produce three levels of units when using lengths and areas as representations of fractional quantities. Ms. Archer’s focus on two levels of units, combined with her connection between a fraction times a number and of the number, were sufficient to use drawn representations to illustrate solutions. Her difficulties with three levels of units, however, precluded further uses of drawn representations discussed above because they require reasoning about length and area quantities to determine products of fractions. Instances in which Ms. Archer compared different ways to represent 1/2 of 1/3 (see Figure 4) and anticipated Theo’s approach to drawing 2/5 of 3/4 (see Figure 6b) suggested that she was learning along side her students how to represent parts of parts with lengths and rectangular areas.

Results on Ms. Reese

The two main results on Ms. Reese also center on her unit structures and knowledge of multiplication and on purposes for which she used drawn representations:

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In contrast to Ms. Archer, Ms. Reese produced three levels of units, but may have been constrained in her response to some students’ alternative strategies because she was not always facile with drawn instantiations of the distributive property.

In contrast to Ms. Archer, Ms. Reese reasoned about both lengths and areas to determine fractions of fractions, led students to the connection between a fraction of a number and times the number, and used drawn representations to infer a computation procedure over six lessons.

Products of Fractions Using Area Representations

Ms. Reese maintained more explicit verbal distinctions between part of a part and part of a whole than did Ms. Archer. Despite this, she may have had difficulty adapting in response to strategies in which students used cross partitioning (partitioning both horizontally and vertically) to represent one fraction.

Attending to Parts of Parts

Ms. Reese introduced fraction multiplication with the example 1/2 of 2/3 and with a drawing similar to that used by Ms. Archer. She introduced squares as representations of brownie pans, asked students how to draw two thirds of the brownie pan, and followed a suggestion to partition the square horizontally (see Figure 7a). She then asked how to represent 1/2 of 2/3, and one student said that half would be one third because “to get two thirds you have to have one third plus one third.” Ms. Reese accepted the solution and said half of two thirds would be one of the two shaded pieces.

She told students that the goal was figure out how to solve this problem using an algorithm and proceeded to divide the square vertically (see Figure 7b). She explained that she cut the whole pan “because I want you to realize that if there were brownies in
the pan, there would be two pieces right there.” Ms. Reese pointed to the two unshaded pieces as she spoke. She then asked students to “shade half” and after a moment shaded half of her pan (see Figure 7c). She said that the double shaded part represented half of two thirds and emphasized that there were no brownies in bottom left hand corner. She then asked what the two pieces stood for as a fraction, and a student said two sixths. Ms. Reese wrote “1/2 of 2/3 is 2/6” across the bottom of her representation, asked if the two solutions were the same, and called on students until one said that 2/6 could be simplified to 1/3. Ms. Reese finished the lesson with a second brownie pan problem that demonstrated 3/4 of 1/2 is 3/8.

Figure 7 About Here

Struggling to Respond to Students

Although Ms. Reese’s discussion of 1/2 of 2/3 demonstrated flexible representation of parts of parts, she had trouble responding to students’ reasoning the very next day. The central issue was how to create the first partition of the brownie pan. Ms. Reese began by reviewing the previous lesson and had students work on the following problems: 1/3 of 5/6, 2/3 of 3/4, 1/4 of 1/3, and 1/4 of 2/5. At one point, she worked with a student whose representation for 1/3 of 5/6 was similar to that shown in Figure 8b. The student explained that the answer was 2/6 because the whole was broken into three pairs of smaller rectangles. If, in producing her drawing, the student took the whole pan as a unit that was divided into thirds each of which was further divided into sixths, she would have produced three levels of units when arriving at her incorrect answer. Ms. Reese responded, “Something is going on here that
is not right. I have got to think about this a minute.” She questioned whether the student had shaded $\frac{5}{6}$ and counted the shaded pieces to check. That Ms. Reese counted the five shaded pieces suggested that she was genuinely stuck. The student continued to explain when Ms. Reese interrupted, “Oh. I gotcha. OK. So you just didn’t divide it up again. You just put like this is one third (pointed to two pieces), that’s one third (pointed to another two pieces), and that’s one third (pointed to the last two pieces). OK. That’s fine.” Apparently, Ms. Reese produced three levels of units, even if she did not address the problem with the student’s work.

Ms. Reese observed that another student had a similar incorrect solution and stopped the class so that she could compare two strategies for determining $\frac{1}{3}$ of $\frac{5}{6}$. The following discussion suggested that she may have figured out, at least in part, what was wrong with the two students’ solution. Ms. Reese began by telling the class that it was alright to have different answers so long as they were equivalent fractions. She demonstrated her solution first (Figure 8a) and pointed out that she partitioned the brownie pan into sixths by drawing lines in just one direction. She shaded five pieces, said she wanted to take a third of just those pieces, and evidenced three levels of units when she continued:

I am thinking out loud. I am acting like a kid. I was thinking at first when I saw this, well, maybe two of these could make up one of the thirds where that would be like one third (pointed to the top two pieces), that would be two thirds (pointed to the next two pieces) but Oh Oh (pointed to the fifth shaded piece) there’s only one left. So that’s not going to work.

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Ms. Reese partitioned her brownie pan vertically into thirds, shaded the first third of the whole, and reminded students that the bottom piece was not included. She counted five double shaded pieces and, with some prompting about the size of the pieces, students said the answer was 5/18.

Ms. Reese then recounted the alternative solution by partitioning a second brownie pan as shown in Figure 8b and told the class that she remembered reading in the teacher’s guide that “it wont always work if you divide like this,” meaning cross partitioning for the first fraction. Ms. Reese did not explain that the first method (Figure 8a) relied on taking a third of each piece (the distributive property). Rather, she had students examine the class solutions thus far: 1/2 of 2/3 is 2/6, 3/4 of 1/2 is 3/8, and 1/3 of 5/6 is 5/18. After students offered a tentative pattern based on multiplying numerators together and denominators together, Ms. Reese asked if the second proposed solution to 1/3 of 5/6 (Figure 8b) fit the pattern. Some students commented that 2/6 was equivalent to 6/18. Ms. Reese pointed out that the two methods gave truly different answers but did not state which one was correct. (Ms. Reese reported working problems in her lessons ahead of time and probably knew the correct answer.)

As Ms. Reese had students work with their partners once more on the remaining problems, one discussion about representing 2/3 of 3/4 suggested constraints on her attention to parts of parts of brownie pans. Jack and Kate had similar work (Figure 9a and 9b). Kate explained that she had divided her brownie pan into fourths (Figure 9b). To us it appeared that Kate had continued by shading two of the three fourths, correctly, but Ms. Reese commented that Kate was going to run into the same problem encountered in Figure 8b and had her start over. Jack partitioned a new brownie pan into fourths all in one direction. As he did so, Ms. Reese commented, “OK. We need to take two thirds of that. You might be able to answer it right there like that. I’m not
sure.” This time when Jack shaded two of the three fourths (Figure 9c), Ms. Reese said, “That will work.” In this case, Ms. Reese did not produce three levels of units with sufficient flexibility to adapt in response to a student’s alternative strategy.

The next day, Ms. Reese and the students used area models to solve several more problems by partitioning horizontally for one fraction and vertically for the second. The day after that, she introduced linear representations for fraction multiplication. I discuss those data next because the final interview with Ms. Reese examined the area examples just discussed and a linear example discussed below.

Ms. Archer was running out of days in the school year and so skipped the main Bits and Pieces II lessons that used thermometers and number lines to develop fraction multiplication. In contrast, Ms. Reese used thermometers and number lines to find several examples of a fraction of a fraction and to infer a computation method. These data contributed to result (4). In so doing, Ms. Reese used recursive partitioning, and thus continued to produce three levels of units. Moreover, she attended to drawn instantiations of the distributive property, even though she did not discuss the property explicitly with her students. These data both contributed to result (3) and raised questions about Ms. Reese’s access to drawn instantiations of the distributive property that we pursued in her final interview discussed below.
Recursive Partitioning and Using Drawn Representations to Infer a Computation Method

As she had done with rectangular areas, Ms. Reese had students use lengths to determine proper fractions of proper fractions and had students consider equivalent fractions in order to resolve apparent counterexamples to the emerging pattern based on products of numerators and products of denominators (e.g., $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$). In subsequent examples, she had students extend the pattern further by converting whole and mixed numbers to improper fractions.

Ms. Reese used lengths in three examples, $\frac{1}{4} \times \frac{2}{3}$, $\frac{1}{3} \times \frac{1}{2}$, and $\frac{2}{5} \times \frac{1}{2}$.

The first problem was couched in a context where students had raised two thirds of their $720$ fundraising goal in four days. Ms. Reese sketched a fundraising thermometer that showed two thirds of $720$ (Figure 10a) and asked students for the fraction that had been raised each day. One student suggested dividing the shaded two thirds into four parts, and Ms. Reese did so (Figure 10b). Ms. Reese pointed out the thermometer was not yet divided into equal sized pieces. Another student said the answer was one sixth because the unshaded part should be divided into two pieces. Ms. Reese partitioned the unshaded part (Figure 10c) and wrote “$\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$” underneath. She then solved the problem a second time using rectangular areas to show that $\frac{1}{4} \times \frac{2}{3}$ was also $\frac{2}{12}$. For the second and third problems, Ms. Reese led the class through a similar sequence of steps that evidenced recursive partitioning. For the second problem she partitioned one half into three parts and had a student explain that the second half should also be divided into three parts to see that $\frac{1}{3} \times \frac{1}{2}$ is $\frac{1}{6}$, and for the third problem she partitioned one half into five parts, shaded two, and had a student explain that the second half should also be divided into five parts to see that $\frac{2}{5} \times \frac{1}{2}$ is $\frac{2}{10}$.
Attention to Drawn Instantiations of the Distributive Property

During the next lesson Ms. Reese’s comments evidenced attention to the distributive property in the context of a fraction times a mixed number. The problem was to determine \( \frac{1}{3} \) of \( 2 \frac{1}{2} \) pounds of cheese. Ms. Reese drew three thermometers, shaded two and half, and divided each piece into thirds (Figure 11a). A student suggested dividing everything into sixths, which Ms. Reese did. Ms. Reese then commented that she needed “to get a third of each thing.” She shaded a third of the half and, as a rhetorical move to invite students’ contributions, said that she was not sure how to get a third of the wholes. A student suggested shading two pieces, and Ms. Reese shaded two pieces of each whole (Figure 11b). Ms. Reese counted five double shaded pieces, and students said the pieces were sixths. Although Ms. Reese attended to a drawn instantiation of the distributive property in this context, subsequent data suggested that she may still have not attended to the distributive property in the context of area representations. When reviewing the use of lengths and rectangular areas for determine products of fractions two weeks later, she stopped a student who began working on \( \frac{1}{3} \) of \( \frac{5}{6} \) by cross partitioning a brownie pan into sixths. Ms. Reese said, “Remember we had a long discussion in class about why you cannot cut them this way and this way because your answer might be close to the right answer but it is not going to be the right answer.”
Ms. Reese Explains Her Lessons and the Distributive Property During Interviews

I generated two hypotheses for why Ms. Reese directed students away from cross partitioning brownie pans for just one fraction even in cases where, from my point of view, students appeared to be progressing towards correct answers. The first was that the incorrect $1/3$ of $5/6$ is $2/6$ solution, combined with comments in the teacher’s edition, were sufficient for Ms. Reese to understand that partitioning horizontally for one fraction and vertically for the second was more reliable, even if she was still figuring out why. The second was that Ms. Reese understood that the distributive property could be used to make either approach to partitioning work but did not want to have that discussion with her students. That Ms. Reese demonstrated a drawn instantiation of the distributive property in the subsequent cheese problem context, combined with comments that she wanted to direct students toward multiplying numerators and multiplying denominators with a minimum of confusion raised the possibility that she had, but chose not to use, relevant knowledge.

Data from the final interview did not allow clear discrimination between these two hypotheses but did provide evidence for three claims. First, Ms. Reese was unsure how to draw representations of the distributive property in some contexts. The interviewer asked about two examples from lessons in which students multiplied mixed numbers incorrectly by only multiplying the whole numbers together and the fractions together (e.g., $3 \frac{1}{3} \times 4 \frac{2}{3} = 12 \frac{2}{9}$). Ms. Reese recognized the mistake as one that several students in each of her classes had made and at one point asked, “So how could we represent that with pictures to come up with a …?” These data demonstrated constraints on Ms. Reese’s understandings of drawn representations for the distributive property, at least in cases of the $(a + b)(c + d)$ pattern that require more than one
application of the property. At other points in her interview, Ms. Reese stated that students should be able to understand cases that fit the $a(b + d)$ pattern. For instance, when asked about the cheese problem discussed above, she commented:

I would hope that [students] would see that, first of all, if you’ve got two wholes and a half more, that they, a third of that amount total, so you could make it easy and do a third of the half and figure that out, and then you could do a third of the two wholes and figure that out and combine those two.

Second, by the time of the interview, Ms. Reese could describe the role of the distributive property in $1/3$ of $5/6$ example but expressed reservations about discussing this with students. When the interviewer showed the clip in which a student used brownie pans to determine that $1/3$ of $5/6$ is $2/6$, Ms. Reese identified the problem quickly, “So she’s not actually taking a third of each piece of the five sixths.” She restated this explanation at a few other points during the same interview. Although Ms. Reese was clear on why the student’s approach did not work by the time of her last interview, she also commented, “I wasn’t really sure what she was thinking. I was thinking well this isn’t $5/6$, well yes it is $5/6$. Well why, you know, well she’s taking two pieces of the whole thing but not of the $5/6$.” This comment lent some weight to the hypothesis that during the initial lesson Ms. Reese did not use her understanding of the distributive property to diagnose the student’s difficulty.

Third, Ms. Reese appeared still to be developing flexibility with three levels of units when the interview replayed the lesson excerpt in which Jack and Kate worked on $2/3$ of $3/4$ as shown in Figure 9. When asked if the students could cross partition the brownie pan to show fourths, Ms. Reese looked at Jack’s initial work shown in Figure 9a and stated, “As long as they had made each piece into thirds and shaded two of each
piece. But see, he just shaded all of that.” Ms. Reese described taking two pieces from each fourth several other times. When the interviewer asked if one needs to divide each fourth into three pieces, Ms. Reese first said “Yes,” then asked for clarification of the question, and finally pointed out that Jack could have taken just two of the three shaded pieces. These data suggested that Ms. Reese’s ability to produce three levels of units, through sufficient to infer numeric methods, was not sufficiently flexible to adapt, the fourth use of representations discussed above.

Discussion

Case studies of Ms. Archer and Ms. Reese informed a description of mathematical knowledge for teaching in the domain of fraction multiplication. The description emphasizes teachers’ unit structures and knowledge of multiplication and their purposes for using drawn representations. Ms. Archer produced two levels of units during instruction but did not produce three levels in several situations where, if she could, one might expect her to do so. She was still able to reason about parts of parts, however, by applying two-level structures twice in succession to find part of one part and by then applying multiplication or repeated addition to the result (using her connection between a fraction of and a fraction times a number). This allowed her to use drawn representations for illustrating solutions to fraction multiplication problems also arrived at through computation but apparently precluded using drawn representations to deduce or adapt. Attention to just two levels of units also constrained her ability to respond to students, for instance when they compared 2/3 and 3/4 incorrectly and when Jeff expressed confusion about one fifth on the number line.

Ms. Reese did produce three levels of units in several situations and this allowed her to use lengths and rectangular areas to determine solutions to fraction
multiplication problems which, in turn, allowed her to use drawn representations to infer a computation method. Data on Ms. Reese’s instruction also made clear that explicit attention to drawn instantiations of the distributive property is central for a teacher to have flexibility sufficient for adapting in response to the range of ways that students might assemble three-level unit structures using drawings. That Ms. Reese apparently attended to drawn instantiations of the distributive more readily when using linear than area representations suggested that she did not engage identical sets of understandings when reasoning with lengths and with areas.

The case studies provided examples in which teachers used drawn representations to illustrate and to infer and suggested that flexible, three-level structures are necessary for the fourth use discussed above, adapt. A small amount of data not presented above made clear that both teachers could also think about partitioning using their knowledge that products of rectangular dimensions yield areas. In her final interview, Ms. Archer determined \( \frac{3}{5} \) of \( \frac{4}{7} \) by drawing a brownie pan that showed a 3-by-4 array nested inside a 5-by-7 array and by multiplying the dimensions to determine the correct product, \( \frac{12}{35} \). Ms. Archer had mentioned connections among whole-number multiplication, dimensions, and areas once during an earlier lesson and during two previous interviews. This was the one time, however, that I saw her use these connections to reason about products of fractions. Ms. Reese evidenced similar connections when talking about one example she saw at a student’s desk. Thus, both teachers had further understandings of multiplication that they connected in isolated cases to partitioning rectangular areas. A connected set of understandings about multiplication, coordinated with flexible three-level unit structures, could support the deduce or adapt use of drawn representations discussed above.
Results of the present study contribute to research on mathematical knowledge for teaching in three principal ways. First, they demonstrate that knowledge for teaching fraction multiplication with drawn representations is complex and that the absence of research in this area is an important omission in the research base. The present study and that by Behr et al. (1997) are the only two of which I am aware that examine teachers’ understandings of nested unit structures and both have found that teachers struggle to reason distributively across conceptual units. These results raise questions about how many teachers lack explicit, flexible three-level unit structures that appear to be a central piece of knowledge for teaching fraction multiplication, and fraction arithmetic more generally, with drawn representations. Understanding the prevalence of phenomena reported in these studies is an important next step for research.

Second, results of the present study and those of Behr et al. (1997) demonstrate that unit structures are useful for gaining insight not only into students’, but also into teachers’ reasoning about fractional quantities. Furthermore, these examples suggest that results from the significant bodies of research on students’ cognition in various domains can be useful for research on mathematical knowledge for teaching not only because teachers need to know how students might reason in particular situations, but also because results on students’ reasoning may provide insight into teachers’ reasoning. In the present study, examining the nested unit structures that two teachers produced when reasoning about fraction multiplication has provided, in turn, more detailed explanations of teacher cognition than past work discussed above that reported teachers’ difficulties with tasks involving products and quotients of rational numbers.

Third, results of the present study and others from the CoSTAR project (Author, Date) raise the possibility that researchers, curriculum developers, and teachers may not
be communicating clearly about the role drawn representations can play in reform-oriented mathematics instruction. Recall that the CMP materials suggested three of the four purposes for using drawings discussed above. Ms. Archer’s comment that drawn representations “might help some students to understand things better” echoed comments made by other teachers working with the CoSTAR project, including Ms. Reese (see Author, Date). Such comments suggest that some teachers interpret the purpose of multiple representations in reform-oriented materials to be for providing alternatives from which each student can understand and use one method. The notions of illustrating, inferring, deducing, and adapting may serve as useful tools for analyzing the design of reform-oriented materials and for future research on teaching and learning with such materials.
References


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Integrating research on teaching and learning mathematics (pp. 177-198). Albany, NY: State University of New York Press.


Footnotes

1. All names are pseudonyms.

2. A graduate student conducted three interviews in my presence with one of the student pairs during the first round of data collection and two more interviews in my presence with the same pair of students during the second round.

3. These students were the ones the graduate student interviewed in my presence.

4. We observed a handful of instances during Ms. Archer’s *Bits and Pieces I* lessons in which she told students that a fraction of a number meant a fraction times a number.

5. We did observe a *Bits and Pieces I* lesson in which students used fraction strips to locate fractions on number lines.

6. The *Bits and Pieces II Teacher’s Guide* does state that cross partitioning “does not lead to the kind of partitioning that suggests multiplication of numerators and denominators” (p. 8, Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006).

7. Author (date) presented further examples in which Ms. Reese evidenced recursive partitioning during her fraction addition lessons.

8. Author (Date) examined the central role recursive partitioning, and hence three levels of units, played in teaching fraction addition and subtraction in Ms. Reese’s classroom.
Figure Captions

Figure 1. Ms. Archer’s white board when comparing $\frac{3}{4}$ and $\frac{2}{3}$.

Figure 2. Number line solution to $\frac{1}{5}$ of $\frac{2}{3}$ copied from the prepublication version of *Bits and Pieces II Teacher’s Guide* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006).

Figure 3. Area representations for $\frac{1}{2}$ times $\frac{2}{3}$. (a) Ms. Archer’s representation. (b) A student uses two squares.

Figure 4. Ms. Archer uses brownie pans to show the $\frac{1}{2}$ of $\frac{2}{3}$ in two ways.

Figure 5. Reproductions of students’ drawn solutions for $\frac{3}{4}$ of $\frac{1}{2}$. (a) Angie. (b) Damien. (c) Theo.

Figure 6. Damien and Theo’s drawn solution to $\frac{2}{5}$ of $\frac{3}{4}$. (a) Damien divides the pan into four and shades three parts (reproduction). (b) Damien divides each fourth into five parts (reproduction). (c) Theo shades two parts in each shaded fourth (students’ original work).

Figure 7. Ms. Archer (a) Shows $\frac{1}{3}$ of a brownie pan. (b) Divides vertically. (c) Shows $\frac{1}{2}$ of $\frac{2}{3}$ is $\frac{2}{6}$.

Figure 8. Ms. Reese demonstrates two approaches to $\frac{1}{3}$ of $\frac{5}{6}$. (a) Her approach. (b) Her presentation of two students’ approach.

Figure 9. Three brownie pan representations of $\frac{2}{3}$ of $\frac{3}{4}$. (a) Jack’s initial work (reproduction). (b) Kate’s initial work (my reproduction). (c) Jack’s revised work (reproduction).

Figure 10. Ms. Reese drawings showing $\frac{1}{4}$ of $\frac{2}{3}$ is $\frac{1}{6}$. (a) Ms. Reese draws a thermometer and shades $\frac{2}{3}$. (b) Ms. Reese divides the shaded $\frac{2}{3}$ into four parts. (c) Ms. Reese divides the unshaded region into two parts.

Figure 11. (a) Ms. Reese draws $\frac{1}{3}$ of $\frac{2}{1/2}$ pounds of cheese and divides each piece into thirds. (b) Ms. Reese divides each whole into sixths and shades five pieces.

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Fraction Multiplication and Adaptive Representation

Figure 1

\[
\begin{align*}
3 \times \frac{3}{4} &= \frac{9}{12} \\
4 \times \frac{2}{3} &= \frac{8}{12}
\end{align*}
\]
Fraction Multiplication and Adaptive Representation

Figure 2

TOP
Fraction Multiplication and Adaptive Representation

Figure 3

TOP

(a)  (b)
Figure 5

(a)  
(b)  
(c)
Figure 6

(a)  (b)  (c)
Fraction Multiplication and Adaptive Representation

Figure 8

(a) \[ \frac{1}{3} \text{ of } \frac{5}{6} = \frac{5}{18} \]

(b) \[ \frac{1}{3} \text{ of } \frac{5}{6} = \frac{2}{6} \]
Fraction Multiplication and Adaptive Representation

Figure 9

(a) 

(b) 

(c)
Fraction Multiplication and Adaptive Representation

Figure 10

TOP

(a)  (b)  (c)
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2 A graduate student conducted three interviews in my presence with one of the student pairs during the first round of data collection and two more interviews in my presence with the same pair of students during the second round.

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