

CHAPTER 7

Perspective on "Radical Constructivism and Mathematics Education"

Leslie Steffe and Thomas Kieren's contribution to mathematics education research was a deeper understanding of the implications of holding a constructivist perspective consistently. The majority of American Piagetian studies that had been done up to 1975 completely missed the point of Piaget's genetic epistemology, which was to develop an understanding that both respected the clear evidence that children's realities were unlike adults' and yet spoke to what those realities might be like and how children might construct them, so that they contained the germs of adult mathematics and science.

Today, the idea that children construct their own mathematics is taken for granted in mathematics education research and in mathematics education at large. But the ideas that the mathematics children construct is impossible to "know," that researchers can build only theoretical models of what children might know, and that children's mathematics is fundamentally unlike adult mathematics are less widely accepted. People find it difficult to understand how a field whose *raison d'être* is to design and improve the mathematics that students learn could, at the same time, say that students' mathematical knowledge is fundamentally unknowable. This radical stance is in contrast with what von Glasersfeld (1995) calls "trivial constructivism," which acknowledges that children construct their own knowledge but assumes that it is unproblematic for researchers to know what the children construct.

The emergence of radical constructivism in mathematics education paralleled a more general

postmodern movement, away from ideas of truth and being and toward greater understanding of situated existence, but the change in mathematics education research also had deep roots in new perspectives on the foundations of mathematics and in Piaget's genetic epistemology. Work in the foundations of mathematics had led to an understanding of mathematics as contingent on the perspectives of the people making it. The logicians (e.g., Russell & Whitehead), formalists (e.g., Hilbert), and intuitionists (e.g., Brouwer) showed that fundamentally different mathematical systems could be built coherently. Piaget's genetic epistemology, especially its focus on knowledge as structured actions and operations and its growth through generalization and reflection, offered a way to think about people building mathematical knowledge out of their material and social experience. Steffe and Kieren contributed to the evolution of constructivism in mathematics education research by specifying the development of conceptual operations that might underlie students' mathematical knowing, and the necessity of doing so within the constraint that we cannot take for granted what that knowledge is. Their account of that evolution unpacks one solution to the paradox: How can a field whose *raison d'être* is to design and improve the mathematics students should learn say simultaneously that students' mathematical knowledge is fundamentally unknowable?

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Radical Constructivism and Mathematics Education

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OUR intention in this article is to provide an interpretation of the influence of constructivist thought on mathematics educators starting around 1960 and proceeding on up to the present time. First, we indicate how the initial influence of constructivist thought stemmed mainly from Piaget's cognitive-development psychology rather than from his epistemology. In this, we point to what in retrospect appears to be inevitable distortions in the interpretations of Piaget's psychology due primarily to its interpretation in the framework of Cartesian epistemology. Second, we identify a preconstructivist revolution in research in mathematics education beginning in 1970 and proceeding on up to 1980. There were two subperiods in this decade separated by Ernst von Glasersfeld's presentation of radical constructivism to the Jean Piaget Society in Philadelphia in 1975. Third, we mark the beginning of the constructivist revolution in mathematics education research by the publication of two important papers in the JRME (Richards & van Glasersfeld, 1980; van Glasersfeld, 1981). Fourth, we indicate how the constructivist revolution in mathematics education research served as a period of preparation for the reform movement that is currently underway in school mathematics.

CARTESIAN EPISTEMOLOGY

There are several classic documents that mark the beginning of the influence of constructivist thought on mathematics educators.¹ We start with the report of the Woods Hole Conference, *The Process of Education*, by J. S. Bruner, because of the emphasis by mathematics educators on the structure of the subject in 1960 and its alleged relation to Piaget's cognitive-development psychology. We see the separation between the structures of mathematics and Piaget's genetic structures in *The Process of Education* as an expression of the classical dualism in the view of mind in Cartesian epistemology—an endogenic (mind centered) vs. an exogenic (world centered) view (Konold & Johnson, 1991; Gergen,

1994). The structures of mathematics were thought to be attained by capacities for reason, logic, or conceptual processing. In this, mathematical structures were regarded as having a mind-independent existence, and the function of rationality was to come to know these fundamental structures.²

Readiness to Learn

A long quotation from a memorandum prepared by Bärbel Inhelder for the Woods Hole Conference defined the capacities for reason and logic of young children within the framework of Piaget's cognitive-development psychology (Bruner, 1960, pp. 41-46). Inhelder is cited by Bruner (1960) as follows:

Basic notions in these fields are perfectly accessible to children of seven to ten years of age, provided that they are divorced from their mathematical expressions and studied through material that the child can handle himself. (p. 43)

Based on Piaget's genetic structures, the spirit was that concrete operational children were ready to learn and indeed could learn fundamental structures of mathematics. This was the foundation for Bruner's (1960) famous concept of readiness to learn the fundamental structures of mathematics: "Any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (p. 33).³

Bruner's concept of readiness to learn seemed to be quite sweeping at the time. A reason why it may

¹ There are important books, journal articles, conferences, and events that we do not mention. Although acknowledging these contributions, we can make no attempt to be exhaustive.

² In science education, knowledge was taken to be achieved when the inner states of the individual represented states of the external world—when mind served as a mirror of nature—and scientific objectivity was reified.

³ It is from remarks such as the two citations from Inhelder and Bruner that Piaget's work, and later constructivism with its emphasis on experience, has been frequently tied to the notion of an individual (usually a child) working with physical materials. Such limitations are not necessary conditions for using constructivism in mathematics education.

have seemed to make readiness to learn a nonissue is that genetic structures were in the main ignored by the developers of the modern mathematics programs. In his report on the School Mathematics Study Group (SMSG) at the Conference on Cognitive Studies and Curriculum Development, Kilpatrick (1964) caught the spirit of the times:

In a sense, the mathematicians who have guided the recent curriculum reforms have been waiting to be shown that psychological theories of learning and intelligence have something relevant to say about how mathematics shall be taught in the schools. These reformers (and I speak now not only of SMSG) have been so successful in teaching relatively complex ideas to young children, and thus doing considerable violence to some old notions about readiness, that they have become highly optimistic about what mathematics can and should be taught in the early grades. (p. 129)

Kilpatrick provided an ad hoc analysis of those features of SMSG that were most harmonious with the results of Piaget's studies (which included an emphasis on the structure of the subject matter by SMSG), but the prevailing attitude of some curriculum developers was that Piaget was an observer rather than a teacher (Goals for School Mathematics, 1963). The thinking was that if Piaget had observed the mathematical thought of children who participated in the modern mathematics programs, he would have realized the elasticity of the limits in their cognitive processes.

Because we interpret Bruner's concept of readiness as maintaining the distinction between fundamental structures of mathematics and Piaget's genetic structures, we believe that such thinking only seemed compatible with Bruner's concept. Not all of the developers of modern mathematics programs ignored Piaget's genetic structures, even though they may have interpreted them in terms of Cartesian epistemology. In these cases, Bruner's concept of readiness to learn was not a nonissue. Rather, it was a hypothesis.

We find it interesting that Bruner at times seemed to conflate fundamental structures of mathematics and Piaget's genetic structures, as indicated by the following quotation: "Good teaching that emphasizes the structure of the subject is probably even more valuable for the less able students than for the gifted ones" (Bruner, 1960, p. 9). In this, we see Bruner entangled in Cartesian anxiety because he turned to the structure of the subject to secure a foundation for children's mathematical knowledge when capacities for reason and logic were only minimally present.

[Cartesian anxiety] is an anxiety that permeates all metaphysical and epistemological questions concerning the existence of a stable and reliable rock upon

which we can secure our thoughts and actions. As Bernstein explains: "Either there is some support for our being, a fixed foundation for our knowledge, or we cannot escape the forces of darkness that envelope us with madness, with intellectual and moral chaos" (p. 18). (Konold & Johnson, 1991, p. 2)

The fundamental structures of mathematics served as the fixed foundation for most of the developers of the modern mathematics programs. In those cases, we believe that the curriculum developers were, perhaps unintentionally, entangled in Cartesian anxiety because they conflated mathematical structures and capacities for reason and logic. They apparently felt no necessity to look beyond mathematical structures to investigate what the mathematical knowledge of students might be like. Even Freudenthal (1973) showed little appreciation in his critique of Piaget's mathematics that the genetic structures he critiqued were formalizations of the organization Piaget observed in children's actions.

The Classical Dualism and Research in Mathematics Education

The Piagetian studies. Henry van Engen, while working at the Research and Development Center for Cognitive Learning at the University of Wisconsin during the last half of the 1960s (Van Engen, 1971), provided leadership for a series of studies in mathematics education that became known as "the Piagetian studies" (e.g., Steffe, 1970; Kieren, 1971; Branca & Kilpatrick, 1972; Johnson, 1974; Carpenter, 1975; Mpiangu & Gentile, 1975; Martin, 1976; Silver, 1976; Adi, 1978; Taloumis, 1979; Days, Wheatley, & Kuhn, 1979; Hiebert, Carpenter, & Moser, 1982). Some of the Piagetian studies were devoted to investigating the readiness of young children to learn mathematics. There were two basic types of readiness studies—correlational and training. The hypothesis of the readiness studies was that capacities for reason or logic could be increased through intensive learning experiences. If the hypothesis was not disconfirmed, then the elasticity of the limits in cognitive processing would be demonstrated and Piaget's cognitive development theory would be falsified as a theory of readiness to learn mathematics.

Further, several mathematical analyses of Piaget's genetic structures were undertaken. The basic intent was to logically demonstrate that basic mathematical structures would serve as well as Piagetian genetic structures as models of the mathematical knowledge of children. The logical analyses were generally followed by an attempt to use the mathematical structures to explain children's mathematical behavior. Finally, Inhelder's proposition cited above, along with the work of Dienes, led to an interest in exploring the role of play and manipulatives in mathematics teaching.

An emerging shift in understanding. It was certainly a purposeful and exciting time for those involved in the Piagetian research program. However, we regarded Piagetian cognitive-development psychology as a hypothetical-deductive system, and the emphasis was on applying it in research in mathematics education. For this reason, the Piagetian research program temporarily went into decline. Fortunately, a shift emerged in understanding Piagetian theory, thanks mainly to the efforts of Myron F. Roszkopf, then chairman of the Department of Mathematical Education at Teachers College, Columbia University. After spending the academic year 1965-66 at the Wisconsin Research and Development Center for Cognitive Learning working with Henry van Engen, Roszkopf felt that closer cooperation between the Piagetians and mathematics educators was needed. Toward this end, he organized the Greyston conference at Teachers College, Columbia University (Roszkopf, Steffe, & Taback, 1971). Roszkopf wrote:

From very small beginnings during the early years of the 1960's, interest by mathematics educators in Piagetian research broadened until at several universities students were working on doctoral dissertations that clearly were almost as closely related to child-development psychology as to mathematical education. Uneasiness over little evidence of close cooperation between psychologists and mathematics educators led to ... a conference on Piaget type research in mathematical education. (Roszkopf et al., p. vii)

An understanding that Piagetian theory could not be simply applied to mathematics education was emerging. But the planning committee for the conference still believed that the theory was a hypothetical deductive system and were primarily interested in Piaget's account of the development of basic mathematical concepts and operations.⁴ This belief and interest was indicated in Roszkopf's letter of invitation to Hermine Sinclair, the Genevan representative:

The committee planning the conference would like you to be one of two principal lectures.... Hopefully, you will direct your presentations to an explication of Piaget's cognitive development investigations with special reference to mathematics.

Sinclair, in opening the conference, took the opportunity to comment on Piaget's genetic epistemology: "At first sight it would seem that a psychological theory that is regarded by its author as a "by-product" of his epistemological research ... is ideally suited to educa-

tional applications" (Sinclair, 1971, p. 1). But she did not provide an explanation of constructivism, nor did she provide any indication about how constructivism might be useful in mathematics education. Instead, she discussed Piaget's main stages, different types of operatory structures, and so on, which is what she was asked to do.

Nativism vs. constructivism. Because of the continuing interest in readiness to learn mathematics, Harry Beilin also was asked to present a major paper at the Greyston conference. The following is an excerpt from Roszkopf's letter of invitation:

An open question is whether training will enable children to reach a Piaget stage at an earlier age than otherwise would be the case. You have carried out some of the most relevant to mathematics training investigations. Thus, we would like to suggest that you choose for your subject a critical review of training research.

Although the planning committee had not anticipated that Beilin would bring the issue of innatism vs. constructivism forth in his paper, in retrospect, it is not surprising that the issue arose in the context of his review. After reviewing studies devoted to the training of logical operations, Beilin (1971) stated the innatist hypothesis in the following way:

The data demonstrating acquisition with a wide variety of training methods give further support to the idea that the logical operational system is under the control of a genetic mechanism that only permits the programmed development of defined cognitive structures through interaction with environmental inputs. (p. 114)

The epistemological conflict between constructivism and innatism raised at the Greyston conference was extensively discussed in a debate between Piaget and Chomsky at the Royaumont conference five years later (Piattelli-Palmarini, 1980). Piattelli-Palmarini (1980) indicated that an allegiance to Cartesian epistemology is recurrent in the Chomskian research program, and that in generative linguistics, "all structure comes from within". Environment reveals this structure; it does not imprint its own patterns on the system" (p. 12). Because nativism stands in contrast to Piaget's constructivism, we find it ironic that the emphasis on the structure of mathematics in the modern mathematics movement of the 1960s found rationale in Piaget's cognitive-development psychology. From an epistemological point of view, the structuralism of the modern mathematics programs was more compatible with Chomsky's program, with its basis in Cartesian epistemology, than it was with Piaget's program.

THE PRECONSTRUCTIVIST REVOLUTION

The Greyston conference marked the beginning of what we call the preconstructivist revolution in

⁴ The conference was cosponsored by the National Council of Teachers of Mathematics, and the planning committee for the conference was appointed by Julius H. Hlavaty, then Past President of the Council. It consisted of M. F. Roszkopf, L. P. Steffe, and S. Taback.

research in mathematics education. It was preconstructivist because most mathematics educators of the time were yet to appreciate the revolutionary implications of Piaget's genetic epistemology. Those of us who were immersed in Piagetian studies had not yet rejected empiricism as normal science in mathematics education despite its rejection by both Piaget and Chomsky. According to Mehler (1980), Piaget and Chomsky shared certain opinions and positions.

One of the most important results that emerged from the confrontation between Piaget and Chomsky at Royaumont was their convergent rejection of positivism and empiricism.... In this case, the environment could ... be described as the independent variable, while the behavior of a subject under study would be the dependent variable. (p. 350)

Mehler's comment concerning independent and dependent variables illuminates the empiricist assumption underlying what we regarded as being normal science in mathematics education research throughout the 1960s and the first half of the 1970s. The preconstructivist revolution was marked by a rejection of this empiricist view and by a reformulation of our understanding of Piaget's genetic structures. We finally came to understand Piaget's genetic structures as models that he made to explain his observations of children's ways and means of operating rather than as a hypothetical-deductive system. What seemed to be a major insight at the time did not occur to us in one fell swoop. Rather, it was a result of our struggles to use Piagetian theory in mathematics education. That we finally came to understand that we needed to make our own models to serve our educational purposes rather than to use Piaget's seemed to be a major breakthrough, and it was quite liberating. In fact, the long-lasting effects of this observation can be seen in contemporary constructivist research in which the researchers seek to observe and describe mechanisms that children and indeed persons of any age use as they, individually or interactively, build up mathematical knowledge in a particular learning space (Pothier & Sawada, 1983; Pirie & Kieren, 1994; Steffe & Wiegel, 1994; Thompson, 1994).

A Turn to Behaviorism

Another revolution in mathematics education was occurring while the preconstructivist revolution in research was underway. The modern mathematics movement of the 1960s ran out of steam in the early 1970s, and the structuralist foundation for mathematical knowledge was replaced by a behaviorist interpretation of rationality. The fundamental structures of mathematics were replaced by long lists of behavioral objectives as the rational bridgehead. This turn to behaviorism in the practice of mathematics education

did not create a major upheaval among researchers operating from an empiricist assumption, because the major influence of the structuralism of Cartesian epistemology was found in the practice of mathematics teaching and curriculum planning rather than in the practice of mathematics education research. So, when a backlash to the modern mathematics movement was manifest in a turn to behaviorism, this back-to-basics movement was felt at the level of the practice of teaching rather than at the level of the practice of research.

More than any other single factor, the separation between the practice of teaching and the practice of research paved the way for the emergence of constructivism in mathematics education. Those of us who were doing research in mathematics education were also mathematics teachers or curriculum developers, so the separation was manifest as a crisis in identity. How could one claim to be a researcher in mathematics education and still maintain his or her identity as a mathematics teacher? Could there be a field of mathematics education where the practice of research and the practice of teaching were not simply compatible, but inseparable parts of a lived experience?

Erlwanger's Benny

If there ever was a "crucial experiment" in mathematics education, the work of Stanley Erlwanger (1973) would have to qualify. In one ingenious stroke, Erlwanger was able to falsify the behavioristic movement in the practice of mathematics teaching (in Lakatos's sense of sophisticated falsificationism). According to Lakatos (1970):

For the naive falsificationist a 'refutation' is an experimental result which, by force of his decisions, is made to conflict with the theory under test. But according to sophisticated falsificationism one must not make such decisions before the alleged 'refuting instance' has become the confirming instance of a new, better theory. (p. 122)

By concentrating on the beliefs of mathematical rules and answers of a child, Benny, who participated in the program Individualized Prescribed Instruction (IPI) produced by the Pittsburgh Research and Development Center, Erlwanger was able to demonstrate how Benny's understanding of mathematics conflicted with any "common sense" understanding of what would be regarded as "good mathematics." This was a crucial aspect of Erlwanger's work, because by demonstrating what a "common sense" view of mathematics should not be, Erlwanger was able to falsify (naively) the behavioristic movement in mathematics education at that very place where behaviorism has its greatest appeal—at the level of common sense.

But Erlwanger was able to accomplish more. His study, conducted under the direction of Jack Easley at the University of Illinois—one of the forerunners

of the constructivist movement in science education in the U.S.—focused on the mathematical thinking of an individual child. Erlwanger interpreted that thinking in a constructivist framework and was able to demonstrate an understanding of a science of mathematics education different from empiricism. In his work, statistics derived from results on measures did not form the core. Rather, Erlwanger tried to show in detail how Benny “made sense” of his experiences in IPI. In doing so, he considered the corpus of Benny’s interactions within the IPI world in which Benny found himself.

Erlwanger demonstrated the power of interpretative research as well as the need for alternative methodologies. Although he did not enunciate its constructs nor its associated problems or methodologies, in retrospect, Erlwanger’s work was a confirming instance of an emerging “new, better theory.” This work, in fact, was also one of the first to focus on both the structural dynamics of an individual, as interpreted from the actions and words of Benny, and on the interactional dynamics between Benny and the ways in which the IPI environment occasioned his actions. In this, the IPI environment was changed by Benny through his actions and through his interactions with others in this environment. Studies of both types of dynamics became the hallmark of later constructivist research in mathematics education.

A Shift in Normal Science

At the same time that Erlwanger was working with Jack Easley at the University of Illinois, the late Charles Smock of the Department of Psychology at the University of Georgia was working to formulate a constructivist research and development program in mathematics education, including the development of a methodology for research that was an adaptation of Piaget’s clinical interview. It was difficult, however, to overthrow the tyranny of the empiricist view of normal science in mathematics education and to emerge from the stranglehold that empiricism had on the practice of research.⁵ Perhaps this struggle is best illustrated by the fact that it wasn’t until 1983 that an article was published in the *JRME* with “constructivist” in the title (Cobb, & Steffe, 1983). There, it was argued that the constructivist researcher needed to be a teacher as well as a model builder, which pushed research methodology beyond the clinical interview.

⁵ The preconstructivist revolution was buttressed by the work of Izaak Wirszup of the University of Chicago, who led the Survey of Recent East European Mathematical Literature. Through Wirszup’s leadership, mathematics educators became familiar with the work of L. S. Vygotsky and with what Menchinskaya (1969) called the “genetic method of research” (p. 6) along with the new methods of research that it implied—the “teaching experiment.” Our claim is not that Vygotsky’s work caused the preconstructivist revolution. Rather, it served an influencing and supporting role.

The nature and the philosophical intent of the teacher-researcher was summed up as follows:

It is not the adult’s interventions per se that influence children’s constructions, but the children’s experience of these interventions as interpreted in terms of their own conceptual structures.... The adult cannot cause the child to have experience qua experience. (p. 88)

In a teaching experiment, it is the mathematical actions and abstractions of children that are the source of understanding for the teacher-researcher. The teacher, to use the ideas of Varela, Thompson, and Rosch (1991), helps provide occasions for children’s mathematical activity, but it is the children’s way of making sense that determines their own knowledge.

The struggle to reformulate “normal science” in mathematics education research during the 1970s found expression in the *JRME* in the latter part of the decade. Coburn (1978) published a set of criteria for judging research proposals and reports that had been written by the Research Advisory Committee of the National Council of Teachers of Mathematics. These criteria generated controversy concerning the nature of research in mathematics education (Fennema, 1978; Wheeler, 1978; Lester & Kerr, 1979; Webb, 1979)—between “laboratory” and “naturalistic” studies. The controversy, however, did not center on the perspectives of the researchers doing or evaluating research until Thompson’s (1982) landmark piece was published:

Webb’s remark suggests the mistake of the RAC (Research Advisory Committee) was in not taking a broader view of research in mathematics education, meaning a view that would encompass both laboratory and naturalistic studies ... as valid instances of scientific research. I will argue instead that what is called for is not a broader view, but an acknowledgement of a multiplicity of views ..., quite possibly each being irreconcilable with the others. (Thompson, 1982, p. 149)

It is probably fair to say that the *JRME*, over the last 12 years, has provided for this multiplicity of views. In doing so, it provided for ongoing discussion between constructivist views and other views on mathematics education theory and research. It is interesting to note that constructivism was often implicit in the discussions (Hiebert et al., 1983; Steffe & Cobb, 1983; and Gagné, 1983; Steffe & Blake, 1983). For example, the two key constructivist points—

- using “conservation” and mathematics performance as variables does not provide a way of seeing how children build up mathematical ideas;
- children with different developmental backgrounds may well be able to get the same

answers on an arithmetical task, but the ways in which they do so might differ significantly

—are only implicit in the Steffe and Cobb critique of the article by Hiebert et al. (1982). The explicit nature of Piagetian constructivism (as opposed to Piagetian “developmental psychology”) and its potential contribution to research and theory in mathematics education did not stand out in the critique.

Constructivism Made Explicit

A fertile ground had been prepared for an emergence of Piaget's constructivism in research in mathematics education, but it took the work of Ernst von Glasersfeld (1984; 1987) to bring it forth through his work in the project Interdisciplinary Research on Number at the University of Georgia. Historically, it may seem that von Glasersfeld was present from 1960 forward, waiting for the right moment to etch his brand of constructivism into the collective consciousness of mathematics educators. However, this was not the case at all. We leave it to his forthcoming book on radical constructivism (von Glasersfeld, 1995) for an elaboration of just how it happened that he became a member of the Department of Psychology at the University of Georgia in 1969 after having arrived in the United States from Italy in 1966 to continue work on a machine translation project sponsored by the U.S. Air Force. It was only through what von Glasersfeld calls “unexpected breaks with the past” that he became introduced to the work of Jean Piaget by Charles Smock. With this introduction, the way was opened for a new revolution in mathematics education of a magnitude no less than the modern mathematics movement of the 1960s. In contrast to the modern mathematics movement, the revolution first occurred in research in mathematics education, and only then did it begin to influence the practice of mathematics teaching.

Von Glasersfeld presented his “radical” interpretation of Piaget's genetic epistemology to the Jean Piaget Society in Philadelphia in 1975. This presentation, along with the establishment of the Georgia Center for the Study of the Learning and Teaching of Mathematics in the same year, marked a separation in the 10-year period of the preconstructivist revolution in research. The basic idea of The Georgia Center was to establish a community of researchers in mathematics education working on problems of interest to the community, where the experience of the researcher, conceptual analysis, and social interaction replaced the controlled experiment as “normal science.” No longer did it seem necessary to use the controlled experiment with its emphasis on statistical tests of null hypotheses and empirical generalization to claim that one was working scientifically.

THE CONSTRUCTIVIST REVOLUTION

The members of the Editorial Board of the *JRME* were aware of von Glasersfeld's work and brought it to the readership of the *JRME* in two quite important publications (Richards & von Glasersfeld, 1980; von Glasersfeld, 1981). In the first, von Glasersfeld and Richards were able to successfully differentiate the radical aspect of Piaget's genetic epistemology from what von Glasersfeld (1989) 9 years later called trivial constructivism—a form of constructivism that asserts that children gradually build up their cognitive structures while maintaining that the cognitive structures being built up are reflections of an ontological reality. Trivial constructivism was what mathematics education began with in 1960 with Jerome Bruner's *The Process of Education*, and it took 20 years for it to be publicly challenged in a mathematics education journal. In hindsight, the editor of the *JRME* demonstrated a sense of an important historical movement in research in mathematics education, and the field of mathematics education owes a great deal to his foresight.

In the second publication, von Glasersfeld opened a new way of doing science in mathematics education through what he calls conceptual analysis. What came to be known as the constructivist teaching experiment had already emerged from the preconstructivist revolution at the time of von Glasersfeld's two publications, but the conceptual analyses involved in building second-order models of children's mathematics were underspecified.⁶ In its original form, the constructivist teaching experiment was an attempt to bridge the gap between research and practice and was a hybrid of Piaget's clinical method and mathematics teaching. In its later form, it included conceptual analysis as a way to build models of children's mathematical knowledge and its construction.

The Construction of Reality

Two key ideas emerged from the Richards and von Glasersfeld (1980) and the von Glasersfeld (1981) papers. The first is the notion that individuals construct their own reality through actions and reflections on actions. Although individuals adapt to remain “viable” in action in their world of experience, the individuals' concepts of reality are not some mirror.

⁶ First-order models are those models the observed subject constructs to order, comprehend, and control his or her experience. Second-order models are those models the observer constructs of the subject's knowledge in order to explain their observations or experience of the subject's states and activities (Steffe, von Glasersfeld, Richards, & Cobb, 1983, p. xvi).

There is no ultimate progress and there is no "congruity" of knowledge and reality. There is adaptation but adaptation means that there are viable organisms and viable theories. (p. 35)

In this radical version of constructivism,

Reality in some absolute sense [lies] beyond the sphere of experiential justification. (Richards & von Glasersfeld, 1980, p. 35)

Thus, as Richards and von Glasersfeld (1980) claimed for the work of Piaget, in constructivism one is not studying reality, but the construction of reality. In mathematics education, this means that one is studying the construction of mathematical reality by individuals within the space of their experience. In this construction, although there may be well-defined tasks or spaces for experience, there are no pre-given prescribed ends toward which this construction strives. There is no optimal selection of the individual's actions or ideas by the environment, nor is some perfect internal representation or match against an external environment the test of the constructed "reality." The constraints experienced by the constructing person are proscriptive and not prescriptive in nature, and the "reality" constructed is "good enough" for allowing effective action by the person, which need not be physical action, in the space of experience.

Modeling the Constructive Process

As constructivist mathematics education researchers, we became oriented toward studying the construction of mathematical concepts and the operations by which children attend to and organize their experiences. In this, we were guided by Piaget's genetic epistemology rather than by his cognitive-development psychology. Piattelli-Palmarini (1980) regarded the Piagetian program to be based on the following comment made by Piaget:

Cognitive processes seem, then, to be at one and the same time the outcome of organic autoregulation, reflecting its essential mechanisms, and the most highly differentiated organs of this regulation at the core of interactions with the environment (p. 26). (p. 4)

Understanding cognitive processes as an outcome of autoregulation and interaction served as a basis of our neo-Piagetian research program in mathematics education, but we were not committed to the proposition that Piaget's genetic structures were the necessary outcome of the functioning of mind (Kieren, 1976; Steffe, von Glasersfeld, Richards, & Cobb, 1983). We were committed to building models of the constructive process and understood that process as being an outcome of individual-environment interaction.

Modeling became the primary means by which we made a distinction between our activities as teachers and our activities as researchers in constructivist teaching experiments. A model consists of coordinated schemes of actions and operations that the researcher constructs out of his or her experience of children's actions. Such models had both general and specific qualities:

On one hand, the model should be general enough to account for other children's mathematical progress. On the other hand, it should be specific enough to account for a particular child's progress in a particular instructional setting. (Cobb & Steffe, 1983, p. 91)

This was to be accomplished through a dialectical interaction between the theorizing and the observing of researchers in teaching experiments. Such models not only provided an explanation of recurrent patterns in children's mathematical behavior; because the researcher was also a teacher, the models also provided an explanation of progress children were observed to make under the influence of constructivist teaching.

The possibility for a "children's mathematics" emerged from the teaching experiment in the broader community of constructivist researchers. Children's mathematics consists of changing and growing sets of coordinated schemes of action and operation related to central mathematical topics (e.g., whole numbers; addition and subtraction) or to topics of a larger scale such as additive and multiplicative structures (Vergnaud, 1982). Such "second-order" mathematics is the creation of the researcher based on intensive analysis of children's mathematical behavior and can be constructed only through social interaction. So, in constructivist mathematics education, the activities of the teacher-researcher and those of the children are co-implicative (cf. Confrey, 1990; Steier, 1995). It is perhaps this phenomenological consideration of children's mathematics arising in interaction with a teacher in very particular spaces of mathematical possibilities that has led to whatever influence constructivist research has had on reformers in mathematics curriculum and teaching. Observing and listening to the mathematical activities of students is a powerful source and guide for teaching, for curriculum, and for ways in which growth in student understanding could be evaluated.

Discussion and Debate

A view of mathematical knowledge and mathematics education research that differed radically from the traditional conceptualizations could be expected to generate discussion and debate. In a frequently charged discussion at the 1987 conference on the Psychology of Mathematics Education (PME) in Montreal, the "radical" assumption of constructivism

was questioned in mathematics education (Kilpatrick, 1987):

We need an epistemology that takes ontology into account. 'We must keep metaphysics and epistemology tied together so that (a) our explanation of Knowledge does not leave us committed to there being things we cannot account for in our theory of Being, and (b) our theory of Knowledge (thus restricted) can accommodate our claim to know what Being is' (McClellan, 1981, p. 265). (p. 19)

Constructivism was also criticized as a solipsistic position at the Montreal meeting (Vergnaud, 1987; Wheeler, 1987) even though von Glasersfeld (1984) and von Foerster (1984) had already made counter arguments. These critics of radical constructivism seemed to be caught up in the throes of Cartesian anxiety. From the debate, however, it was clear that constructivists needed to somehow, in the words of Konold and Johnson (1991), "address questions about the ultimate medium ... that serves as the setting for constructive activity" (p. 4). This was an interesting turn, because the problem of the relation between research and practice reemerged in that case where the "ultimate medium" referred to the practice of mathematics teaching.

Rather than resurrect the classical duality, constructivist mathematics educators instead chose to concentrate on mathematical realities constructed through interaction among human beings. Following Maturana's (1978, pp. 45-46) analysis that asserting the existence of an object is tantamount to bringing it forth through "language" and giving it form in the domain of consensual coordination of action in which we exist as human beings, the study of practice in a constructivist mathematics education is centering on interactive mathematical communication and the consensual domains produced (Cobb et al., 1991).

Empiricism vs. constructivism. The Montreal debate, perhaps more than any other social event, served to clarify the nonsolipsistic but radical assumption of constructivism in mathematics education. Prior to the Montreal debate, a heated exchange had taken place between Brophy and Confrey in a 1986 issue of the *JRME*. Confrey (1986) used constructivist arguments to question the effectiveness of various outcome-based practices toward which Brophy was directing his educational research. In his response to Confrey's criticisms, Brophy made the following comments about constructivist research:

If it is to be of much practical use, however, such input will have to become much more specific, prescriptive, and empirically based.... It will have to come to grips with the challenge facing the typical K-12 teacher (teach 20 to 40 students to preset curriculum objectives while working within time and resource constraints), and it will have to include process-outcome

data that allow for a scientific assessment of the hypothesized effects of recommended procedures, Until they begin to produce such information ..., constructivists ... who have not yet done so can continue to expect scholars to ask "Where are the data?" (Brophy, 1986b, p. 367)

Brophy's comments are best interpreted in terms of an empiricist view of normal science, where the environment is the independent variable and the behavior of the subjects under study is the dependent variable (Mehler, 1980). The same separation between research and practice was implicit in Brophy's comment that plagued mathematics educators in the two previous decades, an issue that was, not surprisingly, left unresolved in the debate.

The explicit necessity to include the actions and operations of the researcher in the research enterprise is a key in bridging the gap between the practice of research and the practice of mathematics teaching. This necessity has been explicitly elaborated by Confrey (1995):

When we seek to speak of cognition, education, problem solving, mathematics, or learning and teaching, we must take particular care to recognize the role of the observer in the description and analysis of the problem. In the radical constructivist research program, this has meant establishing clear methodological guidelines concerning the importance of "close listening." (p. 196)

The necessity to include the observer in the research enterprise drove the early development of the constructivist teaching experiment in which the constructing mathematical child was of central concern. An extensive corpus of "data" concerning constructive activity is available from such teaching experiments, though perhaps not the kind of "data" acceptable by those of an empiricist and representationist bent (e.g., Confrey, 1991; Kieren & Pirie, 1991; Thompson, 1994).

Interactive mathematical communication. From the background of working in teaching experiments where the goal was to specify a mathematics of children, Cobb and Wheatley mounted a research program at Purdue University where the goal was to focus on social interaction in the classroom. As reported in the *JRME* (Cobb et al., 1991), the researchers on the project took the results of their work as an indication that a constructivist "problem centered instructional approach in which the teacher and students engage in discourse that has mathematical meaning as its theme is feasible in the public school classroom" (p. 25).

But such research went beyond "empirical results." The research and related teaching activities showed in detail how constructivism might manifest

itself through problem challenges, small-group work, and classroom discussions using what appeared to be "ordinary" curriculum materials with ordinary classes. As seen in Yackel, Cobb, & Wood (1991) and particularly in Cobb, Yackel, & Wood (1992), evidence and arguments were made supporting a constructivist, nonrepresentationist view of mind useful for researchers and teachers as they think about mathematics education. They claim that an "approach that views mathematics as both an individual and a collective activity transcends the contradictions of the representational view and offers an account of truth, certainty, and intersubjectivity" (p. 3). At the heart of their approach stands a discussion of how social interaction and individual construction of mathematics fit together:

Both explicit problems and conflicts that arise in the course of social interactions and the generally unnoticed mutual appropriations of meanings that occur in any communicative interaction serve as occasions for individual students' constructive activities.... an account of a student's mathematical learning in a classroom should consider the development of both the taken-as-shared, communal meanings and practices and the individual student's personal meanings and practices. (p. 18)

A competing paradigm. A discussion of the debates involving radical constructivism in mathematics education research would not be complete without at least mentioning the debates that arose as a result of the influence of information-processing psychology in mathematics education. The debates clarified the differences between the innatist models of number development provided by those working in information processing and the models of the construction of number of those working in radical constructivism. For example, in the proceedings of the Wingspread Conference, *Addition and Subtraction: A Cognitive Perspective* (Carpenter, Romberg, and Moser, 1982), Starkey and Gelman (1982) commented as follows:

It appears that coming to know about number is much like coming to know about language (Gelman, 1979). The ability to learn language is rule governed; the ability to count verbally is rule governed.... These findings, taken as a whole, support the view that some number abilities are natural human abilities in the same sense that some language abilities are natural human abilities (Gelman & Gallistel, 1978). (p. 113)

Starkey and Gelman proceeded on, criticizing Piaget's developmental model of children's construction of number, apparently unaware of the neo-Piagetian work in mathematics education that focused on children's construction of counting schemes. This point is important because Starkey and Gelman appeared to believe that a Piagetian view would exclude a study

of children's construction of counting schemes: "As such, this view (in contrast to the Piagetian view) emphasizes the importance of counting." (p. 113).

The innatist hypothesis also served as a basic but perhaps implicit tenet for other models of number development influenced by information-processing psychology (e.g., Riley, Greeno, & Heller, 1983; Briars & Larkin, 1984; De Corte & Verschaffel, 1985). Cobb (1987) provided an extensive analysis of the work of Briars and Larkin, and of Riley, Greeno, and Heller, and contrasted these two information-processing models with a neo-Piagetian model of the construction of counting schemes based in radical constructivism (Steffe et al., 1983). Cobb based his analysis on three implicit principles. The first was that the innatist hypothesis served as a basis for the information-processing approach, which is to explain "how children build up problem representations from the information in a problem statement.... The explanation of this process should ideally take the form of an executable computer program whose output matches that of humans" (Cobb, p. 171). Cobb commented that "Steffe et al. developed a model that cannot be expressed in the precise formalisms of a computer language" (p. 176). The second principle implicit in Cobb's comment is that the constructivist model was based on a particular interpretation of Piaget's reflective abstraction. Because reflective abstraction cannot be an operation of an executable computer program, Cobb pointed out that information-processing models do not account for the construction of mathematics by human beings. To be comprehensible as developmental models, they have to be at least implicitly based on the innatist hypothesis.

The third point implicit in Cobb's analysis supersedes the first two. The model of children's construction of counting schemes was a result of a conceptual analysis of children's mathematical language and actions as they participated in teaching episodes. As such, it was a product of reflective abstraction by the model builders and consisted of a constellation of conceptual constructs that they found useful in constructing children's counting schemes. It was not an objective model that could be simply applied in different contexts. Rather, it would be useful for a teacher or any other adult interested in children's mathematics only to the extent that it reemerged in interactive mathematical communication with children as spontaneous and independent contributions of the children.

So, the conflict between innatism and constructivism that served as a basis for the debate between Chomsky and Piaget was essentially replayed in the debates in mathematics education concerning information-processing models of number development and models of children's construction of number. But the

key for Cobb lay not in the artifacts of this conflict—such as whether being able to develop computer models of mathematical behavior was a marker of successful research—nor in showing whether or not the brain had certain features or properties. Cobb was pointing to a critical difference in perception of what persons engaged in mathematical activity were doing: Were persons building up mental representations that in some way pictured or matched a pregiven world? Or were these persons building up their own schemes for successful action determined by their own structures and histories of interaction and action in an environment and by their abstract reflections on such action and interaction? That is, Cobb was pointing to the critical distinctions between representationist accounts of personal mathematical activity and nonrepresentationist accounts developed by constructivist researchers. In his conclusion, Cobb (1987) acknowledged the conflictive nature of the differing assumptions:

Adherents to anyone of the research programs will find the choice between the three models relatively unproblematic. Others who are less committed will not find things so straightforward.... Uncommitted readers will probably agree with Kuhn's (1977) observation that when criteria are "deployed together, they repeatedly prove to be in conflict" (p. 322). (p. 177)

In both the information-processing models and the constructivist models, there was significant attention given to children's numerical operations. In fact, it was Kieren (1980) who dubbed these operations "constructive mechanisms." Studies involving such mechanisms that go beyond early number construction by children appeared in the *JRME* as well (e.g., Hunting, 1983; Pothier & Sawada, 1983; Behr et al., 1984; Noss, 1987; Thompson & Dreyfus, 1988; Clements & Battista, 1989; Davis & Pitkethly, 1990; Olive, 1991; Williams, 1991; Lamon, 1993). Although some of these studies focused mainly on physical actions by students of various ages and the interpretation of these actions, some considered the nature and products of reflective abstraction. For example, in a report on college students working on algebraic word problems, Clement (1984) saw distinguishing numerical concepts from an object background and inventing "hypothetical operation on the variables that creates an equivalence" as keys to successful work by students. The fact that studying mathematical knowledge construction in the work of individuals is not equivalent to studying physical actions and work with concrete materials is highlighted in this comment. Thus, at least in some instances in reports in the *JRME* throughout the 1980s, the vision of constructivism as tied to physical activities—perhaps a misinterpretation of earlier remarks by Bruner or Inhelder for that matter—is dispelled.

FINAL COMMENTS

Although we have not begun to tell the whole story, what we have written above is one portrayal of the development of constructivism as a living force in mathematics education over the last 30-some years. As we have suggested, some of the contributors to this story of constructivism in mathematics education did so intentionally, but many, including many of its critics, did not. We tried to show that constructivism in mathematics education and in its reportage in the *JRME* moved from an almost hidden, still dualistic phenomenon in the 1960s and 1970s, to a more defined, evolving, and seemingly individually oriented but seriously challenged system for mathematical knowing in the mid 1980s, to an interactionist but non-representationist view of mathematical knowing and teaching today.

Such research focused on building up the mathematics of children—second-order models of how children construct personal mathematical concepts and operations that are stripped of their physical content, not needing the actions that brought them forth. These researcher models arose out of interpreting children's actions in particular environments. But rather than thinking that the environment caused the children's thoughts and actions or that such thoughts were mirrors of the environment, the creators of constructivist models thought it clearer and better to think of the person's actions to be determined by that person's own conceptual structures. The creators of such models did not see such structures either as innate nor as simply emergent phenomena arising out of brain activity. They saw such personal knowledge structures as plastic and arising out of reflective abstraction on action and interaction in a world that such structures allow the person to bring about or construct. In that sense the person's knowledge structures and the world of action out of which they arise are co-implicative.

The actions and operations of the children were not seen to simply arise from the children themselves. They were observed to be occasioned by the environment, including the actions and language of peers and teachers, as well as by the taken-to-be-shared meanings that arose in the setting of classroom communities. Because of this emphasis in constructivist research on interaction or coimplication of personal knowledge and environmental possibilities, the very nature of the activities and products of this research provide meaning for mathematics teaching and learning that teachers can use to build their own models for their own actions in practice. Because constructivist research has sought means by which persons can construct their own mathematical knowledge structures, one of the products of such research has

been descriptions of a variety of constructs (e.g., various kinds and levels of units pertaining to natural numbers, rational numbers, and ratios) and constructive mechanisms (e.g., unitizing operations, partitioning operations, proportionality operations, unit compositions and decompositions). Such descriptions can serve the practicing teacher in two ways. First, they provide guides for listening to and observing students. Second, they provide potential sources both for the content and organization of various mathematical curricula. Further, because of the priority given to children's activities as an occasioning source for constructivist models of mathematical activity, the very actions of constructivist teachers in listening to, in questioning, and in modeling children's structures as well as in providing spaces for children's mathematical activity provide provocative examples for the practice of mathematics teaching. Given such potential, it is perhaps not surprising that influences of constructivist approaches to mathematical learning and teaching are apparent in both the curriculum and evaluation and the teaching standards of the National Council of Teachers of Mathematics (1989, 1991).

Although by no means the only journal or venue for constructivist thought in the 1980s and the 1990s, the *JRME* proved supportive to radical constructivism in that it published theoretical and research papers by its key exponents as well as preconstructivist and related papers on mechanisms for mathematics knowing over the last 15 years. As well as reviewing numerous recent hooks related to constructivism in mathematics education, the *JRME* has served as a continuing forum for debate and criticism on both theoretical and research issues related to constructivism in mathematics education.

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