

Constructive Resources for Algebraic Reasoning:
Middle School Students' Construction of Fraction Composition Schemes
Amy J. Hackenberg, ahackenb@uga.edu
Erik S. Tillema, eriktil@uga.edu

Purpose

The purpose of our year-long constructivist teaching experiment was to understand how middle school students can construct algebraic reasoning out of their evolving quantitative reasoning. We taught four pairs of sixth graders at a rural middle school in Georgia from October 2003 to May 2004. One area of investigation was the students' multiplicative reasoning with fractions, including their construction and use of fraction composition schemes (Steffe, 2003). A fraction composition scheme is the way of operating used by a student to produce the part of a fractional whole that is constituted by, say, one-third of two-fifths of the whole. The purpose of this paper is to analyze how two pairs of students constructed and used fraction composition schemes during the experiment. We focus on these schemes because they are crucial in beginning to reason algebraically: We argue that such schemes are a key resource in the construction and solution of basic linear equations of the form $ax = b$, and also that these schemes are themselves in the province of algebraic reasoning.

Theoretical Framework

Mathematical Learning and Quantitative Reasoning

Following Piaget (1970) and von Glasersfeld (1995), as well as scholars who rely on them (e.g., Confrey, 1995; Steffe, 2002; Thompson, 1994), we view learning as the process by which a person makes modifications and reorganizations in her or his ways of operating in response to perturbations brought about by these current ways of operating. Perturbations are awakenings or disturbances that a person experiences when, for example, her or his ways of operating bring about an unexpected result. Eliminating perturbations in one's experiential

environment opens possibilities for learning, because eliminating perturbations often requires adapting one's current ways of operating or constructing new ways of operating. Although perturbations are in some sense "controlled" or determined by a person's current conceptual structures and expectations, they are occasioned (Kieren, 2000) by interaction of various kinds: among people, between people and objects, and of ideas within a person.

This view of learning is consistent with an *ontogenetic* approach to doing research on learning, in which researchers attempt to understand how students' mathematical ways of operating grow from modifications and reorganizations students make in their previous ways of operating during mathematical interaction with others (cf. Steffe & Thompson, 2000). Researchers who take ontogenetic approaches investigate broad, basic tendencies in students' spontaneous activity that observers might call mathematical—such as counting and measuring activity (e.g., Steffe, 1994; Steffe & Tzur, 1994). These researchers seek to explain how such activity can be brought forth in interaction with teachers and other students as a basis for engendering further mathematical activity, such as multiplicative reasoning with whole numbers (e.g., Steffe, 1988, 1992), fractional reasoning (e.g., Olive & Steffe, 1999, Steffe, 2002, 2003), and conceptions of ratios and rates (e.g., Thompson, 1995; Thompson, A. G., & Thompson, P. W., 1996; Thompson, P. W., & Thompson, A. G., 1994).

In our study, we conceived of students' algebraic reasoning to be rooted in their ways of reasoning with known and unknown quantities. Our view of *quantitative operations* (Piaget, 1970) includes actions performed mentally in building and analyzing relationships between two known quantities or between a known and an unknown quantity. Operations are components of *schemes*, goal-directed ways of operating that include a situation, activity, and result. In our view, *quantitative reasoning* is the purposeful functioning of schemes in the context of quantities

and quantitative relationships. Reasoning quantitatively involves elaborating the relationships between unknown and known quantities with a goal of modeling these relationships and determining unknown quantities (cf. Thompson, 1993, 1994). Quantitative reasoning often entails multiplicative and fractional reasoning since relationships among quantities are often multiplicative and fractional: For example, my height is $\frac{5}{6}$ of your height (cf. Thompson & Saldanha, 2003). Quantitative reasoning can be used to solve problems that can be solved by basic linear equations of the form $ax = b$.

Algebraic Reasoning

Because of our ontogenetic approach, we do not take as given the use of basic linear equations such as $(\frac{2}{3})x = \frac{3}{4}$ to solve the following problem:

Task 1, the Box Problem: Two groups of students, the Cobras and Lizards, have a box-stacking contest. The Cobras' tower is $\frac{3}{4}$ of a decameter tall, and that's $\frac{2}{3}$ of the Lizards' tower. How tall is the Lizards' tower?

Instead, we are interested in how students construct a scheme to solve such a problem, become aware of quantitative operations and relationships implicit in the functioning of such a scheme, and symbolize these quantitative operations and relationships with drawings (or other private symbols), as well as with conventional (or publicly accepted) algebraic symbols.

Although there are no sharp boundaries between quantitative and algebraic reasoning, we believe that students' quantitative reasoning can become more algebraic by operating explicitly on unknowns, reasoning with reciprocal relationships, and reasoning with standard algebraic notation. By drawing and operating on diagrams that represent unknown quantities as well as known quantities, students move from operating solely on knowns (to produce unknowns) to operating on both. For example, students may draw a picture of both the known and unknown

heights in the Box Problem, creating and operating upon the unknown height in relation to the known. In doing so, students may also transition from operating in “one” direction with multiplicative relationships (from known to unknown) to operating with bi-directional (reciprocal) relationships between known and unknown quantities. For example, students may conceive of the height of the Lizards’ tower as $\frac{3}{2}$ of the height of the Cobras’ tower, based on the given multiplicative relationship. This characteristic underscores the importance of multiplicative, fractional reasoning in the construction of algebraic reasoning. Finally, of course, a well-known hallmark of algebraic reasoning entails operating on conventional algebraic symbols in lieu of operating on other representations like drawings.

The Box Problem is just one example of a problem that can be solved by writing and solving a linear equation of the form $ax = b$. From our point of view, such problems are statements of division. We call them reversible multiplicative relationship (RMR) problems to reflect aspects of quantitative reasoning that we believe are entailed in solving them. RMR problems can involve relatively “simple” statements of whole number division, such as:

Task 2, Juice Problem: Twenty-eight ounces of juice is four times the amount that you drank; how much did you drink?

Middle school students may be unlikely to see any need to use a linear equation to solve such problems, since they can rely on their whole number multiplication knowledge to do so. Furthermore, problems like the Juice Problem do not seem to involve fractions overtly.¹ But to solve many RMR problems that involve fractional relationships and fractional or whole number quantities, such as the Box Problem, composing fractions multiplicatively is critical.

¹ One could argue that multiplicative, fractional reasoning is in fact *not* implicit in solving the Juice Problem, since taking $\frac{1}{4}$ of 28 involves fractional reasoning. Although students may solve the problem without using their fraction schemes, they may also assimilate the problem to their fraction schemes in conceiving of $\frac{1}{4}$ of 28 as 7 ounces.

Fraction Composition as Algebraic

A fraction composition scheme is involved in solving RMR problems like the Box Problem because students need to form a goal to find one-half of $\frac{3}{4}$ of a decameter. That is, since $\frac{2}{3}$ of the unknown height is a known quantity ($\frac{3}{4}$ of a decameter), determining half of the known height will be $\frac{1}{3}$ of the unknown height. So finding half of the known height is crucial. To form this goal seems to require an understanding of equivalence (that $\frac{2}{3}$ of the unknown quantity is identical to $\frac{3}{4}$ of a decameter) and being able to reverse making fractions (that to make $\frac{3}{3}$ of a quantity involves finding $\frac{1}{3}$ of it and iterating that amount 3 times). Enacting the goal of finding half of the known height transforms the initial relationship between the two quantities into a state that produces the solution. Thus we believe doing so is a fundamental “tool” in the solution of RMR problems, and in the construction of linear equations. Our experience with sixth grade students (as well as with pre-service and in-service teachers) indicates that enacting this goal is a significant achievement.

Enacting a fraction composition scheme entails partitioning a fractional quantity recursively (cf. Steffe, 2003). That is, to find $\frac{1}{2}$ of $\frac{3}{4}$ of a decameter, students may form a goal to find $\frac{1}{2}$ of each $\frac{1}{4}$ decameter and partition all the fourths into two equal parts—even the “missing” fourth (they may perform these operations solely mentally or also materially). In so doing, they may see that 8 little such pieces make up the entire decameter. Being able to determine that $\frac{1}{2}$ of $\frac{1}{4}$ of a decameter is $\frac{1}{8}$ of a decameter in this way is an example of a *unit* fraction composition scheme. In this scheme, students effectively use “halving” recursively on the “fourthing” they performed on the 1 decameter (to make the original fourths of a decameter). So the activity of the scheme can be seen as a composition of fractions-as-functions: One-half becomes a “halving function” that operates on $\frac{1}{4}$, which in turn was created by a “fourthing

function” operating on 1 decameter (cf. Steffe, 2001). Note that this recursive activity is only the first “tier” of activity necessary to accomplish the overarching goal, finding $1/2$ of $3/4$ of a decameter. That is, to take one-half of *three-fourths* requires taking one-half of *each* of the three one-fourths—and recognizing that this result is one half of *all* three fourths. This activity is clearly distributive. For these reasons, we view the construction of a fraction composition scheme *itself* as in the realm of algebraic reasoning—as a kernel of function composition that involves distributivity.

Methodology and Methods

Learning from Students and Conceptual Analysis

Ontogenetic approaches to research on mathematical learning are frequently linked to constructive teaching experiment methodology in which teacher-researchers use teaching as a method to engender and explain students’ learning (Steffe & Thompson, 2000). In a constructivist teaching experiment, teacher-researchers are interested in justifying the mathematical constructions that students make, not in justifying their own first-order mathematical knowledge or some *a priori* notions of mathematics. Teacher-researchers “remain aware that we may not, and probably cannot, account for students’ mathematics using our own mathematical concepts and operations” (p. 268). Instead, each teacher-researcher needs to learn new mathematics to understand students’ mathematics, attempting “to put aside his or her own concepts and operations and not insist that the students learn what he or she knows” (p. 274). To this end, a teacher-researcher acts responsively and intuitively in learning to think like her students—in merging with the students’ experiences to the extent that is possible (Leslie P. Steffe, personal communication, April 25, 2002) and in giving students’ mathematical ways of operating an independent “life” (Cobb & Steffe, 1983).

Simultaneously, teacher-researchers engage in conceptual analysis of how students *might* operate in the context of mathematical interactions. This conceptual analysis is based on teacher-researchers' mathematical knowledge, previous interactions with these students, and interactions with previous students. Out of the interplay between learning mathematics from students and engaging in conceptual analysis, teacher-researchers design and modify problem situations, formulate and test conjectures, and develop models of students' mathematical ways of operating. As teacher-researchers refine their models of the students, they can increasingly tailor problem situations to these students' ways of operating. Teaching practices include presenting students with problem situations, assessing students' responses as indications of students' current schemes and operations, and determining new problem situations that might allow students to construct potentially more powerful schemes and operations.

On-going and Retrospective Analysis

At least one witness-researcher, a critical component of teaching experiment methodology, is present during each teaching episode in a constructivist teaching experiment. The witness-researcher assists in videotaping the episodes with two cameras: One camera captures the interaction between the teacher-researcher and the pair of students, while the other camera focuses on the students' computer or written work. However, the more important role of a witness-researcher is to provide alternative perspectives during the actual episodes, in on-going analysis that occurs between episodes, and in retrospective analysis that occurs after teaching episodes have ceased. This triangulation of interpretations during analysis is critical in the establishment of viable explanatory models of students' activity.

Central activities of on-going analysis involve making local conjectures about students' current ways of operating, designing new problems and problem sequences for the next teaching

episode, and constructing *working* (often intuitive) models of students' mathematical ways of operating. The central activity of retrospective analysis is to refine working models into analytic ones. To this end, retrospective analysis usually starts with reviewing videotaped records to create chronologies of the path the teacher-researcher and students took through the experiment (cf. Cobb & Whitenack, 1996). In this process teacher-researchers look for consistencies and significant shifts in students' ways of operating, refine and discard conjectures, and search for both corroborating and disconfirming evidence in relation to evolving explanatory constructs. Looking across many episodes with the same student, and looking across many students' ways of operating, either in actual interaction or from previous research, are key tools a teacher-researcher has for constructing reliable, viable analytic models (cf. Clement, 2000).

Our Teaching Experiment

During September and October of 2003, we selected 8 sixth grade students out of a pool of approximately 100 students, all of whom had the same classroom mathematics teacher. We observed four of this teacher's five mathematics periods, which were organized by level of achievement, and consulted with this teacher to identify students for 20-minute selection interviews that were neither audiotaped nor videotaped. We then conducted interviews with 20 students. Our main goals were to select students who demonstrated a range of multiplicative reasoning and school achievement, who would be willing to communicate verbally with the research team, who were relatively stable in the school (i.e., were not planning to move soon from the district), and who had relatively strong attendance records. We also selected an equal number of girls and boys, although in this paper we report only on some of the girls' activity.

At the start of the teaching experiment, Sara and Amber, one pair of girls, were just beginning to reason multiplicatively in that they consistently coordinated at most two levels of

units prior to activity (cf. Steffe, 1988). For example, Sara and Amber could solve problems like this one:

Task 3, Splitting Problem: This stick represents my string and it's 5 times longer than yours. Draw your string.

Solving the Splitting Problem entails reversible reasoning—students need to posit their string in relation to the given string. Doing so seems to require using at least two levels of units, because the given string has to be partitioned into a unit of five equal units, any of which can be iterated five times to make the given string. Being able to solve this kind of problem underlies solving problems like the Juice Problem, Task 2.

Deborah and Bridget, another pair of girls, were coordinating *at least* two levels of units prior to activity at the start of the teaching experiment. So they could solve problems like the Splitting Problem, but they could also solve problems like this one:

Task 4, Money Problem: Tanya has \$16, which is $\frac{4}{5}$ of David's money. How much money does David have?

Solving the Money Problem requires more advanced operations than solving the Splitting Problem because of the explicit involvement of fractions. That is, students have to conceive of $\frac{4}{5}$ as four times $\frac{1}{5}$ in order to form a goal to split the \$16 into four equal parts, each of which is $\frac{1}{5}$ of David's amount of money. So the given amount, \$16, has to be conceived of as a unit that can be partitioned into four equal units, any of which can be iterated five times to make the whole (David's money), which may itself be taken as a unit. Note that the Money Problem is an RMR problem that we view as more complex than the Juice Problem but less complex than the Box Problem.

Teaching episodes were thirty minutes long and occurred biweekly during school hours for two to three weeks, followed by a week off. Thus the full data set for the experiment consisted of videotapes from 60 to 70 teaching episodes for each pair of students from October 30, 2003 to May 12, 2004. The two videotapes from each teaching episode were mixed electronically into a single video file where the video of the computer or written work was inset into the video of the interaction between teacher-researcher and students. Most teaching episodes involved the use of computer software—TIMA: Sticks and Javabars. These programs allow students to draw line segments (sticks) or rectangles (bars), respectively, and operate upon these “wholes” in various ways: partition the whole into an equal number of parts, partition partitions, break a whole into its parts, pull a part out of a whole without destroying the whole, copy and join parts, color parts or wholes, and repeat parts so as to join them together in succession.

Results and Discussion

Sara and Amber

At the beginning of the teaching experiment, Sara and Amber had not constructed a *unit* fraction composition scheme. So in this section we examine Sara and Amber’s activity in situations intended to engender this construction. Through contrasting the two students’ ways of operating in these situations, we highlight how distributive activity is involved in students’ construction of a unit fraction composition scheme, and how in the activity of such a scheme we can see a kernel of function composition. In addition, we examine the close relationship between students’ multiplicative structures and the distributive nature of their mathematical activity.

The Cake Problem

Amber’s activity. Sara was absent during the teaching episode on February 11th, which was the first time Amber encountered a problem intended to provoke the construction of a unit

fraction composition scheme. Amber had already made a “cake” (a stick) and had partitioned it into 15 equal parts using the buttons in the TIMA: Sticks microworld. Then the teacher-researcher posed this problem:

Cake Problem, Task 5: At a party, you and fourteen people share a cake fairly. Then you share your piece of cake fairly with a latecomer. How much of the cake does the latecomer get?

A witness-researcher asked Amber to pull out one part—one piece of the cake—and to share that piece with him. Amber pulled out one part from the cake (1/15-stick) and partitioned it into two equal parts (see Figure 1).



Figure 1, One-fifteenth (1/15-stick) disembedded from the cake and shared with two people.

Protocol I: Amber’s solution of the Cake Problem on 2/11/04.²

W: Can you pull out my share? [A pulls out one-half of the 1/15-stick.] I wonder how much of that cake I have?

A: Let’s see that’s... [looks off into space for approximately 25 seconds]. How much of it is yours from like this? [A uses the mouse to point to the 1/15-stick.]

W [attempts to redirect A’s statement of the question]: That’s my piece [referring to half of the 1/15-stick], right? I wonder how much of the whole cake I have?

A: Okay mine was one fifteenth, so you would have half of one fifteenth. So it would be...[continues to think for 15 seconds, indicating that she believes there may be another fractional name for the part]. I am not sure.

T: Could you use his piece to figure it out?

A [drags the half of the 1/15-stick next to the whole cake]: Well you could fit two of his pieces in one, so...[subvocally utters fifteen times two is thirty]...

T: I think I heard her say it. Do you remember what you said?

A [isn’t sure what T is referring to]: Oh, I said fifteen times two.

² In this protocol, A stands for Amber, S for Sara, T for the teacher-researcher (one of the authors), and W for a witness-researcher. Comments enclosed in brackets describe students’ non-verbal action or interaction from the teacher-researcher’s perspective. Ellipses (...) are used to indicate a sentence or idea that seems to trail off. Four periods (....) are used to indicate omitted dialogue.

- T: Oh and why would it be fifteen times two?
 A: Because two of his pieces can fit into one fifteenth and there's fifteen pieces in the whole cake so...
 W: So how much would it be?
 A: It'd be... [stares at the ceiling] thirty.
 T: Thirty of his would fit in the whole cake? Is that what...so how much of—
 W: So how much of the whole thing is my piece?
 A: Okay so thirty of his pieces would fit in the whole cake, so that would be thirty fifteenths. [She wrinkles her forehead and looks at T.] No I don't think that's right.
 T: How many pieces has he got?
 A: He has one.
 W: And how many little pieces in the whole cake?
 A: Thirty.
 W: You got it solved. It's thirty little pieces, then what would one little piece be?
 A: One thirtieth.

In order to solve the Cake Problem, we conjecture that Amber mentally partitioned *each* of the fifteenths into two equal parts. She contributed this activity independently—that is, she engaged in partitioning a partition to serve a non-partitioning goal (determining the size of the smallest part in relation to the unit bar). In conceiving of the situation in this way, Amber's multiplying schemes were activated and she knew that she needed to multiply two by fifteen in order to find out how many parts were in the whole cake. Steffe (2002, 2003) has referred to such activity as *recursive partitioning* and has discussed its importance in the construction of a unit fraction composition scheme. He has also commented that in engaging in this activity, he sees the echoes of the composition of two functional processes (2001). We concur, but we would like to emphasize the distributive, not just the recursive, nature of Amber's activity. That is, Amber effectively enacted “halving” on the “fifteenthing” she had first performed on the cake in an effort to determine the size of the smallest part. In this sense she distributed halving across the fifteen parts of the cake. We see a kernel of a distributive operation in her activity, although we don't claim that she was aware of a distributive pattern in her activity.

As is seen in Protocol I, Amber needed support from the witness-researcher and the teacher-researcher in order to arrive at the final answer of one-thirtieth. However, because of Amber's independent use of recursive partitioning in the situation, we inferred that the situation had significant meaning for her. Furthermore, Amber seemed to remain aware of the three levels of units in the situation (Steffe, 1988); that is, she was aware of the whole cake as a unit that was partitioned into fifteen units, each of which she had mentally partitioned into two units. Because she was aware of all three levels of units in the situation, she was able to compare half of the one-fifteenth piece to the whole cake by figuring out how many such parts made up the whole cake. Amber's coordination of the three levels of units in her *activity* in the Cake Problem is what we refer to as *experientially* creating three levels of units. Our current conjecture is that being able to conceive of the cake as a unit of fifteen units each containing two units *during her activity* was an essential basis for her ability to view one-half of one-fifteenth and one-thirtieth as the same amount. Her subsequent activity with similar problems during March and April indicates that her solution to the Cake Problem marked a relatively permanent way of operating with fraction composition—i.e., the initial construction of a unit fraction composition scheme.

Sara's activity. Because Sara was not present at the episode when Amber solved the Cake Problem, the same problem was posed to Sara during the next teaching episode on February 18th. In contrast to Amber's way of operating in the situation, Sara did not seem aware that there was a name for the smallest part besides "one-half of one-fifteenth." That is, when Sara partitioned one of the fifteen parts of the cake into two equal parts, she did not mentally partition all of the fifteenths into two equal parts, and did not create a unit composed of fifteen units, each containing two units. Rather, Sara appeared more experimental in her solution. She repeated her piece of cake (the smallest part) until she had made a cake the same size as the original cake.

After doing so she counted the number of parts in this “new” cake and determined that, since there were 30 parts, each part was one thirtieth of the original cake.³

Sara’s mathematical activity was certainly creative! But it indicated that she did not see the relevance of multiplying fifteen by two in the situation. Furthermore, she did not relate the result of producing thirty parts to the initial situation of having one of fifteen parts partitioned into two equal parts. So her multiplying scheme did not seem to be activated. Although she could say that the smallest part was one thirtieth of the cake, she did not find this result by using multiplication, which indicates to us that she did not insert units of two into units of fifteen. This lack of inserting units into other units means that Sara’s way of operating to determine half of one-fifteenth involved neither the notion of composition of processes nor a distributive aspect similar to Amber’s ways of operating. In short, solving the Cake Problem did not appear to be a basis for Sara’s construction of a unit fraction composition scheme.

Sara’s Sequential Use of Schemes

Because Sara solved the Cake Problem without using her multiplying scheme, the teacher-researcher posed a new problem to her during the same teaching episode on February 18th:

Sub Problem, Task 6: Can you share this sub sandwich among 17 people? Now *each* of the people shares their piece with three people. Could you figure out how much one little piece is of the whole sandwich?

Sara immediately partitioned each of the seventeen parts into three equal parts. Then she said she didn’t know how many pieces were in the whole sandwich. The teacher-researcher encouraged her to try, asking if she could figure it out if he gave her a piece of paper. She nodded and

³ To make this solution, Sara relied heavily on Amber’s suggestions.

immediately wrote down seventeen times three in a vertical format, computing the result with her standard computational algorithm for whole number multiplication. She said there would be fifty-one pieces in the sub sandwich and that one piece would be “one fifty-oneth.”

In contrast to her activity on the Cake Problem, Sara seemed to assimilate the Sub Problem to her multiplying scheme. We infer that by partitioning each of the seventeen parts of the sandwich into three equal parts, Sara knew that she had made seventeen three's. For her, finding the resulting number of parts was accomplished by multiplying seventeen and three using her computational algorithm for whole number multiplication. Then, after she had figured out the total number of parts, Sara certainly established the sandwich as a unit of fifty-one units, and she used her fraction scheme to determine the size of one of the parts in relation to the whole. However, a clear difference between the Sub Problem and the Cake Problem is that the former indicates that partitioning each part of the initial partition should occur. So we cannot infer that Sara engaged in recursive partitioning as Amber had—in fact, partitioning a partition in service of a non-partitioning goal seemed unavailable to Sara for much of the teaching experiment.

In addition, based on Sara's work in both the Cake and Sub Problems, we cannot infer that Sara “saw” the sandwich as a unit of seventeen units each containing three units. In fact, Sara's solutions to the Cake and Sub Problems prompted us to conjecture that Sara was coordinating only two levels of units at that time. That is, she could first work with seventeen as a unit of seventeen units. Instead of counting by ones to seventeen, she could think of each “one” as a “three” and keep track of them until she had counted seventeen three's. Furthermore, this situation was multiplicative for her, and she knew that her computational algorithm for whole number multiplication would allow her to find the result. Once she found the result, fifty-one, she could then conceive of the sub sandwich as a unit that contained fifty-one units, but in doing

so we cannot infer that she maintained a conception of the sandwich as a unit of seventeen units, each containing units of three. It was as if she could coordinate two levels of units at two different times, but could not hold the two structures together.

So, in operating in situations of fraction composition, determining the number of parts in the whole and identifying the fractional size of the part in relation to the whole appeared to be separate but associated problems for Sara. To solve them Sara seemed to use her multiplying and fraction schemes *sequentially*, rather than embed her multiplying scheme into her fraction scheme as Amber had done. Because she could not yet make this kind of coordination, she seemed unable to construct a unit fraction composition scheme at this time.

Amber's Progress

Taking a non-unit fraction of a unit fraction. During the rest of February and early March, the teacher-researcher investigated whether Amber could extend her unit fraction composition scheme into taking any proper fractional amount of a unit fraction. Samples of the problems the teacher-researcher posed to Amber are the following:

Task 7: You are at a party and a cake is cut into nine pieces. Two people show up to the party late and you decide to share your piece of the cake with them. What fraction of the whole cake do the latecomers get together?

Task 8: Can you make $\frac{2}{5}$ of $\frac{1}{3}$ of the cake? How much is that of the whole cake?

Amber solved these problems by partitioning the cake into fractional parts, partitioning one fractional part into the required number of “mini-parts,” pulling out the number of these mini-parts requested in the problem, and identifying the fractional size of the result in relation to the whole cake. For instance in Task 7, Amber partitioned the first ninth of the cake into three parts, pulled out two of them, determined that each one was one twenty-seventh of the original

cake, and therefore concluded that the latecomers would get two twenty-sevenths of the whole cake. Although Task 7 and Task 8 do not require significantly different ways of operating, we posed Task 8 to engender Amber's awareness of making a fraction composition. We believed this goal could be achieved through the explicit use of fraction language. We conjecture that the explicit use of fraction language could contribute to Amber becoming more aware of using fractions like two-fifths as an operation. That is, using her concept of two-fifths on quantities other than a whole meant two-fifths had an operational meaning as opposed to solely the meaning of a resulting quantity.

Taking a unit fraction of a non-unit fraction. The teacher-researcher wanted to investigate whether Amber could construct a more general fraction composition scheme by asking her to take a unit fraction of a non-unit fraction (e.g. take $1/5$ of $2/3$). Such problems are more challenging than Tasks 7 and 8 because they require students to take a unit fractional part of a quantity consisting of multiple fractional parts. Thus such problems require students to distribute their concept of taking a unit fraction, like one-fifth, across more than one fractional part, like two-thirds. Using their unit fraction composition scheme, students also need to keep track of the relationship of the "mini-parts" (one-fifth of one-third) to the whole in order to find the resulting fractional part of the whole. So on March 10th, a month after Amber solved the Cake Problem (Task 5), the teacher-researcher posed this problem:

Task 9: Can you make $1/3$ of $3/4$ of that cake and find out how much of the whole cake that is?

Amber solved this problem by partitioning each of the three fourths into three equal parts and taking one part from each of the three fourths. She then named the resulting fraction three twelfths of the whole cake. In solving this problem, she appeared to distribute her concept of

“taking one-third” across each of the three fourths in order to find one third of *all* three of the fourths. Furthermore, we infer that she used her unit fraction composition scheme in determining the size of the smallest parts in relation to the whole cake. Although here again we don’t claim that Amber was aware of the distributive pattern of her activity, we do see her distributive activity here as an advance over her previous ways of operating with fraction composition situations. Amber’s solution to this problem was novel for her, but her difficulty with similar such problems in subsequent episodes indicates that she had not yet constructed a general scheme for taking a unit fraction of a non-unit fraction.

Deborah and Bridget

At the start of the teaching experiment, both Bridget and Deborah, another pair of students, could solve problems like the Cake Problem (Task 5) and like Tasks 7 and 8. That is, both had constructed at least a unit fraction composition scheme. We have already argued that such schemes can be considered in the realm of algebraic reasoning for two reasons: because we can see the kernel of a composition of functions in implementing a unit fraction composition scheme, and because distributive activity is implicated in the recursive partitioning necessary to enact such a scheme. In this section we demonstrate our second major point about fraction composition schemes—that they are a critical tool in solving reversible multiplicative relationships (RMR) problems. In doing so, we elaborate on the distributive activity involved in making fraction compositions to further demonstrate how such schemes may be considered in the province of algebra.

The Box Problem

The teaching episode on February 18th was the first time both Bridget and Deborah encountered a complex RMR problem, involving a fractional quantity and a fractional

relationship. This episode was the last of seven (i.e., approximately a month's time, from mid-January to mid-February) during which Deborah and Bridget had worked on operating with unknown quantities and notating their operations using paper-and-pencil drawings and algebraic symbols. Our main goal at the start of the episode was to generate a need to notate operations. Toward this end the teacher-researcher engaged the girls in problems with different measurement units, such as finding their heights, and eventually any person's height, in inches, centimeters, millimeters, and meters. Near the end of the episode the teacher-researcher posed The Box Problem, Task 1:

Task 1, the Box Problem: Two groups of students, the Cobras and Lizards, have a box-stacking contest. The Cobras' tower is $\frac{3}{4}$ of a decameter tall, and that's $\frac{2}{3}$ of the Lizards' tower. How tall is the Lizards' tower?

Deborah's response. Deborah, in part because of the earlier work with different measurement units, immediately wanted to convert decameters to meters. But the teacher-researcher indicated that they could just use decameters for this problem. Deborah stated flatly that she couldn't figure it out, while Bridget drew a picture to represent the situation. Although the teacher-researcher suggested twice to Deborah that drawing a picture might help, she persisted in trying to work computationally and didn't seem to generate her own "private" imagery on which to operate. In general, Deborah did not like to draw pictures (especially when she thought she already knew the answer to a problem). She seemed to enjoy calculating, and her calculational facility with whole numbers was quite sophisticated: She had embedded a

distributive operation in her multiplying schemes with whole numbers so as to regularly and impressively reason strategically.⁴

From such activity we infer that she likely had a rich set of images when operating with whole numbers, but they were probably entirely implicit in her thinking. Her response to the Box Problem indicates that she had not yet developed that level of sophistication—or private imagery—with her fractional calculation (and did not know the algorithm for fraction division, or if she did, she did not recognize the Box Problem as a potential situation of fraction division). Over the approximately five minutes we spent on the problem she appeared to be in a state of perturbation that was consciously conflictive and quite uncomfortable, and she did not find a way of operating to eliminate the perturbation.

Deborah had entered the teaching experiment with sophisticated fraction schemes and operations as shown by her ability to solve Task 4, the Money Problem at the start of the experiment. Furthermore, she modified her schemes and operations swiftly during our interactions. Yet in working on the Box Problem, these ways of operating seemed almost entirely blocked or suppressed, and she was unable to initiate activity that satisfied her, let alone that solved the problem. Seeing Bridget achieve some success with the problem via reasoning with a drawing may have only exacerbated Deborah’s relative paralysis in the situation and persistence in *not* drawing.

Bridget’s response. Unlike Deborah, Bridget seemed to find drawing pictures useful, and she independently drew a picture right after the teacher-researcher posed the Box Problem. To represent the Cobras’ height,⁵ $\frac{3}{4}$ of a decameter, Bridget drew a rectangle partitioned

⁴ For example, to determine 78 divided by 13, Deborah reasoned that 7 would be too big because 7 times 13 was 70 + 21. She determined that the answer was 6, because 6 times 13 was 60 + 18 = 78.

⁵ I use “Cobras’ height” to refer to the height of the Cobras’ tower, and similarly for the Lizards.

horizontally into three equal parts. She stated that the other tower was bigger because the Cobras' height was two-thirds of the Lizards' height. Then she partitioned the middle of the three equal parts of the Cobras' height into two equal parts and crossed out the lines that indicated the fourths of a decameter. To represent the Lizards' height she drew another rectangle consisting of three rectangles, each equal to half of the Cobras' height (see Figure 2).

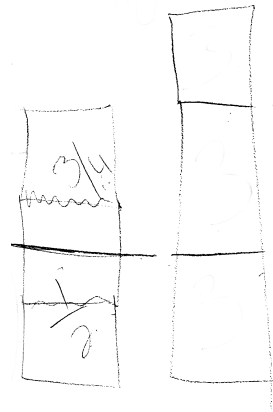


Figure 2, Bridget's drawing of the Cobras' height ($\frac{3}{4}$ dm), which was two thirds of the Lizards' height.

Upon questioning, she articulated that one of the two equal parts of the Cobras' height was one-third of the Lizards' height. We infer that she used her reversible fraction scheme to draw the height of the Lizards' tower in relation to the height of the Cobras' tower.

The teacher-researcher then challenged her to find the Lizards' height. She said that it was one decameter "because you've got one extra fourth." But immediately she thought that was wrong, and then she seemed stumped. The teacher-researcher tried to focus her on the three fourths of a decameter and what she had done to it in making the Lizards' height. Then the teacher-researcher intervened more strongly by asking Bridget about half of the Cobras' height. Bridget responded that "you can't divide three fourths into half—well you can, but you'd get one

point five over two.” The teacher-researcher probed to see if Bridget could “figure it out as a fraction.” After a few moments, Bridget drew a rectangle to represent a whole decameter and partitioned it into fourths (see Figure 3).

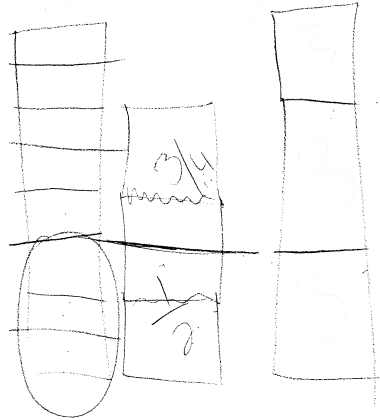


Figure 3, Bridget’s determination of half of three fourths of a decameter.

She then partitioned each fourth into two equal parts and said “three eighths.” Although Bridget’s response was impressive, simultaneously Deborah was experiencing considerable discomfort in not finding a way to act in the problem. So the teacher-researcher began to curtail the episode, complimenting both girls on a good start to a hard problem. “Wait a minute,” Bridget said excitedly, pointing to the Lizards’ height, “This is—this is nine eighths!”

Bridget’s work here is significant because she partitioned $\frac{4}{4}$ of a decameter recursively to determine half of the $\frac{3}{4}$ of a decameter, obviously with considerable interventions from the teacher-researcher. These interventions assisted her in forming a goal to find half of $\frac{3}{4}$ of a decameter, which she had indicated was relevant through how she made the Lizards’ height, but which she had not explicitly articulated. But she independently contributed the drawing of the decameter and the further partitioning of it to accomplish this goal. At the time of this episode, the research team inferred that although Bridget’s reversible fraction scheme was quite solid, she

had not yet constructed a fraction composition scheme that would allow her to enact her reversible fraction scheme in a situation like the Box Problem.

Distributive Reasoning

Because Bridget had difficulty determining half of $\frac{3}{4}$ of a decameter and Deborah seemed almost entirely blocked from operating in the Box Problem, during the next teaching episode on March 1st we returned to the computer (using JavaBars) to work on fraction composition problems. Making fraction compositions seemed necessary for solving problems like the Box Problem, but we soon came to understand their algebraic significance with respect to distributive reasoning.

Bridget's coordination of whole numbers of parts. When making fraction compositions, Bridget tended to coordinate whole numbers of parts of bars. Her solution to a problem near the end of the March 3rd episode exemplifies these characteristics.

Task 10: I need $\frac{3}{4}$ of a yard of ribbon to tie up a package. My sister needs $\frac{2}{5}$ of my piece. Can you make it and tell how much she needs?

Bridget said the problem was easy. She copied the unit yard and colored three fourths of it. Then she expressed an idea about “dividing into fifteen.” After some discussion with Deborah and the teacher-researcher, she pulled out one fourth of the unit yard, used it to make three fourths, and partitioned each fourth into five equal parts (see Figure 4).

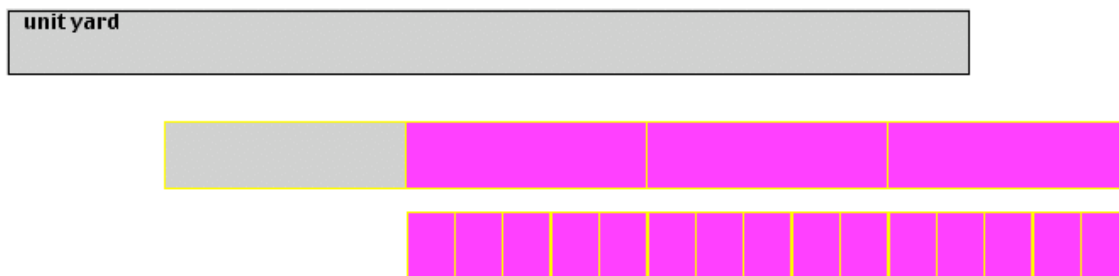


Figure 4, Each fourth of $\frac{3}{4}$ of a unit yard partitioned into five equal parts.

Protocol II: Bridget's coordination of whole numbers of parts on $3/3/04$.⁶

B: And then, that would be fifteen, three, she'd have six.

T: Six! Six what?

B: Six—[pauses and laughs] six fifteenths.

D [almost simultaneously with B]: Six twentieths.

B: Or twentieths or something.

D: Six twentieths of the whole bar, six fifteenths of yours.

B: Yeah. Six fifteenths of yours, that's what I mean.

T: Oh. Six fifteenths of mine. And why is it six twentieths of the whole bar?

B: Because they're all divided by five.

D [simultaneously with B]: Because you have to add five more. Because you didn't have the whole yard.

In solving Task 10, Bridget's intention to make 15 parts total in the bar representing $3/4$ of a yard allowed her to make 5 equal parts out of a 3-part bar because 15 could be divided by both 5 and 3. By intending to make 15 parts, we infer that she could insert 5 units into each of the 3 units of the $3/4$ -bar and then "regroup" so that the bar consisted of 5 units each containing 3 units. So she seemed to operate with three levels of units with respect to the 15-part bar: It was a unit of three units each containing five units, and she could reorganize it to see it as a unit of five units each containing three units.

However, Bridget did *not* seem to take the $3/4$ -bar as a unit in relation to the unit bar because she called the 15 parts she had made fifteenths. Even though a $4/4$ -bar was in her visual field, it seemed as if, at the moment, her "mathematical "world" was the $3/4$ -bar. She used her multiplying scheme in this world to find that one fifth was 3 parts out of 15, and two fifths was 6 parts out of 15. But for her these parts were not integrally related to the unit bar, which indicates she did not take the unit bar as a unit to which the $3/4$ -bar stood in relation (as a unit). Therefore, we cannot claim that she was operating with three levels of units *in relation to the unit bar* in her solution to Task 10. As a result, her scheme for making fraction compositions did not seem to be

⁶ In this protocol, D stands for Deborah, B for Bridget, and T for the teacher-researcher (one of the authors). Other conventions are as in Protocol I.

“fully fractional” in that she did not determine the length of the composition as an outgrowth of making it.

However, that Bridget made sense of, and agreed with, Deborah’s response of six twentieths indicates that Bridget could make three levels of units in activity, in reviewing the bar she had made. That is, Bridget *could* take the $\frac{3}{4}$ -bar as a unit in relation to the unit bar after she made the composition, and she could use her recursive partitioning operation to relate the small parts to the unit bar. So Bridget’s ways of operating with fraction composition seemed to be a “two step” process: make the composition by coordinating whole numbers of parts, and then determine the measure of it (in relation to the unit bar) as a “separate” problem. In this sense, Bridget’s ways of operating with fraction compositions seemed to be constrained by the levels of units she could coordinate prior to activity, and we cannot claim that at this point she had constructed a general fraction composition scheme.

Deborah’s distributive activity. Like Bridget, Deborah sometimes coordinated whole numbers of parts in making fraction compositions. In addition, perhaps partly as a result of her affinity for computation over making pictures, Deborah frequently made the composition after she had calculated the result, often by iterating individual parts. Doing so could hide the structure that she imputed to the situations. For example, on March 10th, Deborah worked on this problem:

Task 11: Make $\frac{11}{9}$ of a yard of ribbon. Your friend wants $\frac{4}{3}$ of that piece. Make it and tell how much your friend needs.

Deborah divided each of the eleven ninths into three equal parts. Then she pulled out one part and repeated it 44 times to make 44 parts. When the teacher-researcher asked about the size of one of the parts in relation to the unit yard, Bridget responded “one thirty-third.” Deborah insisted that $\frac{1}{33}$ was wrong. After five seconds, she stated that it was $\frac{1}{27}$, so the new piece of

ribbon was $\frac{44}{27}$ of a yard. In explanation of how she had made the composition, she said, “take one out of each of those boxes [the eleven ninths] and that equals eleven and that would be one third, so I multiplied that by four ‘cause she needs four.”

This explanation is significant for two reasons. First, Deborah indicated that she was (at least sometimes) using a distributive operation to make the unit fraction of the given fraction. To make $\frac{1}{3}$ of the entire $\frac{11}{9}$ she imagined taking $\frac{1}{3}$ from each of the $\frac{11}{9}$. So 11 parts was $\frac{1}{3}$ of the whole bar. This way of operating is pivotal in the construction of a general fraction composition scheme because it allows recursive partitioning to be embedded in the scheme for making the composition (i.e., one third of each ninth is $\frac{1}{27}$, so the entire 11 parts is $\frac{11}{27}$.) We conjecture that Deborah’s use of an explicit distributive operation in making $\frac{1}{3}$ of $\frac{11}{9}$ is a central reason why she emphatically rejected Bridget’s idea that each part was one thirty-third and concluded that each part was one twenty-seventh.

Second, Deborah used her iterative fraction scheme to make four thirds by iterating one-third four times, but she did not use the tools of JavaBars to reflect this construction (i.e., she did not repeat one part to make eleven parts and then repeat the eleven parts four times.) In fact, she did not often independently use the tools of JavaBars to demonstrate her “structural” ways of operating with fraction composition problems, perhaps because she found making the bars somewhat tedious—or perhaps because making the bars remained largely an illustration of the end *results* of her mental activity. Nevertheless, from Deborah’s explanation of her solution of Task 11 we infer that she had made an initial construction of a general fraction composition scheme.

Conclusion

A central conclusion of our research is that students' multiplicative structures—the extent to which they coordinate levels of units prior to activity—significantly influence their distributive activity with fractions. As shown in the contrast between Amber's and Sara's ways of operating in fraction composition situations, coordinating three levels of units in activity seems necessary for constructing a unit fraction composition scheme. That is, coordinating three levels of units seems necessary for inserting composite units, such as a unit of two, into a unit of units, such as a whole that has been partitioned to show fifteenths. This description summarizes our interpretation of Amber's mental activity in the Cake Problem (Task 5), and the result was that she had made a unit of fifteen units, each containing two units, which meant that the bar was *both* a unit of thirty units (the result of her multiplying scheme) *and* a unit of fifteen units each containing two units. In contrast, Sara could coordinate at most two levels of units, and so the second “view” of the bar that Amber had was not available to Sara. Thus Sara's activity did not yet include a distributive operation because she could not insert composite units into a unit of units and retain the structure of the number (or bar) as a unit of units of units. This constraint in her ways of operating is corroborated by her activity in the Sub Problem (Task 6).

Our central conclusion gains further support from the activity of Deborah and Bridget. Coordinating three levels of units *prior to activity* seems necessary for constructing a more general fraction composition scheme. In a fully general scheme, students may enact at least two “tiers” of distribution in making a composition and determining the result as an outgrowth of making it. That is, for Deborah to determine $\frac{4}{3}$ of $\frac{11}{9}$ (Task 11), she first took $\frac{1}{3}$ of $\frac{11}{9}$ by taking $\frac{1}{3}$ of each of the eleven ninths. Taking $\frac{1}{3}$ of $\frac{1}{9}$ already involves distributive activity through the use of recursive partitioning, as explained in Amber's activity with unit fraction composition. But Deborah also explicitly used distribution across all eleven ninths by taking $\frac{1}{3}$

of each of the eleven one-ninths in order to take $1/3$ of $11/9$. In some sense, Deborah was able to “keep in mind” the $9/9$ to which the $11/9$ stood in relation as she operated on all eleven parts. In contrast, Bridget did not seem to have both tiers of distributive activity available to her. She could coordinate three levels of units within a unit, as she did in taking $2/5$ of $3/4$ by partitioning each of the three equal parts (the three fourths) into five equal parts (see Task 10). However, she did not hold this unit structure as a unit in relation to the $4/4$ —it was as if, in making the three levels of units coordination within the $3/4$ -bar, Bridget lost the first tier of distributive activity involved in her unit fraction composition scheme.

Amber, Bridget, and Deborah, as a result of their multiplicative structures, all engaged in distributive activity.⁷ However, *within their respective problem situations*, Amber engaged in activity that was more distributive than Sara, and Deborah engaged in activity that was more distributive than Bridget. Another way of articulating this claim is that Amber and Deborah were involved in embedding their multiplying and fraction schemes within their respective problem situations. In contrast, Sara seemed to enacting her multiplying and fraction schemes sequentially. And Bridget, although she demonstrated sophisticated reasoning in making fraction compositions, did not coordinate both tiers of distributive activity in doing so—her unit fraction scheme did not seem to be embedded into reasoning with three levels of whole numbers of units. In short, there is a certain kind of “nesting” involved in Amber’s and Deborah’s activity that was not evident in Sara’s or Bridget’s activity in these respective contexts. This sense of nesting refers to embedding multiplying schemes within fraction schemes, but it can also refer to the resulting composition of fractions as operations on wholes or parts of wholes. We believe that

⁷ Recall that although it is not highlighted in this paper, Bridget had a solid unit fraction composition scheme and could solve all the problems that Amber solved.

this embedment of schemes is one source of students' activity that appears as a root of composing mathematical processes.

From the outset we have argued that fraction composition schemes are important in solving RMR problems—Bridget's solution of the Box Problem is evidence of this claim, although she made her solution under considerable coaching from the teacher-researcher. Still, Bridget's operations to solve the Box Problem are operations that could be used to solve $(2/3)x = 3/4$, where a key step is taking half of both the known quantity ($3/4$ dm) and the unknown quantity ($2/3$ of the other height). For us, this notion of fraction composition schemes as a crucial tool in solving RMR problems is sufficient reason for us to include them in our conception of algebraic reasoning for middle school students. However, in investigating students' construction of fraction composition schemes, we have found that because of their distributive and embedded nature, the schemes themselves can be considered in the realm of algebra.

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