

Prompt

Students are studying multiplying binomials (factoring and roots) $ax + b$ and $cx + d$. A student asks "What if we divide instead of multiply?"

Commentary

In our foci, we are assuming that we have a new function $f(x) = (ax + b)/(cx + d)$. We prove that this is a function within Mathematical Foci 3 (asymptotes). With this new function, we find that there are restrictions on the domain and range, unlike the function $f(x) = (ax + b)(cx + d)$. In this situation, we do not consider complex numbers. We see that the graph of this function is always a hyperbola, with the exception of the degenerative (or undefined) cases. To completely discuss hyperbolas, we found it important to discuss how to represent the roots and intercepts, graphically and algebraically. It is also necessary to show how the coefficients a , b , c , and d affect the graph. In the last two foci, we explain the limits and the inverse of the function.

Mathematical Foci

Mathematical Foci 1: Roots and Intercepts

1. X-intercepts/Roots:
 - a. Graphically: A root or x-intercept occurs when our function crosses the x-axis. We could also think of roots as the x-value such that $f(x)$ intersects the line $y=0$.
 - b. Algebraically: $\frac{ax+b}{cx+d} = 0$ when $ax + b = 0$. Therefore, our x-intercept/root is $x = \frac{-b}{a}$.
2. Y-intercepts:
 - a. Graphically: where does $f(x)$ cross the y-axis? When our x value is 0.
 - b. Algebraically: evaluate $f(x)$ at $x = 0$.

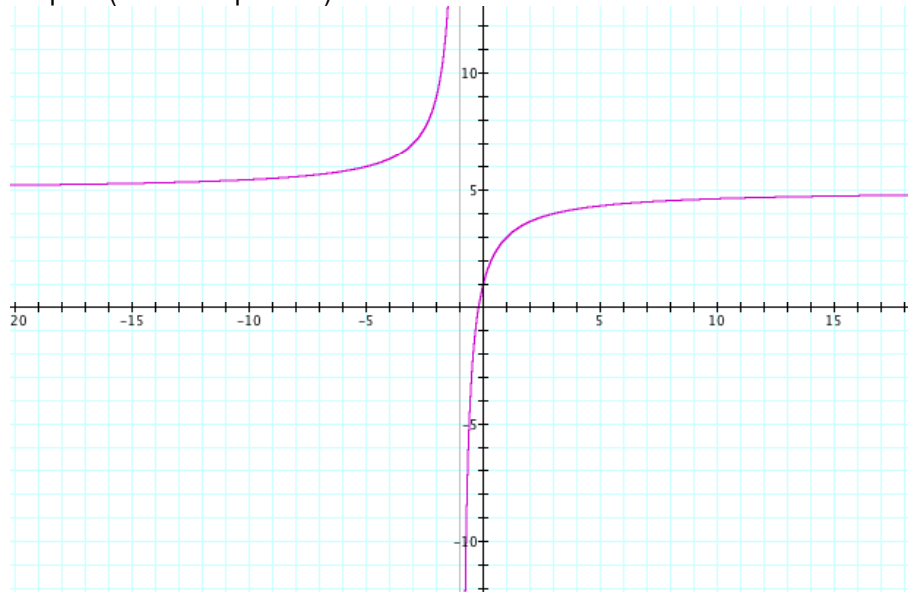
$$f(0) = \frac{a(0) + b}{c(0) + d} = \frac{b}{d}, d \neq 0.$$

Mathematical Foci 2: How the variables a , b , c , and d affect the graph

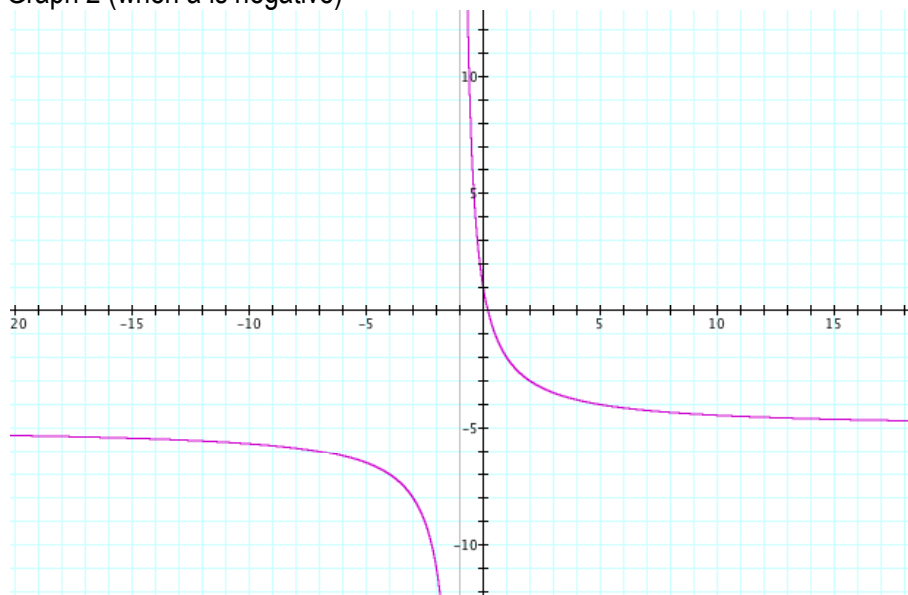
As "a" changes:

- Vertical Asymptote @ $x = -1$
- Horizontal Asymptote @ $y = a$
- The graph shifts horizontally, depending on the value of "a"

Graph 1 (when a is positive)



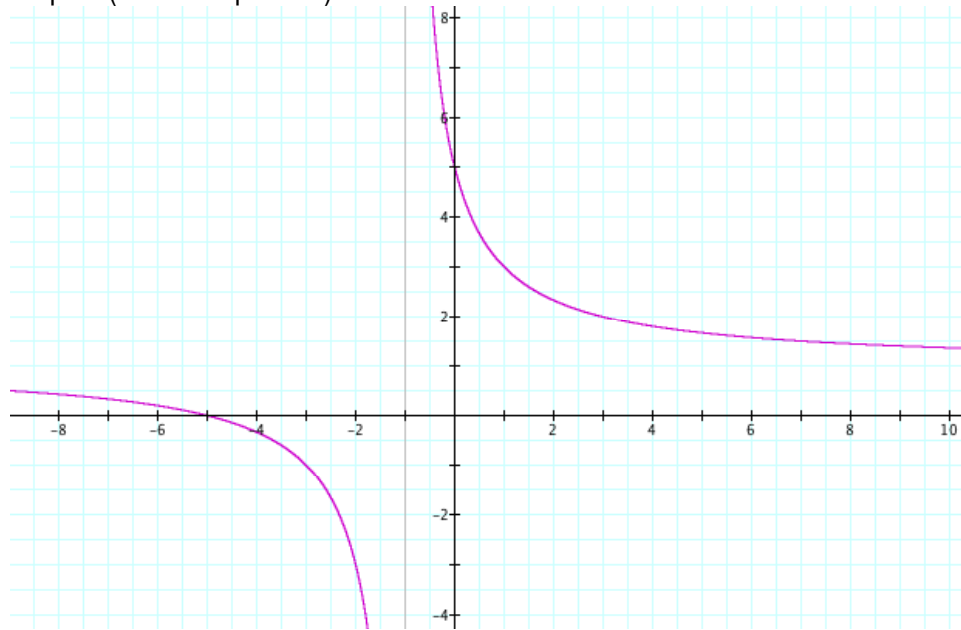
Graph 2 (when a is negative)



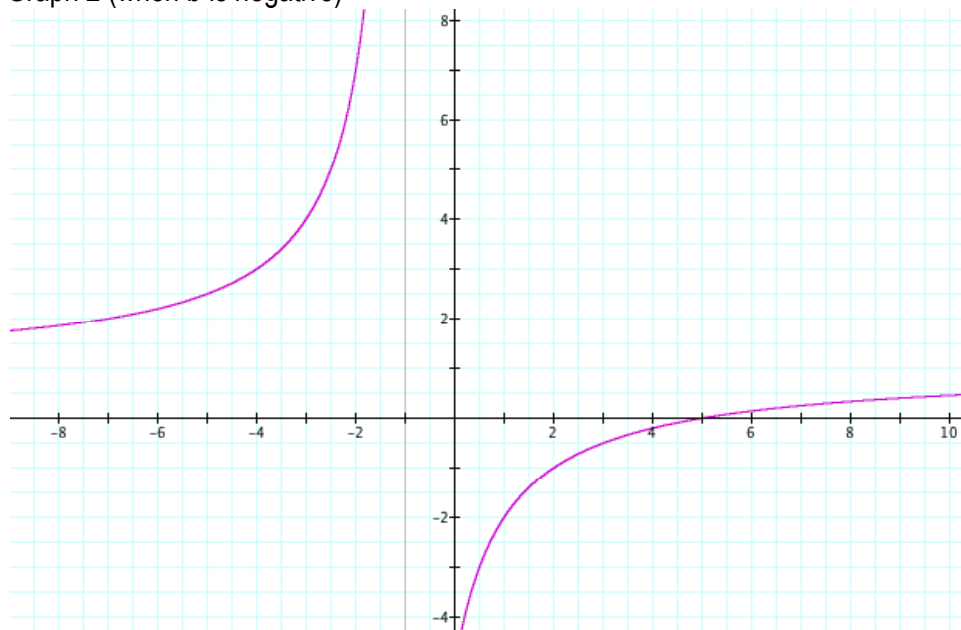
As "b" changes

- Asymptotes same as "a" but graphs are slightly different (where $y = b$)

Graph 1 (when b is positive)



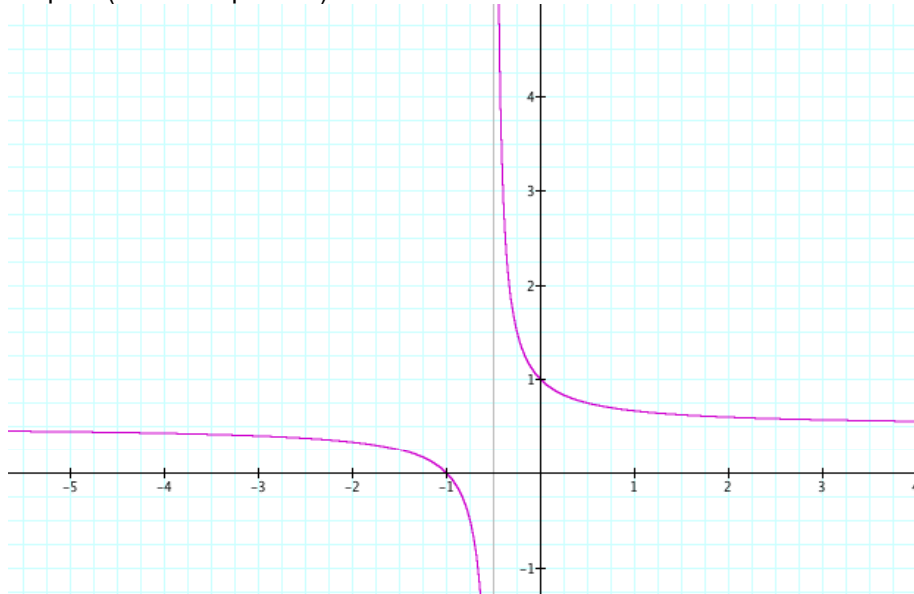
Graph 2 (when b is negative)



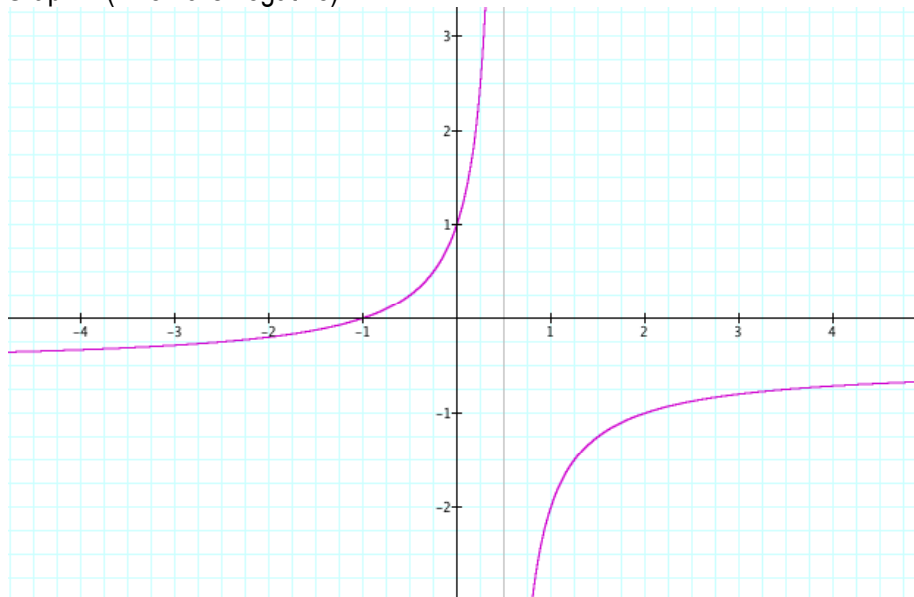
As "c" changes

- Vertical Asymptote @ $x = -1/c$

Graph 1 (when c is positive)



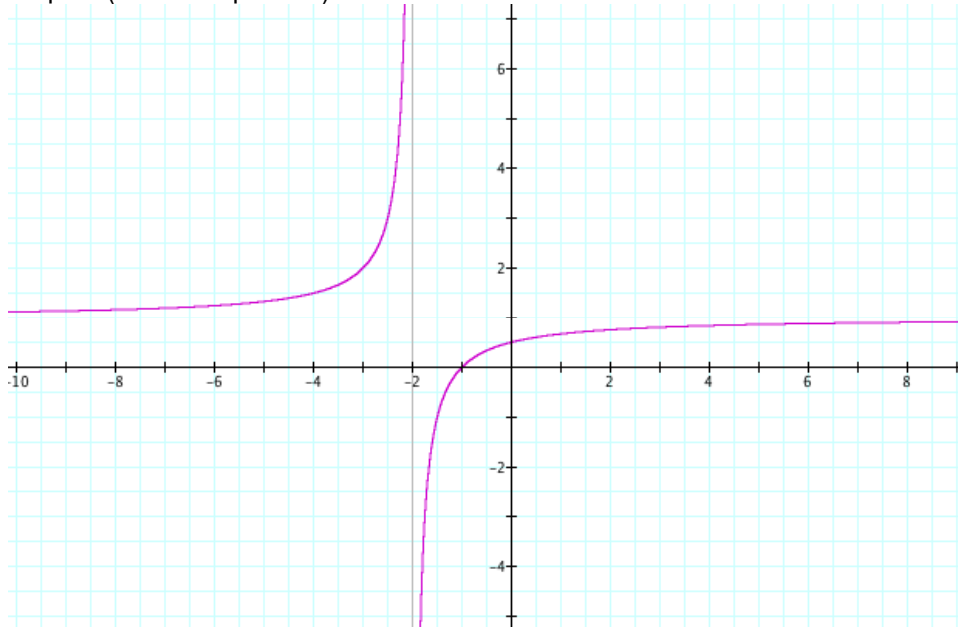
Graph 2 (when c is negative)



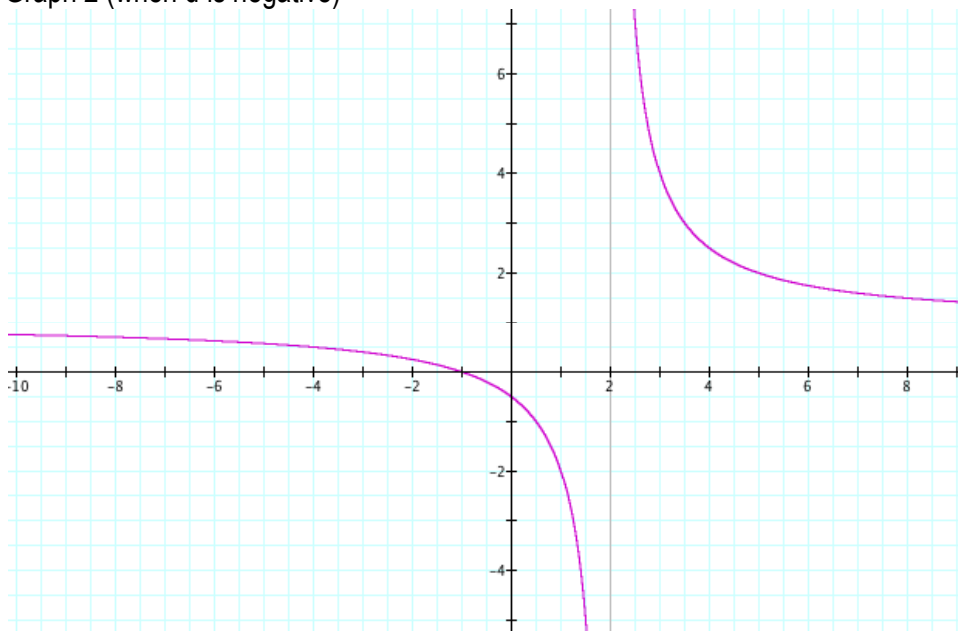
As "d" changes

- Vertical Asymptote @ $x = d$

Graph 1 (when d is positive)



Graph 2 (when d is negative)



In general, as all values approach zero the two parts of the graphs converge upon one another. As the values distance themselves from zero, the two parts move further away from one another.

See [animation](#) (using graphing calculator) as a , b , and d change together

Mathematical Foci 3: Asymptotes

An **asymptote** of a curve is a way of describing its behavior far away from the origin by comparing it to another curve.

Asymptotes occur when you have a value for which the function is undefined, and where the limit (the number the function is approaching as it gets close to that value) is either positive or negative infinity. This could result from a denominator in a function including a variable that cannot be canceled out with something in the numerator.

The line $x = a$ is a vertical asymptote if the graph increases or decreases without bound one or both sides of the line as x moves in closer and closer to $x = a$. The line $y = b$ is a horizontal asymptote if the graph approaches $y = b$ as x increases or decreases without bound. Note that it doesn't have to approach $y = b$ as BOTH increases and decreases. It only needs to approach it on one side in order for it to be a horizontal asymptote.

To find the asymptotes, start with the rational function, $f(x) = \frac{ax^n + \dots}{bx^m + \dots}$ where n is the largest exponent in the numerator and m is the largest exponent in the denominator.

1. The graph will have a vertical asymptote at $x = a$, if the denominator is zero at $x = a$ and the numerator isn't zero at $x = a$.
2. If $n < m$, then the x -axis is the horizontal asymptote. There are some cases where this is not true.
3. If $n = m$, then the line $y = \frac{a}{b}$ is the horizontal asymptote.
4. If $n > m$, there will be a slant asymptote.

In general, rational functions have a horizontal asymptote OR a slant asymptote; not both.

If the degree of the numerator is *exactly one more than* the degree of the denominator (so that the polynomial fraction is "improper"), then the graph of the rational function will be, roughly, a slant straight line. The equation for the slant asymptote is the polynomial part of the rational that you get after doing the long division.

A little more about horizontal asymptotes

Suppose f is a function. Then the line $y = a$ is a **horizontal asymptote** for f if

$$\lim_{x \rightarrow \infty} f(x) = a \text{ or } \lim_{x \rightarrow -\infty} f(x) = a.$$

Intuitively, this means that $f(x)$ can be made as close as desired to a by making x big enough. How big is big enough depends on how close one wishes to make $f(x)$ to a . This means that far out on the curve, the curve will be close to the line.

Another example would be $f(x) = \frac{1}{x^2 + 1}$, which has a horizontal asymptote at $y=0$, as can be seen by the

$$\lim_{x \rightarrow \infty} \frac{1}{x^2 + 1} = 0.$$

A different situation arises in this problem. Generally, most textbooks state that when the degree of the denominator is greater than the degree of the numerator in a rational function, the horizontal asymptote will be $y = 0$. However, for $f(x) = \frac{x - 4}{x^2}$, when $x = 4$, $y = 0$ as well which is supposed to be the asymptote. By the definition of an asymptote, it is a line that the function gets infinitely close to, but does not touch. We should not bother about a point of intersection at $(4, 0)$, or even $(4000000000000, 0)$. If only beyond that point there are no more points of intersection.

The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \rightarrow b$ as $x \rightarrow \pm \infty$.

Mathematical Foci 4: Limits

$$f(x) = \frac{ax + b}{cx + d}$$

What is a limit?

A limit is the output value a function appears to be reaching as it approaches a specific x , or input value, from one side of the graph. Generally it is referred to as the left and right hand limit. Both of these limits may be equal, but they may not. Limits help explain the behavior of functions graphically.

Concerning horizontal asymptotes

When x gets very large (heads towards infinity) or very small (heads towards negative infinity) the limit of the function is equal to a/c . This is true only if we assume A , B , C , and D are non-zero numbers.

Below is an example. If we let $a=6$, $b=1$, $c=3$, and $d=1$, then we observe that the limit of $f(x)$ as it approaches negative and positive infinity = 2 , or a/c .



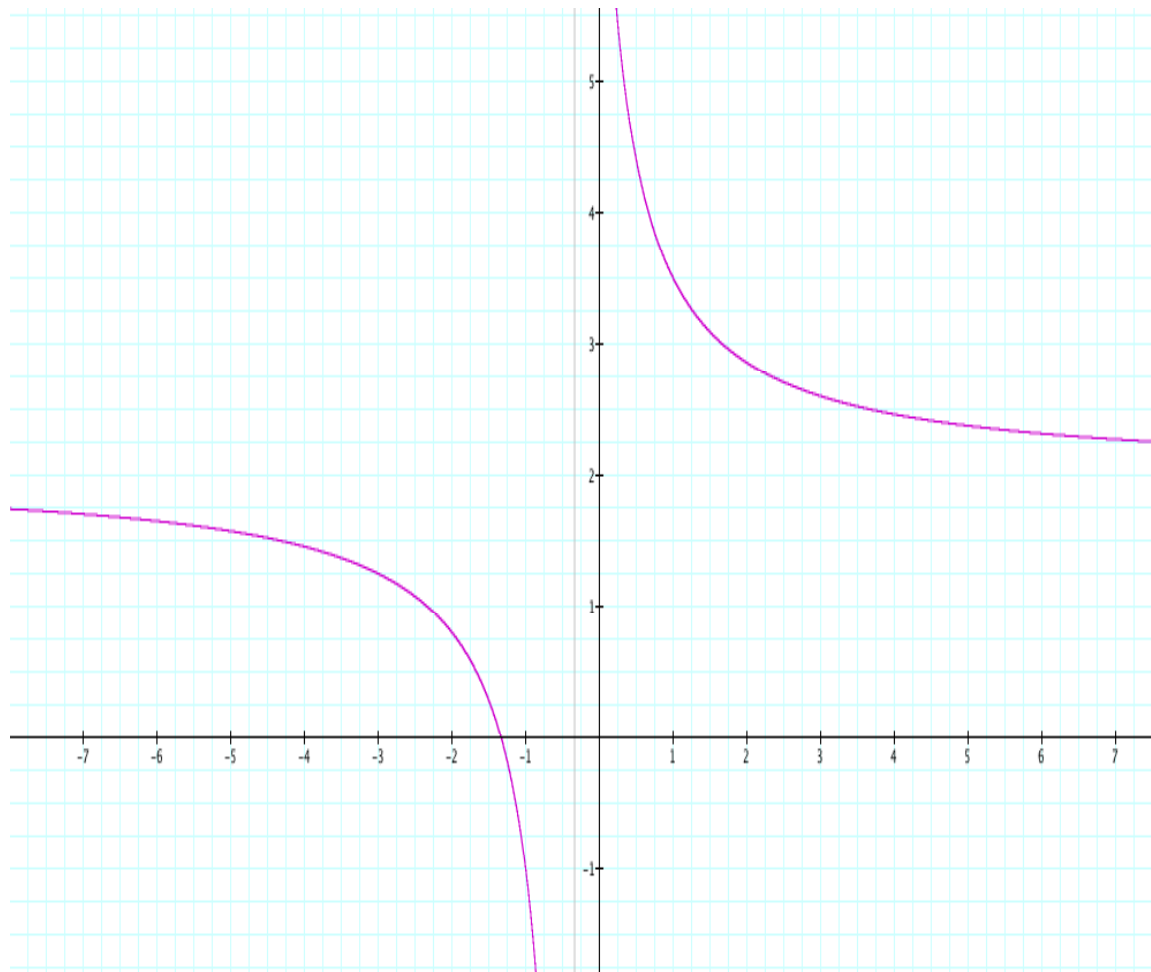
Concerning vertical asymptotes:

When x approaches the value $-d/c$, the limit of the function of $f(x)$ goes toward either positive or negative infinity. It will head towards positive infinity from one side of the graph and negative infinity from the other.

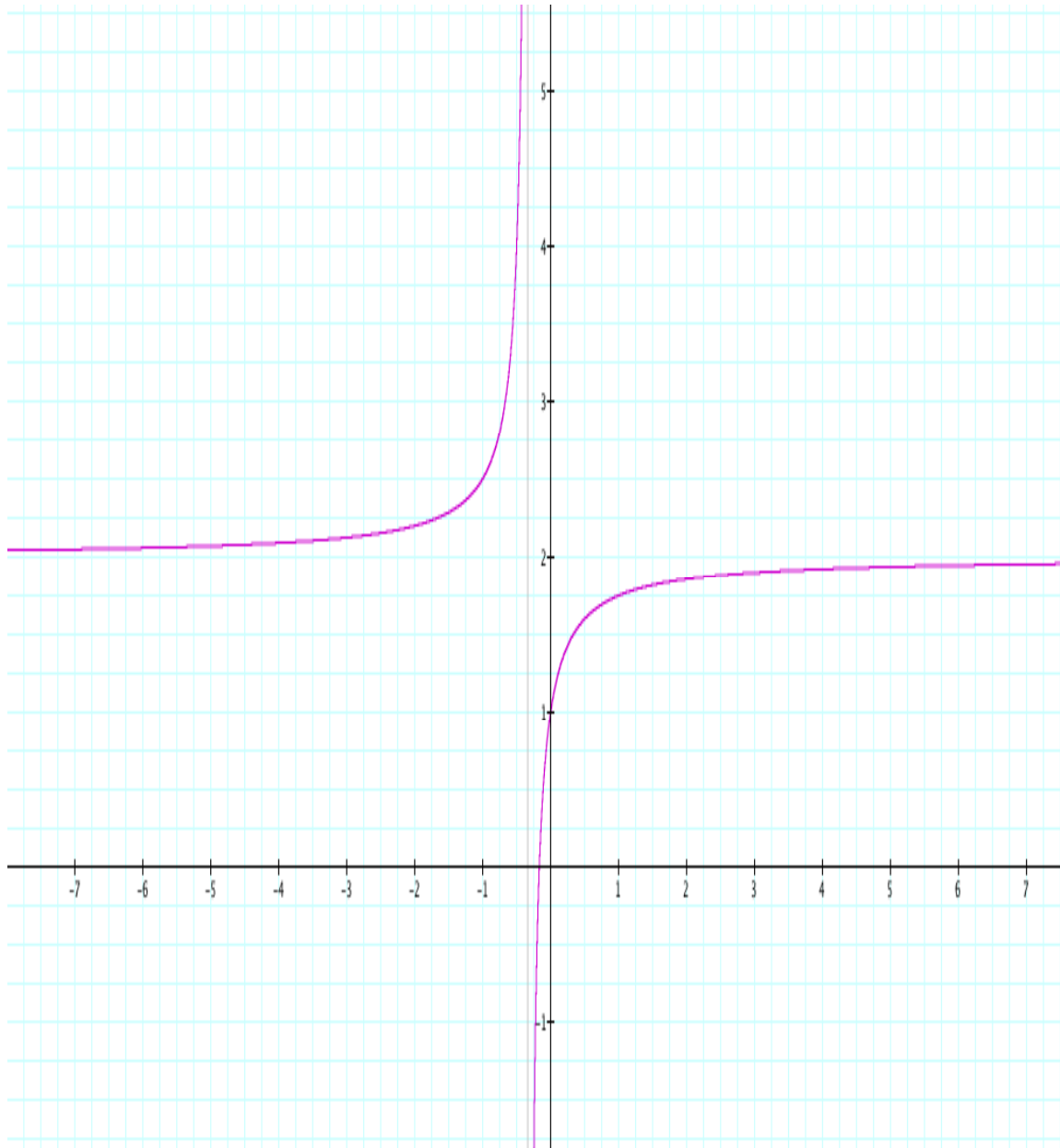
When $a < b$, the limit of the function approaches negative infinity from the left side and infinity from the right side.

When $a > b$, the limit of the function approaches infinity from the left side and negative infinity from the right side.

In example #1 $a=6$, $b=8$, $c=3$, and $d=1$. Observing the graph, there appears to be an asymptote at $x = (-1/3)$ or $(-d/c)$. It also can be observed that the limit from the left side is going towards negative infinity and the limit from the right side is going towards positive infinity. Here $b > a$.



In example #2 $a=6$, $b=1$, $c=3$, and $d=1$. Observing the graph, there appears to be an asymptote at $x = (-1/3)$ or $(-d/c)$. It also can be observed that the limit from the left side is going towards infinity and the limit from the right side is going towards negative infinity. Here $b < a$.



Concerning Holes:

If $a=b=c=d$ and does not equal 0 then where $x=-d/c$, there is a hole. The limit of the function from the right in this instance is equal to a . The same limit is approached from both the right and left sides of the function.

In the graph below $a, b, c,$ and $d = 1$. The graph appears to be continuous. However there is a hole at $f(-1)$.

This is because $-2/0$. When we solve for $f(-1)$ is undefined.

However, going from the left and right side the limit here appears to be equal to 1 or a .

It is important to note that limits can also exist if a function is continuous. A limit is just helping explain what is going on graphically from the left and right as x , or the input value approaches a specific value.

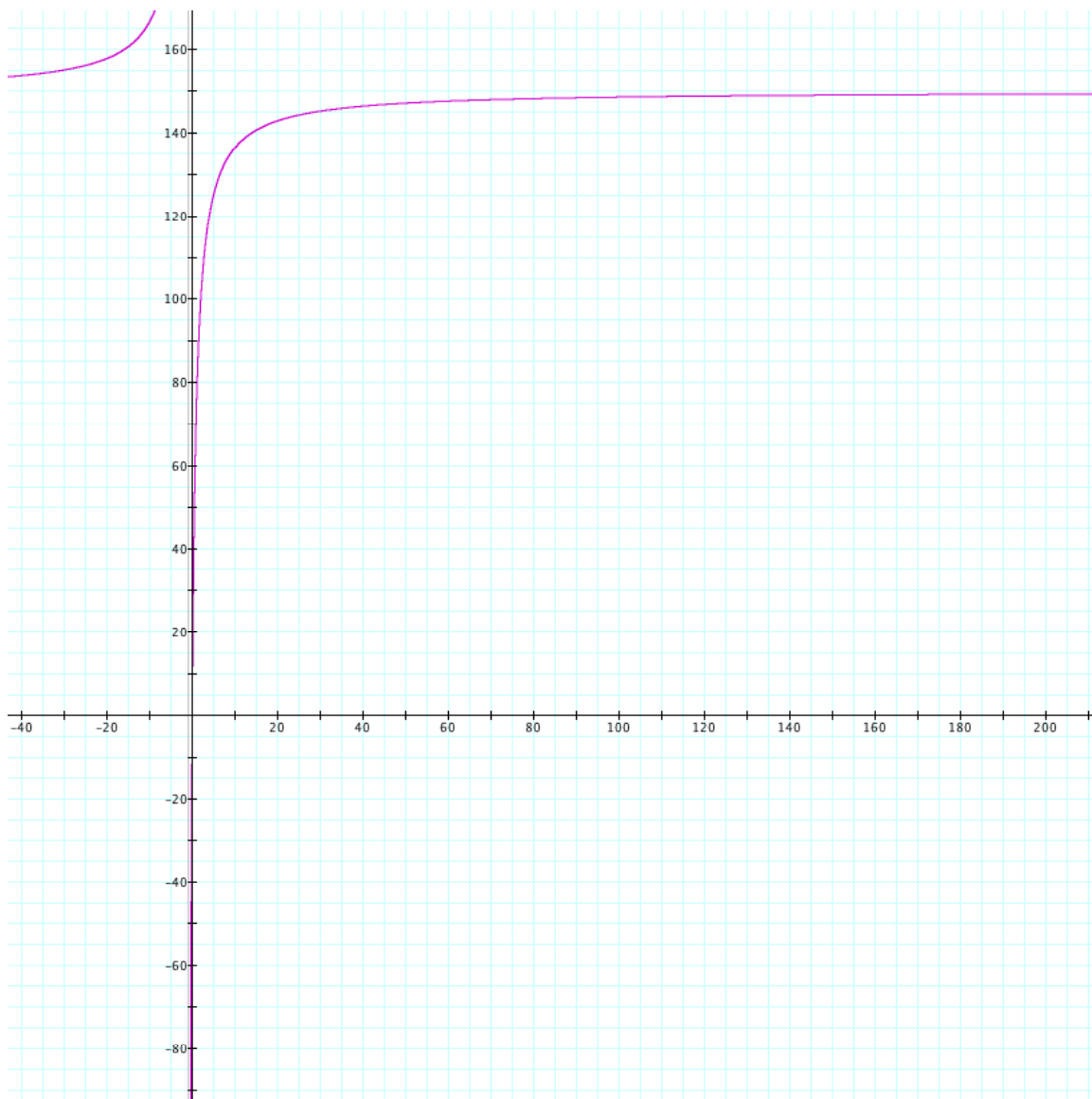


What if instead of looking at the limit of x , we look at the limits of a , b , c , and d in relation to our function?

Limit of a (towards infinity):

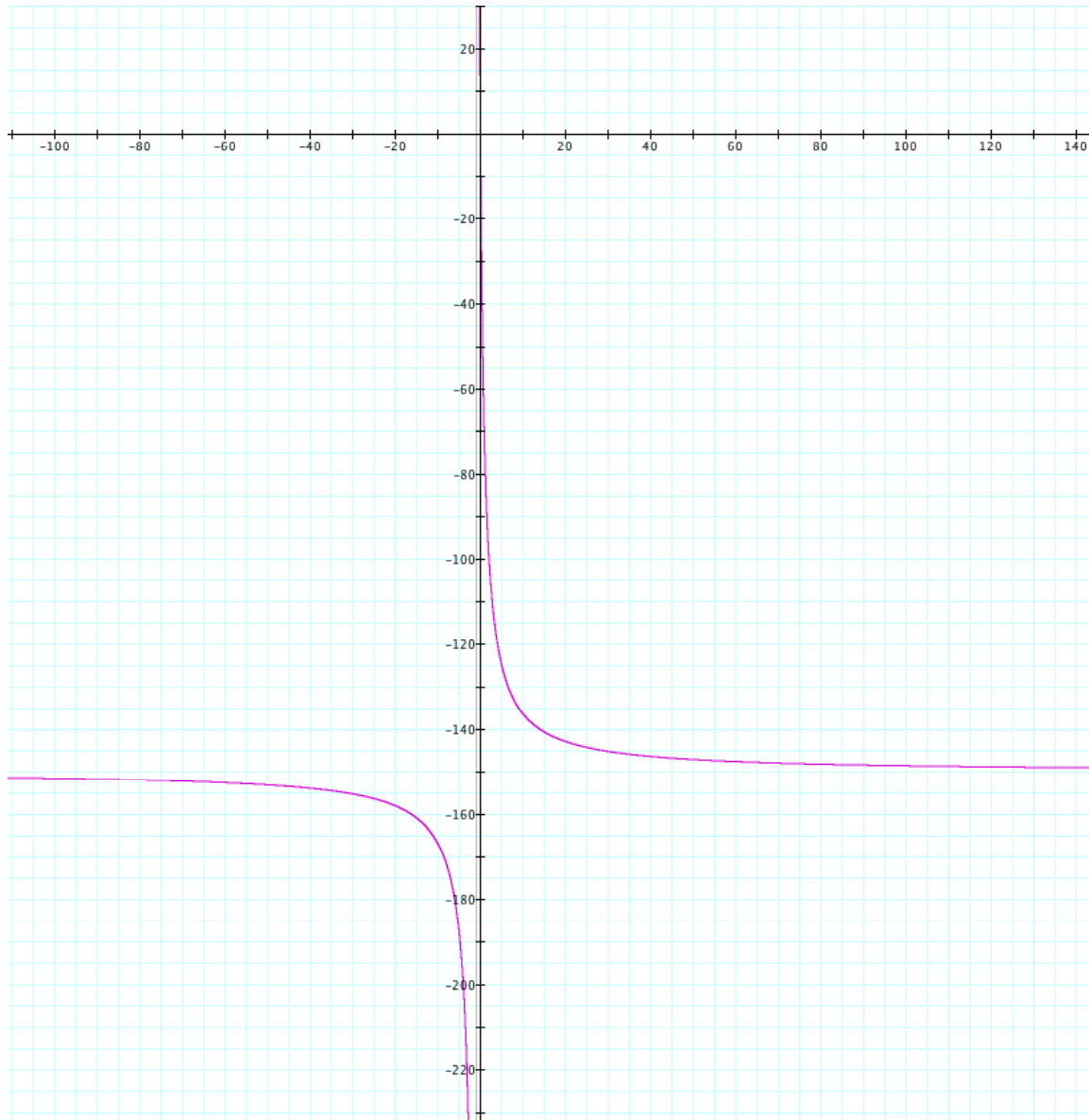
As the limit of a approaches infinity the right hand limit approaches negative infinity and left hand limit of $f(x)$ approaches infinity. This happens very quickly. Below is a graph illustrating this idea. Here, b , c , and $d = 1$ and $n = 150$. While this isn't showing $a = \text{infinity}$ we can generalize the idea.

We also know $f(x)$ cannot exceed the value of infinity since a horizontal asymptote exists at a .



Limit of a (towards negative infinity):

Here as x approaches infinity the left hand limit approaches negative infinity and right hand limit of $f(x)$ approach infinity. Once again, a graph is used to illustrate this point ($a=-150$, $b=1$, $c=1$, $d=1$). In this case, the function approaches negative infinity rather quickly. Additionally we know that the limit of $f(x)$ does not go past negative infinity since a horizontal asymptote exists at a .

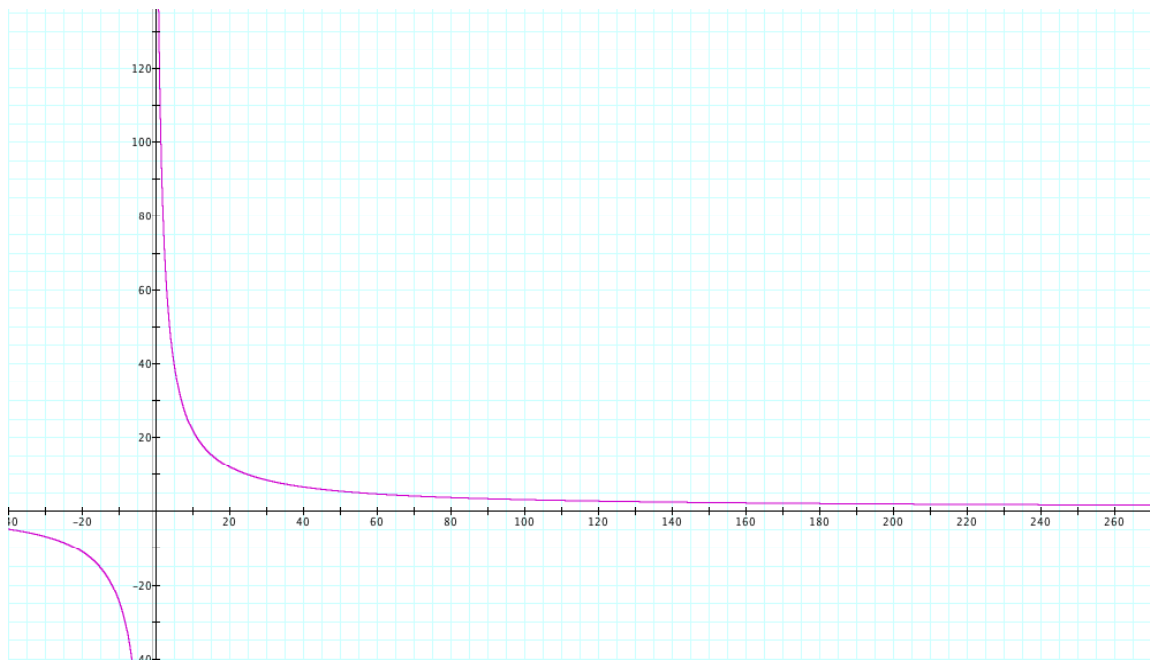


Limit of b (towards infinity and negative infinity):

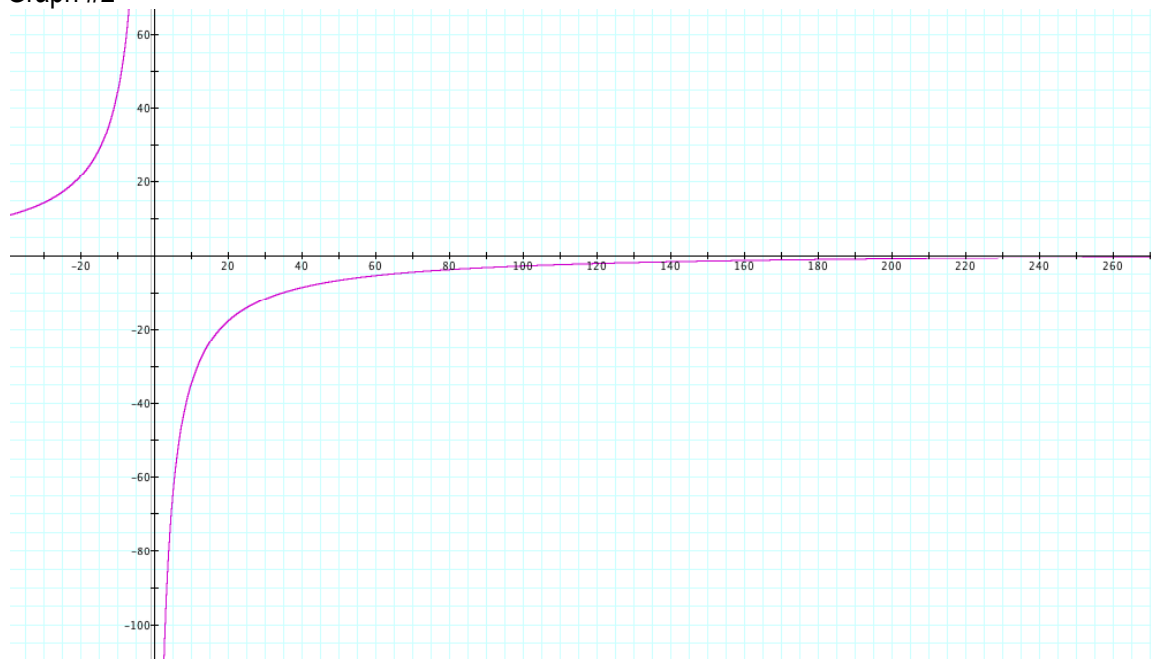
Here I looked at what happens to the function $f(x)$ if the limit of b is examined. First, we can see that as b goes towards infinity, the right hand limit of $f(x)$ goes towards infinity and the left hand limit goes towards negative infinity. Also, if b goes towards negative infinity, the right hand limit $f(x)$ goes towards negative infinity and the left hand limit goes towards positive infinity.

The first graph shows $f(x)$ with $a=230$, $b=1$, $c=1$, and $d=1$ to illustrate what happens when b goes towards infinity. The second graph shows $f(x)$ with $a=-390$, $b=1$, $c=1$, and $d=1$.

Graph #1



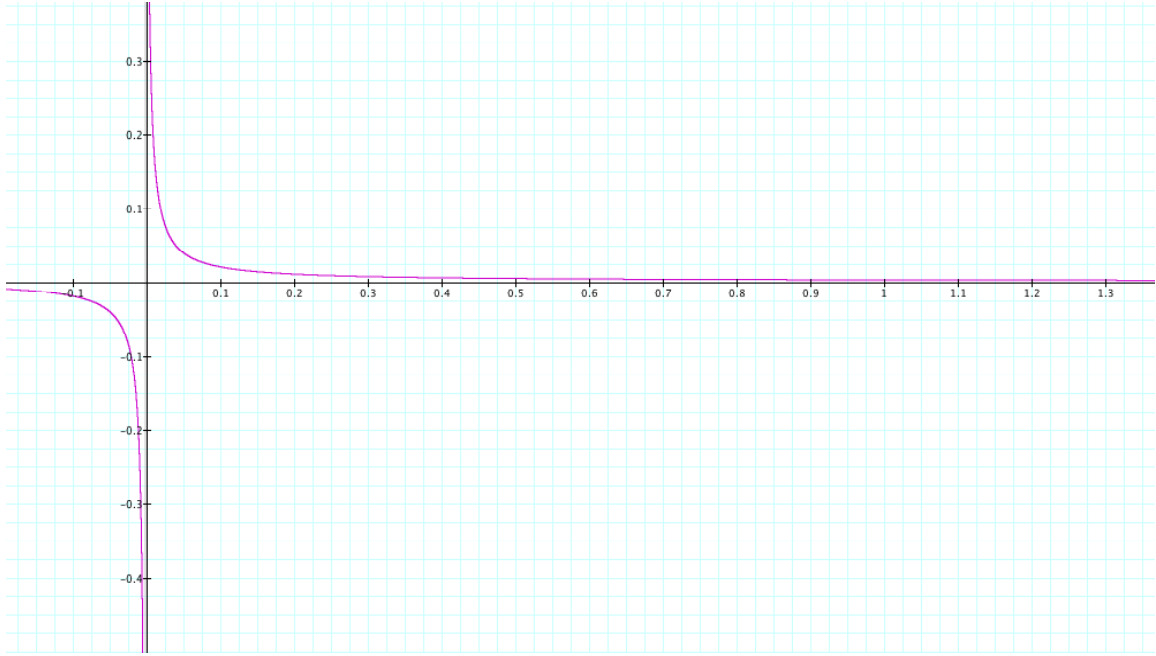
Graph #2



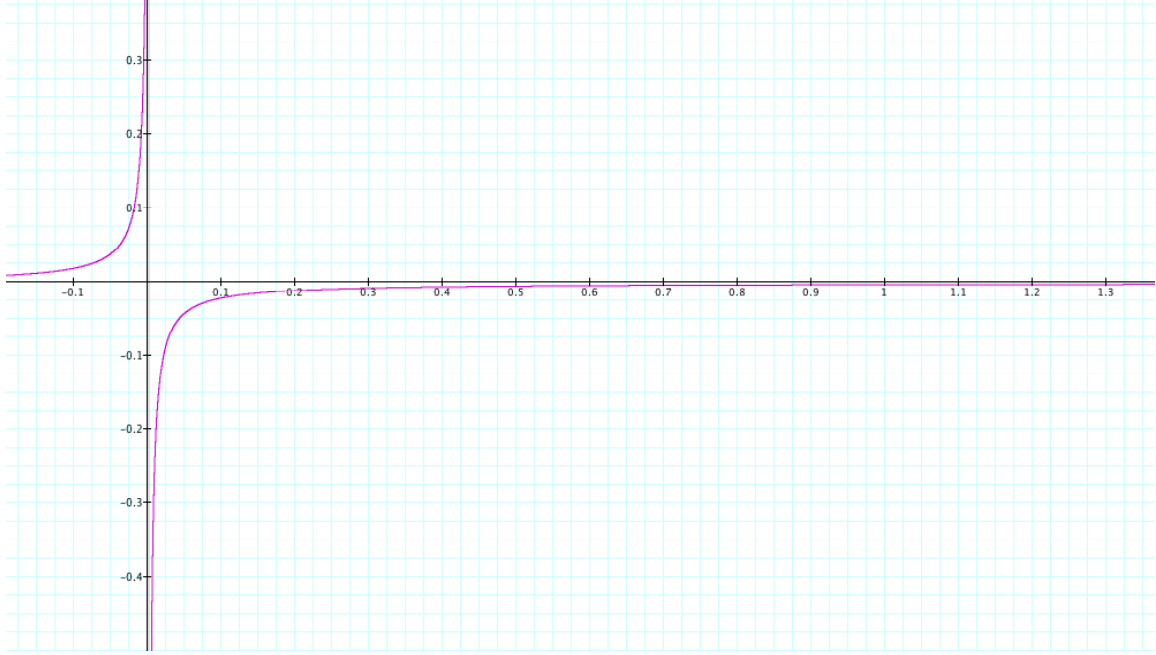
Limit of c (towards infinity and negative infinity):

Here when c approaches infinity the limit of $f(x)$ approaches 0. We can see this illustrated through graphs by observing what happens when $c=500$, and if we let a , b , and d remain as fixed values. When this happens, the limit of $f(x)$ approaches 0 from both sides. This is also true if we allow c to approach negative infinity. In the second graph I allowed $c=-500$ to illustrate this point.

Graph #1



Graph #2



Limit of d (towards positive and negative infinity): When a , b , and c remain fixed and d approaches positive or negative infinity, the limit of $f(x)$ approaches 0. The first graph illustrates this point by showing $d=500$, $a=1$, $b=1$, and $c=1$. The second graph shows that this also occurs when d approaches negative infinity, where $d=-500$, $a=1$, $b=1$, and $c=1$.

Graph #1



Graph #2



a) Algebraically:

Assuming $f: \mathbb{R} \rightarrow \mathbb{R}$ a function and $f(x) = (ax + b)/(cx + d)$.

To find an inverse of a function we assume $f(x) = y = (ax + b)/(cx + d)$ then cross multiplying $(cx + d)$ and y .

We will get $cxy + dy = ax + b$ and if we transfer all the x 's in the same side then we get $x(cy - a) = b - dy$.

Dividing both side by $(cy - a)$ then we get $x = (b - dy)/(cy - a)$. Multiplying both side by -1 then we get $x = (dy - b)/(-cy + a)$.

The last step is replacing y 's with x then we get $f^{-1}(x) = y = (dx - b)/(-cx + a)$.

b) Using linear algebra method

Assume $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix and $ad - bc \neq 0$. Multiplying this matrix by a 2×1 matrix, then we get 2×1 matrix $A[x] =$

To find the inverse of $A[x] =$ we must define identity matrix. Identity matrix is denoted by I .

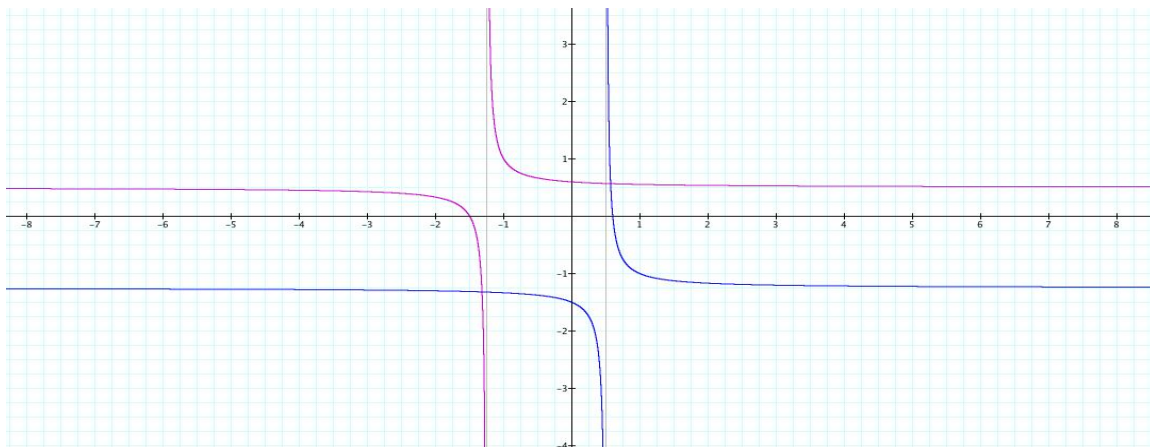
$$I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \dots, I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

We denote the inverse of a 2×2 matrix A is A^{-1} such that, $AA^{-1} = I$, where I is identity matrix. Then the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ where } ad - bc \neq 0. \text{ If we assume } ad - bc = 1 \text{ then } A^{-1}[x] = .$$

The inverse of the function has something to say about the function itself. Here are some that we found out.

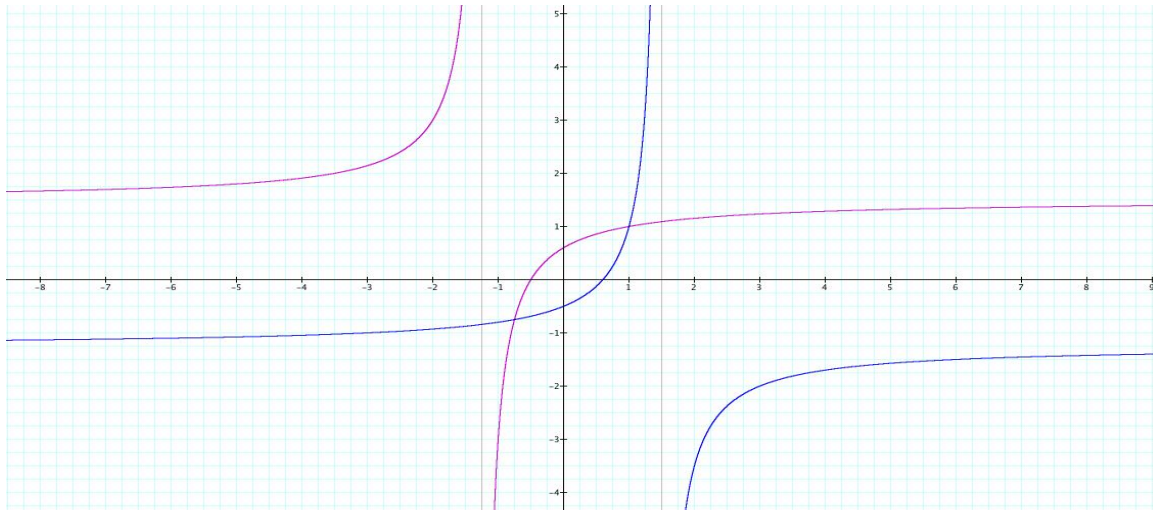
Take for example $f(x) = (2x+3)/(4x+5)$ and its inverse $f^{-1}(x) = (5x-3)/(-4x+2)$. The graphs of the function and its inverse are here below.



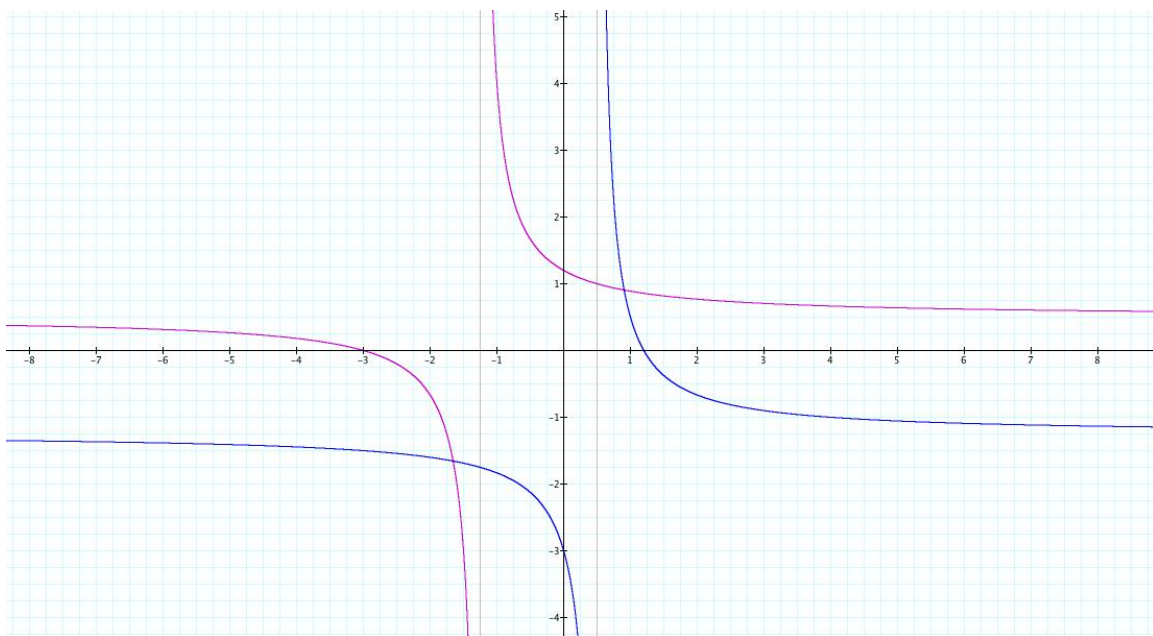
Graph 1

* $f(x)$ is in red while $f^{-1}(x)$ is in blue.

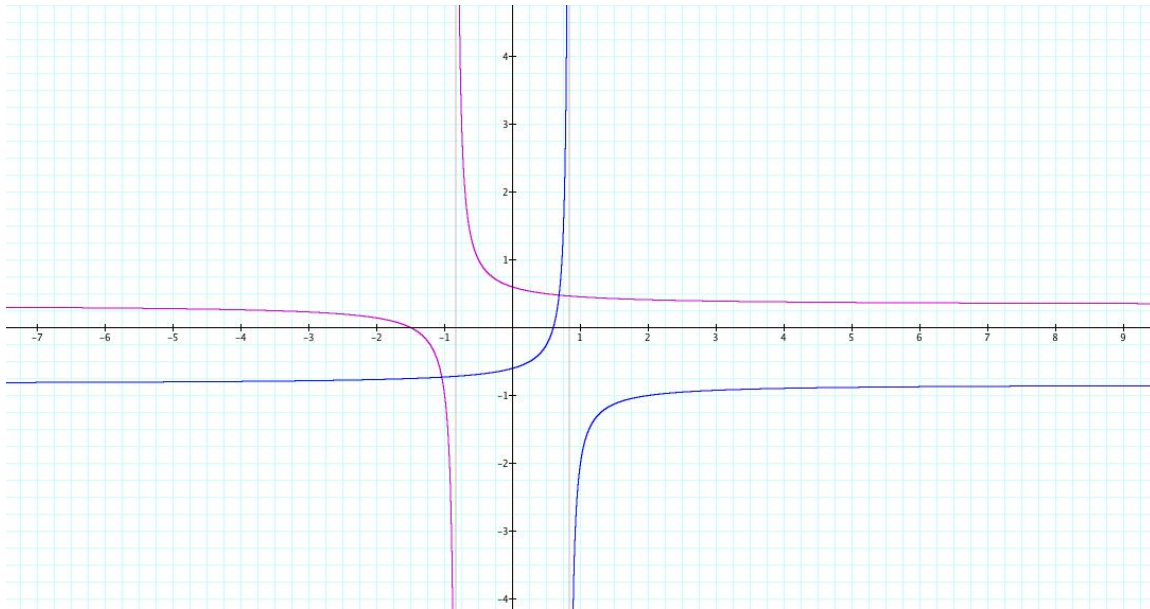
To make a reasonable observation, the graphs below are presented. The same set of function and its inverse is used. The value of b , c , and d are changed in graphs 2, 3, and 4 respectively.



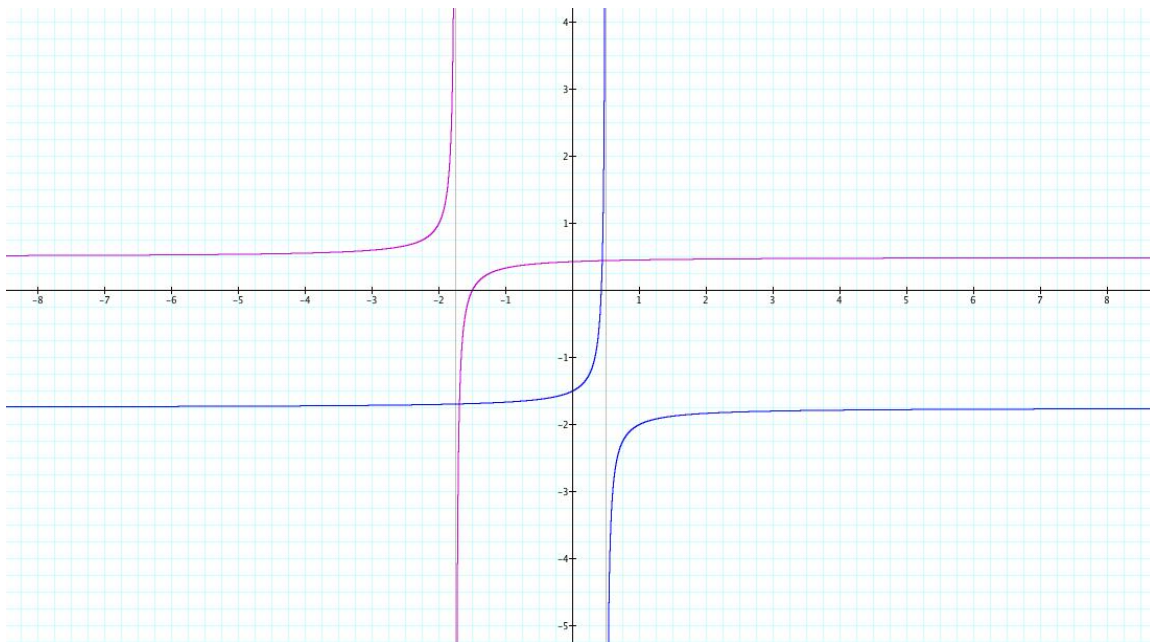
Graph 2



Graph 3



Graph 4



Graph 5

General observation:

The non-corresponding asymptotes of the graphs of the $f(x)$ and $f^{-1}(x)$ are parallel. Furthermore, the graph of $f^{-1}(x)$ is reflected to the opposite direction of $f(x)$.