Non-Euclidean Geometry Topics to Accompany Euclidean and Transformational Geometry

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Chapter 0: Introduction

For centuries Euclid's monumental work *The Elements* was regarded as a systematic discussion of absolute geometric truth. However, *The Elements* contains many assumptions. Euclid states some of these assumptions as Postulates and Common Notions, while others, such as the infinitude of a straight line, are merely implied in his proofs. We will see that by eliminating one or more of these assumptions, we may derive geometries dramatically different from the regular Euclidean geometry.

To arrive at these geometries, the primary assumption to disregard is that of the historically controversial parallel postulate. Euclid's parallel postulate, Postulate 5 of *The Elements*, states:

> "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles."[6]

This postulate garnered much criticism from early geometers, not because its truth was doubted - on the contrary, it was universally agreed to be a logical necessity - but because its complexity left them uneasy about it being a postulate at all, and not a proposition. There were several attempts to prove the parallel postulate, but they often assumed something that turned out to be its equivalent. The result was the discovery of a host of equivalent statements to the parallel postulate. Some of these include:

- If a line intersects one of two parallels, it must intersect the other also (Proclus' axiom).
- Parallel lines are everywhere equidistant.
- Through a point not on a given line there exists a unique line parallel to the given line (Playfair's theorem).
- The sum of the angles of a triangle is two right angles.
- If two parallels are cut by a transversal, the alternate interior angles are equal.
- Similar triangles exist which are not congruent.

By taking all other Euclid's assumptions, and substituting one of the above for the parallel postulate, we arrive at the usual Euclidean geometry.

In the early 18th century, an attempt was made by the Italian mathematician Girolamo Saccheri to prove the parallel postulate without the use of any additional assumptions. In the process, he derived some of the first results in what would be called elliptic and hyperbolic geometry. Saccheri considered a quadrilateral $ABCD$ in which the sides $\overline{AD}$ and $\overline{BC}$ are equal in length and perpendicular to the base $\overline{AB}$ (see Figure 0-1.)

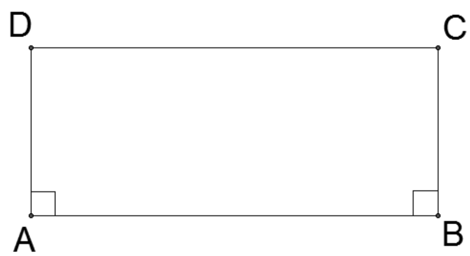


Figure 0-1

He proved correctly that in such a quadrilateral the summit angles $\angle ADC$ and $\angle BCD$ are equal. Proposition 29[1] of *The Elements* may be invoked to prove that the summit angles are right angles, but because Proposition 29 is dependent upon the parallel postulate, Saccheri could not make this claim. Instead, he assumed by way of contradiction (or so he hoped) that the summit angles were either larger or smaller than right angles.

By assuming that the summit angles were larger than right angles, he arrived at the following results:

(i) $AB > CD$.

(ii) The sum of the angles of a triangle is greater than two right angles.

(iii) An angle inscribes in a semicircle is always obtuse.

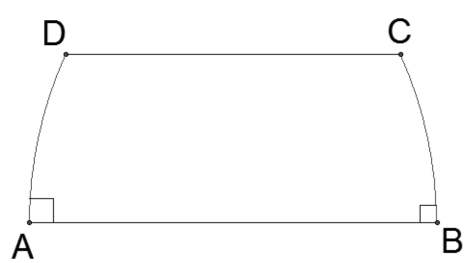


Figure 0-2

[1]See the Appendix for a list of Euclid's propositions, postulates, and other assumptions from Book 1 of *The Elements*.

Figure 0-2 shows such a quadrilateral. In this case, Saccheri was able to derive contradictions to Euclid's propositions 16, 17, and 18, however these propositions use Euclid's unstated assumption that lines are infinite in extent. Later, the three properties above would be shown to hold true in elliptic geometry, in which lines are never infinite.

If the summit angles were smaller than right angles, Saccheri derived the following results:

(i) $AB < CD$ (see Figure 0-3).

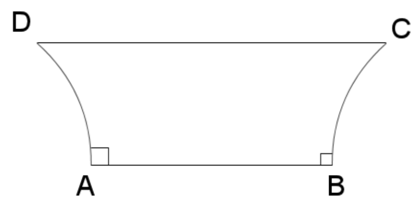


Figure 0-3

(ii) The sum of the angles of a triangle is less than two right angles.

(iii) An angle inscribes in a semicircle is always acute.

(iv) If two lines are cut by a transversal so that the sum of the interior angles on the same side of the transversal is less than two right angles, the lines do not necessarily meet, that is, they are sometimes parallel.

(v) Through any point on a given line, there passes more than one parallel to the line.

(vi) Two parallel lines need not have a common perpendicular.

(vii) Parallel lines are not equidistant. When they have a common perpendicular they recede from each other on each side of the perpendicular. When they have no common perpendicular, they recede from each other in one direction and are asymptotic in the other direction.

In this final property, Saccheri believed he had found a contradiction, namely that lines l and m intersect at some infinitely distant point and therefore had proved the parallel postulate. However, this was not a contradiction at all: the idea of "limit" was yet to be formalized in mathematics, and his eight properties above would become the initial results of what was later termed hyperbolic geometry. Yet the two new geometries stumbled upon by Saccheri would not be actively acknowledged and researched until the early nineteenth century.

The credit for first recognizing non-Euclidean geometry for what it was generally goes to Carl Frederich Gauss (1777-1855), though Gauss did not publish anything formally on the matter. Gauss, as many others, began by desiring to firmly establish Euclidean geometry free from all ambiguities. His objective was to prove that the angle measures of a triangle must sum to $180°$ (recall, this is equivalent to the parallel postulate). He supposed the contrary, and so was left with two possibilities: either the angle sum is greater than $180°$, or the angle sum is less than $180°$.

Using, as Saccheri had done, Euclid's assumption that lines are infinite in length, Gauss arrived at a contradiction in the case where the angle measures of a triangle sum to more than $180°$. However, the case where the angle sum is less than $180°$ did not lend itself to such a contradiction. In a private letter written in 1824, Gauss asserted:

> "The assumption that the sum of the three angles is less that $180°$ leads to a curios geometry, quite different from ours, but thoroughly consistent, which I have developed to my entire satisfaction."[2]

While Gauss may have developed this geometry to his own satisfaction, for whatever reason, he did not see fit to publish any of his results. Instead, this credit goes to two mathematicians in different parts of the world who, unbeknownst to each other, arrived at the same conclusion around the same time: that unless the parallel postulated were someday proven, the geometry in which the angle sum of a triangle is less that $180°$ is entirely valid.

In 1829 a Russian mathematics professor named Nikolai Lobachevsky from the University of Kasan published "On the Principles of Geometry" in the *Kasan Bulletin*. In this article, he described a geometry in which more than one parallel to a given line may be drawn through a point not on the line. He found that this was tantamount to the angle sum of a triangle being less than $180°$. This was the first publication on non-Euclidean geometry, and so Lobachevsky is recognized as the first to clearly state its properties. However, his work was not widely regarded by the mathematical community at the time, and he died in 1856 before his work received wide acceptance. Today, Hyperbolic geometry is sometimes called Lobachevskian geometry.

The same year that Nikolai Lobachevsky published his work on non-Euclidean geometry, a Hungarian officer in the Austrian army named Johann Bolyai submitted a manuscript to his father, Wolfgang Bolyai, a math teacher with ties to Gauss. The manuscript contained the younger Bolyai's discovery of non-Euclidean geometry with many of its surprising results. "Out of nothing, I have created a strange new universe," Bolyai is credited with stating in a letter to his father. His work was published in 1832 as an appendix entitled "The Science of Absolute Space" to his father's book on elementary mathematics. It was in a letter to the elder Bolyai after reading this appendix that Gauss confessed to having come to the same conclusions thirty to thirty-five years prior. Today, Gauss, Lobachevsky, and Bolyai are given some share of credit for discovering the non-Euclidean geometry now called hyperbolic geometry.

While Gauss, Lobachevsky, and Bolyai all focused their attention on the geometry formed by assuming the angle sum of a triangle is less than 180°, a mathematician named Georg Friedrich Bernhard Riemann (1826-1866) discovered that by disregarding the assumption that lines have infinite length, one arrives at a valid geometry in which the angle sum of a triangle is greater than 180°. Euclid's second postulate states that a straight line may be continued in a straight line. However, one might imagine a line as being somewhat like a circle, "continuing" forever yet by no means infinite. Riemann considered this the distinction between "*unboundedness* and *infinite extent*."

Having decided that lines could after all be finite, Riemann took to studying geometry free of the parallel postulate. He found that this eliminated any contradiction in the case where the angles of a triangle sum to more than 180°. Interestingly, he found that in such a geometry parallel lines do not exist. This new non-Euclidean geometry came to be known as elliptic geometry, or sometimes, Riemannian geometry.

Thus, by the mid-nineteenth century there were two competitors with the geometry of Euclid. Unless the parallel postulate could be proven, both hyperbolic and elliptic geometry seemed logically consistent. But it was not until 1868 that an Italian mathematician named Eugenio Beltrami (1835-1900) proved beyond a doubt that these new geometries were every bit as valid as Euclid's own. He showed through clever analysis that if a contradiction existed in either hyperbolic or elliptic geometry, then a contradiction also existed in Euclidean geometry. Therefore, the mathematical community had to accept these new geometries as valid alternatives, and the quest to prove the parallel postulate finally came to an end.

While elliptic and hyperbolic geometry share most of the spotlight for non-Euclidean geometry, there do exist other geometries which are non-Euclidean. A fairly recent development is Taxicab geometry, the beginnings of which were formulated by the mathematician Hermann Minkowski (1864-1909). Taxicab geometry is formed by taking the regular geometry in the Euclidean coordinate plane and redefining the way distance between points is calculated. This means that the assumption that lines of the same length are congruent must be discarded, and with the loss of that assumption goes many of Euclid's most well-known results. Congruence conditions for triangles, for example, do not apply in Taxicab geometry. Research continues to see what other geometries might be formed by defining distance in still different ways.

Today, non-Euclidean geometries are commonly used in mathematics. There are even applications to these geometries outside of pure mathematics. Hyperbolic geometry, for example, is invoked by physicists studying Eistein's General Theory of Relativity to describe the shape of our universe. Spherical geometry, a simple form of Elliptic geometry, is used in navigational calculations for movement on the earth, and taxicab geometry provides a good model of urban geography. While Euclidean geometry still most accurately represents our universe locally, scientists continue to discover surprising new applications for non-Euclidean geometries. As Poincare once asserted: “One geometry cannot be more true than another; it can only be more convenient.”

Chapter 1: Hyperbolic Geometry

We begin our discussion of hyperbolic geometry with the quadrilateral construction done by Girolamo Saccheri in the 18th century. Our goal will be to construct a new geometry in which Euclid's parallel postulate does not hold and in which the angle sum of a triangle is less than 180°. We allow ourselves the use of Euclid's first four postulates and all his assumptions not equivalent to the parallel postulate. (This includes the assumption that lines are infinite.) Since the first 28 postulates of *The Elements* do not require the parallel postulate, these results will be valid in our geometry. Any time a proposition is invoked in our discussion, it will be stated explicitly, and can be referenced in the Appendix.

Now, recall from the introduction the quadrilateral $ABCD$ considered by Saccheri in his attempt to prove the parallel postulate:

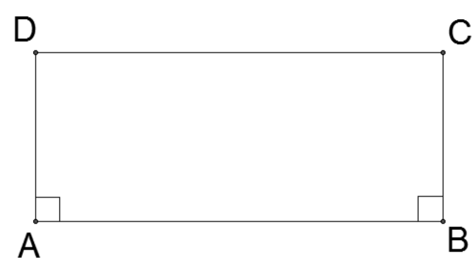


Figure 1-1

We call $\overline{AB}$ the *lower base*, $\overline{CD}$ the *upper base*, $\overline{AD}$ and $\overline{BC}$ the *arms* (which have equal length) and angle $\angle C$ and $\angle D$ the *upper base angles*. Note that $\angle A$ and $\angle B$ are right angles. This is called the *Saccheri quadrilateral*.

Theorem 1-1. *The upper base angles of the Saccheri quadrilateral are equal.*

Proof. By the side-angle-side (SAS) congruency condition, triangles ΔABC and ΔBAD are congruent.

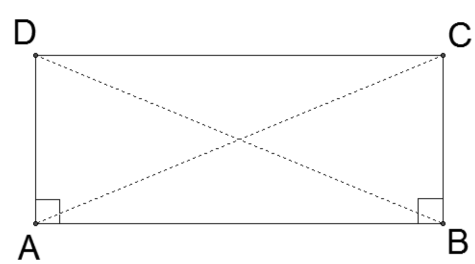


Figure 1-2

Thus, $\overline{AC} \cong \overline{BD}$. Then $\triangle ADC$ and $\triangle BCD$ are congruent by the side-side-side (SSS) congruency condition. Therefore, $\angle D \cong \angle C$. □

Propositions 4 and 8 of *The Elements* give the SAS and SSS congruency conditions. Thus, the above theorem is also valid in Euclidean geometry, since its proof does not require the parallel postulate.

Proposition 27 states that if two lines share a perpendicular, they are parallel. We are thus able to prove the following:

Theorem 1-2. *The line joining the midpoints of the upper and lower bases of the Saccheri quadrilateral (called the altitude) is perpendicular to both. Therefore, the upper base and lower base lie on parallel lines sharing a common perpendicular.*

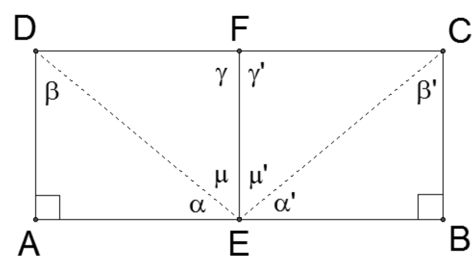


Figure 1-3

Proof. Let E and F be the midpoints of the lower base and upper base respectively. Then let angles α, α', β, β', γ, γ', μ and μ' be as in Figure 1-3. Note $\triangle DEA \cong \triangle CEB$ by SAS. Thus $\overline{DE} \cong \overline{CE}$, $\alpha = \alpha'$, and $\beta = \beta'$. Invoking SSS, it follows that $\triangle DEF \cong \triangle CEF$. Hence $\gamma = \gamma'$, and since these angles are supplementary, each must be $90°$. Also, $\mu = \mu'$, and so $\alpha + \mu = \alpha' + \mu = 90°$, again since these angles are supplementary. Thus $\overline{EF} \perp \overline{AB}$ and $\overline{EF} \perp \overline{CD}$, and it follows from Proposition 27 that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ with common perpendicular $\overleftrightarrow{EF}$. □

In hyperbolic geometry the angle sum of a triangle is always less than $180°$. Without any additional postulates, we are now able to prove that the angle sum does not exceed $180°$. Later we will require the Hyperbolic Parallel Postulate to show that the angle sum is strictly less than $180°$.

The proof of the following theorem uses the fact that a triangle cannot have two angles summing to more than two right angles. This is given in Proposition 17 of *The Elements*.

Theorem 1-3. *The angle sum of a triangle does not exceed* $180°$.

Proof. By way of contradiction, suppose that there exists a triangle $\triangle ABC$ whose angle-sum is $180^\circ + \alpha$, for some $\alpha > 0$. Let D_1 be the midpoint of $\overline{BC}$. Construct line segment $\overline{AD_1E_1}$ so that $\overline{AD_1} \cong \overline{D_1E_1}$. (Observe that we have invoked the assumption that lines may be extended indefinitely.)

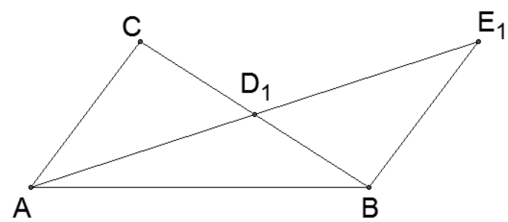


Figure 1-4

Proposition 15 of *The Elements* gives us that vertical angles are congruent, so we have $\angle AD_1C \cong \angle E_1D_1B$. Thus $\triangle AD_1C \cong \triangle E_1D_1B$ by SAS. Now notice that the angle sum of $\triangle ABC$ is the same as the angle sum of $\triangle ABE_1$. Next, notice that $\angle A = \angle CAD_1 + \angle E_1AB$, so either $\angle CAD_1 \leq \frac{1}{2}\angle A$ or $\angle E_1AB \leq \frac{1}{2}\angle A$. Suppose (without loss of generality) that $\angle E_1AB \leq \frac{1}{2}\angle A$. Then $\triangle ABE_1$ is a triangle with the same angle sum as $\triangle ABC$, and has the angle $\angle E_1AB \leq \frac{1}{2}\angle A$.

Now, repeat the above construction on $\triangle ABE_1$ and come up with a triangle whose angle sum is equal to that of $\triangle ABC$, and that has an angle less than or equal to $\frac{1}{4}\angle A$.

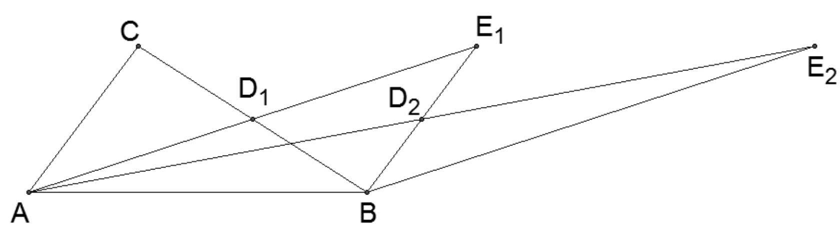


Figure 1-5

Figure 1-5 shows the construction of $\triangle ABE_2$ with the same angle sum as $\triangle ABC$ and with $\angle E_2AB \leq \angle A = \frac{1}{4}\angle A = \frac{1}{2^2}\angle A$. (Note that we have chosen without loss of generality that $\angle E_2AB$ be less than or equal to $\frac{1}{2^2}\angle A$. Had in fact $\angle E_1AD_2$ been less than or equal to $\frac{1}{2^2}\angle A$, the desired triangle would have been $\triangle AE_2E_1$.)

Continuing this construction, we eventually arrive at a triangle $\triangle ABE_n$ such that $\triangle ABE_n$ has the same angle sum as $\triangle ABC$ and contains the angle $\angle E_nAB \leq \frac{1}{2^n}\angle A$. If we let n be such that $\frac{1}{2^n}\angle A < \alpha$, we have that $\angle E_nAB \leq \frac{1}{2^n}\angle A < \alpha$. Then since the angle sum of $\triangle ABE_n$ equals the angle sum of $\triangle ABC = 180° + \alpha$, we get that

$$180° + \alpha = \angle E_nAB + \angle ABE_n + \angle BE_nA < \alpha + \angle ABE_n + \angle BE_nA,$$

and so

$$180° < \angle ABE_n + \angle BE_nA,$$

or that the sum of the above angles is greater than two right angles. Thus $\triangle E_nAB$ is a triangle with two angles summing to more than two right angles, which contradicts Proposition 17 of *The Elements*. □

Notice that Theorem 1-3 implies that the upper base angles of the Saccheri quadrilateral are not obtuse (why?).

Theorem 1-4. *Consider a quadrilateral with a lower base that makes right angles with its two arms.*

(i) If the upper base angles are unequal, so are the arms.

(ii) If the arms are unequal, so are the upper base angles, with the greater upper base angle opposite the greater arm.

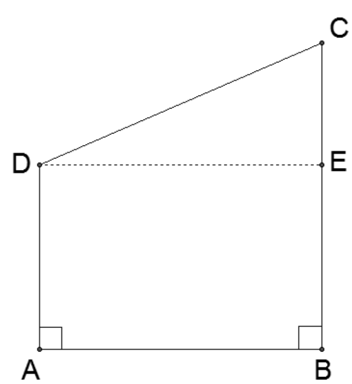


Figure 1-6

Proof. In order to prove (i) we only need to note that it is the contrapositive of Theorem 1-1.

For (ii), suppose that $\overline{BC} > \overline{AD}$ in quadrilateral $ABCD$. Let E be the point on the segment $\overline{BC}$ such that $\overline{AD} \cong \overline{BE}$. Then $ABED$ is a Saccheri quadrilateral, so $\angle ADE \cong \angle BED$ by Theorem 1-1. Note that $\angle ADC = \angle ADE + \angle EDC$. Proposition 16 gives us the Exterior Angle Theorem, namely that, in a triangle, an exterior angle is greater than either of the interior and opposite angles. So we have $\angle BED = \angle ADE > \angle ECD$, and therefore $\angle ECD < \angle ADE + \angle EDC = \angle ADC$. □

Theorems 1-1 through 1-4 belong to *neutral geometry*, because they make no assumptions about parallel lines. This means that they hold in both Euclidean and hyperbolic geometry. In order to examine some results that hold in hyperbolic geometry but not Euclidean geometry we must first state a replacement for the Parallel Postulate.

Axiom 1-1 (Hyperbolic Parallel Postulate)**.** *The upper base angles of the Saccheri quadrilateral are acute.*

Recall that given our assumptions, the upper base angles of the Saccheri quadrilateral are not obtuse. Two possibilities remain: either the angles are right angle, or that they are acute. The Parallel Postulate is equivalent to the angles being right angles. Therefore, the Hyperbolic Parallel Postulate is the negation of the Parallel Postulate. As such, the negation of anything equivalent to the Parallel Postulate will belong to hyperbolic geometry. This gives us, for example, the following:

(i) There exist parallel lines which are not equidistant from one another.

(ii) There exists a line and a point not on the line through which run more than one parallel to the line.

(iii) Similar triangles are always congruent.

Now we will examine some results unique to hyperbolic geometry. Our figures from now on will be drawn in such a way as to approximate the behavior of hyperbolic lines.

Theorem 1-5. *In the Saccheri quadrilateral:*

(i) the altitude is shorter than the arms, and

(ii) the upper base base is longer than the lower base.

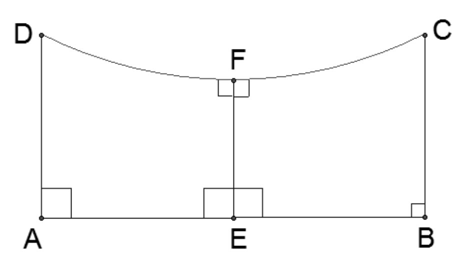


Figure 1-7

Proof. On the Saccherri quadrilateral $ABCD$, let E and F be the midpoints of the lower base and upper base respectively. Theorem 1-2 gives us that $\overline{EF}$ is perpendicular to both the upper and lower base. By the Hyperbolic Parallel Postulate, $\angle C$ and $\angle D$ are acute. Then, by Theorem 1-4, $\overline{AD} > \overline{EF}$ in $AEFD$ and $\overline{CB} > \overline{EF}$ in $EBCF$.

To prove (ii), consider $EFDA$ as having lower base $\overline{EF}$ and arms $\overline{AE}$ and $\overline{DF}$. Theorem 1-4 gives us that $\overline{DF} > \overline{AE}$ and similarly $\overline{FC} > \overline{EB}$. Therefore $\overline{DF}+\overline{FC} > \overline{AE}+\overline{EB}$ or $\overline{DC} > \overline{AB}$. Thus the upper base is longer than the lower base. □

Let us consider the Saccheri quadrilateral from the previous construction (see Figure 1-8).

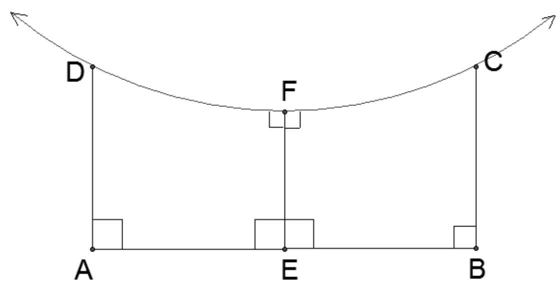


Figure 1-8

In Theorem 1-2 we proved that lines $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$ are parallel with common perpendicular $\overleftrightarrow{EF}$. Theorem 1-5 gave us that $\overline{AD} > \overline{EF}$ and $\overline{BC} > \overline{EF}$. Thus the parallel lines $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$ are not equidistant.

Definition 1.1. A Lambert[1] quadrilateral is a quadrilateral with three right angles.

Theorem 1-6. *In hyperbolic geometry, the fourth angle of a Lambert quadrilateral is acute, and each side adjacent to the acute angle is longer than the opposite side.*

The proof is left as an exercise for the reader in the problem set.

In Euclidean geometry two parallel lines always have a common perpendicular. They have, in fact, infinitely many common perpendiculars. In hyperbolic geometry, however parallel lines either have no common perpendiculars or a unique one.

Theorem 1-7 (Uniqueness of a common perpendicular to parallel lines)**.** *If two parallel lines have a common perpendicular, then they cannot have a second common perpendicular.*

Proof. Let ℓ and m be parallel lines with two common perpendiculars p_1 and p_2.

[1]Named after Johann Lambert(1728-1777), who used its construction in an attempt to prove the Parallel Postulate, much like Saccheri had done.

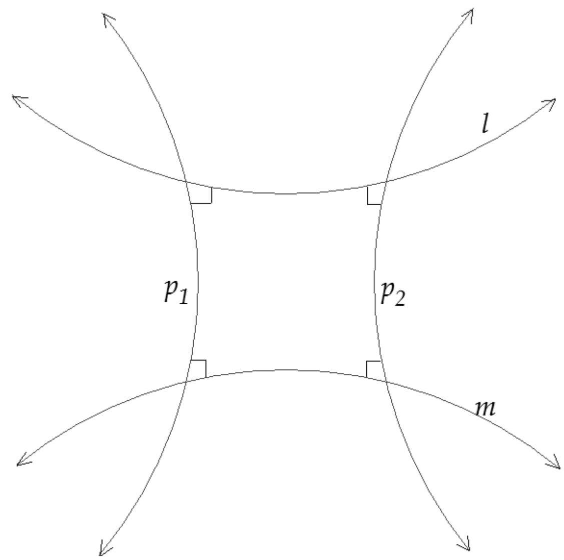


Figure 1-9

Then ℓ, m, p_1, and p_2 define a Lambert quadrilateral with four right angles, which contradicts Theorem 1-6. $\square$

The following three theorems give us an idea of what certain parallel lines "look" like in hyperbolic geometry. They will also be useful in proving that the angle sum of a triangle is always less that 180°.

Theorem 1-8. *Given two lines, if there exists a transversal which cuts the lines so as to form equal alternate interior angles or equal corresponding angles, then the lines are parallel with a common perpendicular.*

Proof. Let ℓ and m be two lines cut by a transversal $\overleftrightarrow{AB}$, where A is a point on m and B is a point on ℓ, so that $\overleftrightarrow{AB}$ makes equal alternate interior angles with respect to ℓ and m. It is enough to prove the case when $\overleftrightarrow{AB}$ makes equal alternate interior angles, because it is equivalent to the case when $\overleftrightarrow{AB}$ makes equal corresponding angles (why?). Proposition 27 of *The Elements* tells us that $\ell \| m$. Let P be the midpoint of $\overline{AB}$. Construct the line perpendicular to ℓ which passes through P. Let C be the point of intersection of this new line with ℓ, so that $\overleftrightarrow{CP} \perp \ell$. Construct the line perpendicular to m through point P. Let D be the point of intersection of m with this line, so $\overleftrightarrow{DP} \perp m$. We will show that $\overleftrightarrow{CP} = \overleftrightarrow{DP}$, which, based on Proposition 14, is true if the sum $\angle CPB$ and $\angle BPD$ equals 180°. Now, $\angle APD$ and $\angle DPB$ are supplementary, so they sum to two right angles by

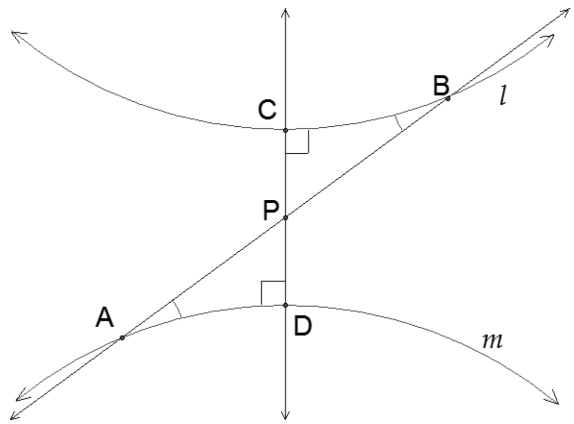


Figure 1-10

Proposition 13. So, if we show that $\angle BPC \cong \angle APD$, we will also have that $\angle CPB$ and $\angle BPD$ sum to 180°. By hypothesis we have $\angle CBP \cong \angle DAP$. By construction, we know $\angle BCP$ and $\angle ADP$ are both right angles. Finally, since P is the midpoint of $\overline{AB}$, we have $\overline{AP} \cong \overline{PB}$. Thus, by Proposition 26 (AAS), $\triangle BCP \cong \triangle ADP$, and so $\angle APD \cong \angle BPC$. Hence, $\overleftrightarrow{CP} = \overleftrightarrow{DP} = \overleftrightarrow{CD}$, where $\overleftrightarrow{CD} \perp \ell$ and $\overleftrightarrow{CD} \perp m$. That is, ℓ is parallel to m with common perpendicular $\overleftrightarrow{CD}$. □

Corollary 1-1. *If two lines are perpendicular to the same line, then they are parallel.* (We see that this intuitive property of Euclidean geometry also holds in hyperbolic geometry.)

We already mentioned that Euclid often relied on unstated assumptions to prove his propositions in *The Elements*. One such assumption was the infinite extent of a straight line. Another was what is now called the Plane-Separation Axiom. This axiom states that any straight line in a plane splits the plane into two disjoint (nonempty) sets, and if any two points lie in separate halves of the split plane, they determine a line which intersects the original straight line. This ensures that if a line has two points on different sides of another line, the lines intersect somewhere. The Plane-Separation Axiom is required for a proof of the following Theorem. It is left as an exercise in the problem set.

Theorem 1-9. *If two lines have a common perpendicular, there exist transversals, other than the perpendicular, which cut the lines so as to form equal alternate interior angles (or equal corresponding angles). Moreover, the only transversals with this property are those which go through the point on that perpendicular which is midway between the lines.*

Theorem 1-10. *The distance between two parallels with common perpendicular is least when measured along that perpendicular. The distance from a point on either parallel to the other increases as the point recedes from the perpendicular in either direction.*

Proof. Let ℓ and m be parallel lines with common perpendicular $\overleftrightarrow{AB}$, which intersects m at A and ℓ at B. Let C be any point on ℓ other than B and construct a perpendicular to m through C. Let D be the point of intersection of the perpendicular with m (D is called the *projection of C on m*).

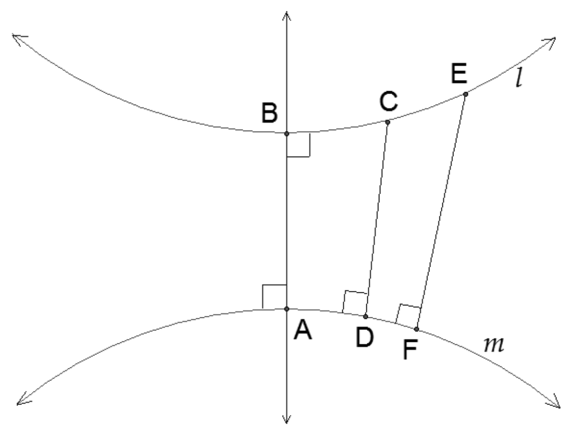


Figure 1-11

Then $ABCD$ is a Lambert quadrilateral, so $\angle BCD$ is acute and $CD > BA$ by Theorem 1-6. Therefore the distance is least when measured along the common perpendicular than when measure along any other perpendicular.

Now choose a point E on ℓ so that C is between E and B. As before, construct the perpendicular to m through point E. Call the intersection with m point F. Since $\angle BCD$ is acute, $\angle DCE$ is obtuse. Also, $AFEB$ is a Lambert quadrilateral, so $\angle CEF$ is acute. Thus $\angle DCE > \angle CEF$. By Theorem 1-4 this means that $\overline{EF} > \overline{DC}$, and it follows that the distance between the lines is increasing as we recede from the common perpendicular. □

Playfairs's Theorem is often used by geometers as an equivalent to the Parallel Postulate. It states that given a line and a point not on the line there exists a unique parallel to the line through the point. The following theorem is a blatant contradiction of this statement.

Theorem 1-11. *Given a line and a point not on the line there exist infinitely many parallels to the line through the point.*

Proof. Let ℓ be a line and L a point not on ℓ. Let M be the projection of L on ℓ. So $\overleftrightarrow{LM} \perp \ell$. Let k be the line through L perpendicular to $\overleftrightarrow{LM}$ (Proposition 11 of *The Elements*). Then k is a line parallel to ℓ with common perpendicular $\overleftrightarrow{LM}$.

Let L' be any point on k (WLOG) to the right on L. Let M' be the projection of L' on ℓ. By Theorem 1-10, we know $\overline{L'M'} > \overline{LM}$, so let P' be the point on $\overline{L'M'}$ such that $\overline{P'M'} \cong \overline{LM}$.

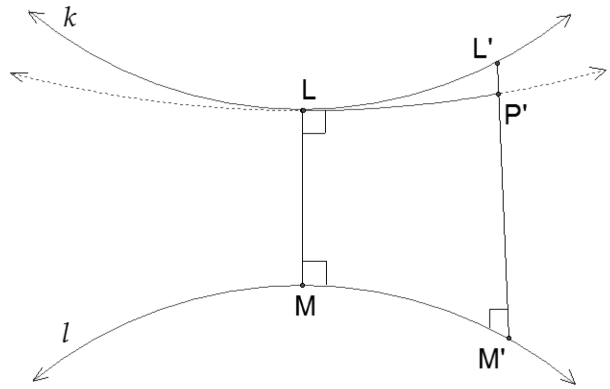


Figure 1-12

Then $MM'P'L$ is a Saccheri quadrilateral, so by Theorem 1-2 we know that the upper base $\overline{LP'}$ is on a line parallel to the lower base $\overline{MM'}$ with a common perpendicular. (Where is this common perpendicular?) Thus we have also $\overleftrightarrow{LP'} \| \ell$.

Now k and $\overleftrightarrow{LP'}$ are two lines through L parallel to ℓ. The line $\overleftrightarrow{LP'}$ was determined by our choice of L' to the right of L on k. Since L' was an arbitrary point, we want to conclude that if we choose a different point on k, say L'', that this will give us another parallel to ℓ through L different from k and $\overleftrightarrow{LP'}$. This would imply that there exist an infinite number of lines parallel to ℓ through L. We leave the rest of the proof as a homework problem. $\square$

So far we have discussed only parallel lines that have a common perpendicular. Two lines that are parallel with common perpendicular are said to be *hyper-parallel.* There are parallel lines in hyperbolic geometry, however, that are not hyper-parallel, that is, they do not have a common perpendicular. Such parallels are asymptotic in one direction and are called *limiting parallel lines*, or *horoparallels.* It can be proved that for any line ℓ and any point P not on ℓ, there are exactly two lines through P horoparallel to ℓ. Figure 1-13 shows the lines h_1 and h_2 horoparallel to a line ℓ through a point P.

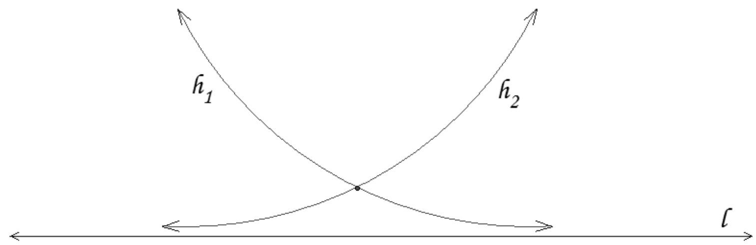


Figure 1-13

It is a famous result of hyperbolic geometry that the angle sum of a triangle is always less than 180°. We now offer a proof of this, and then state some other interesting theorems regarding hyperbolic triangles.

Theorem 1-12. *The angle sum of a triangle is always less than* 180°.

Proof. We show first that the theorem holds for right triangles and then prove the general case.

Let $\triangle ABC$ be a right triangle with right angle at A. Let $\overleftrightarrow{CD}$ be the line through C such that $\angle ABC \cong \angle BCD$ (Proposition 23 of *The Elements* gives us this construction).

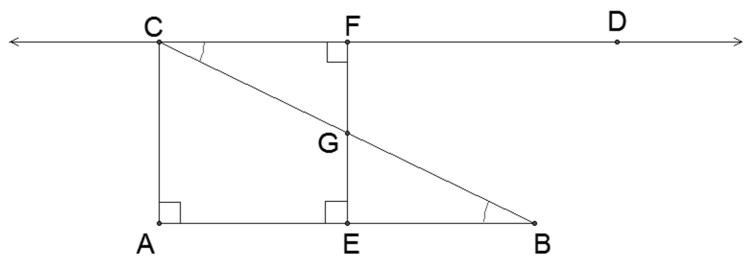


Figure 1-14

So $\overleftrightarrow{BC}$ is a transversal making equal alternate interior angles with respect to $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$. Then by Theorem 1-8, $\overleftrightarrow{CD} \| \overleftrightarrow{AB}$ with common perpendicular. Let this perpendicular be denoted by $\overleftrightarrow{EF}$ where E is the intersection of the perpendicular with $\overleftrightarrow{AB}$ and F is the intersection of the perpendicular with $\overleftrightarrow{CD}$. Let G be the midpoint of $\overline{EF}$. Then by Theorem 1-9, $\overleftrightarrow{CB}$ passes through G, and thus we may conclude $\overline{CG} \cong \overline{GB}$ (why?). Notice that $AEFC$ is a Lambert quadrilateral, so $\angle ACF$ must be acute. Therefore $\angle ACF = \angle ACB + \angle BCF \cong \angle CBA + \angle BCF < 90°$. It follows that the angle sum of $\triangle ABC$ is less than 180°, that is, in any right triangle the angle sum is always less than 180°.

Now consider any non-right triangle $\triangle PQR$.

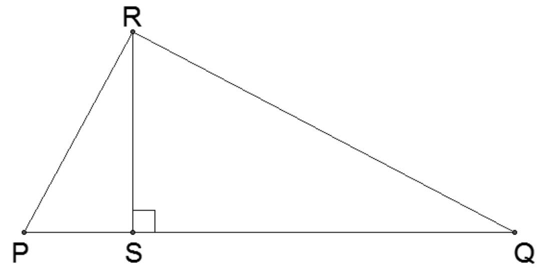


Figure 1-15

Since by Theorem 1-3, the angle sum of a triangle is at most $180°$, $\triangle PQR$ can have at most one obtuse angle, that is, it must have at least two acute angles, say at P and Q. Let S be the projection of R onto $\overleftrightarrow{PQ}$. Then S lies between P and Q (why?), so $\overline{RS}$ divides $\triangle PQR$ into two right triangles. Each right triangle has angle sum less than $180°$, and $\angle PSR + \angle QSR$ form a straight line, so the angle sum of $\triangle PQR$ is the sum of the angle sums of $\triangle PSR$ and $\triangle SQR$ minus $180°$. It follows that the angle sum of $\triangle PQR$ is less than $360° - 180° = 180°$. $\square$

Theorem 1-13 (AAA)**.** *If two triangles have the three angles of one congruent to the three angles of the other respectively, then the triangles are congruent.*

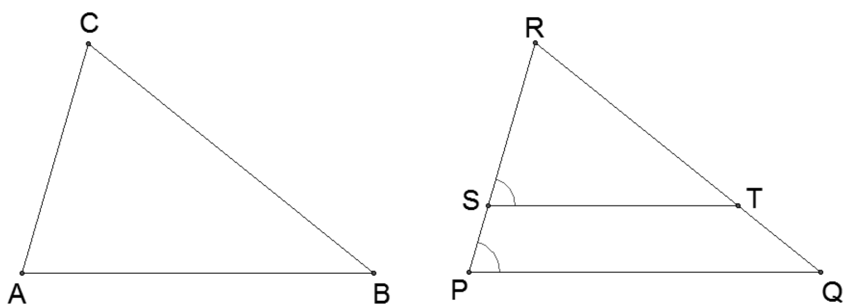


Figure 1-16

Proof. Let $\triangle ABC$ and $\triangle PQR$ be triangles such that $\angle A \cong \angle P$, $\angle B \cong \angle Q$, and $\angle C \cong \angle R$. By way of contradiction, suppose that the triangles are not congruent. Then there is a side of one triangle that is longer than the corresponding side in the other triangle, so suppose (WLOG) that $\overline{PR} > \overline{AC}$. Let S be on $\overline{PR}$ such that $\overline{SR} \cong \overline{AC}$, and construct

$\angle RST \cong \angle P$ (Proposition 23). Then by ASA, $\triangle$STR$\cong \triangle PQR$. The rest of the proof is left as an exercise in the problem set. □

Corollary 1-2. *Not all triangles have the same angle sum.*

While we already proved that the angle sum of a triangle is always less than 180°, we now know that this angle sum can vary. In fact, for any positive real number $x < 180°$, there exists a hyperbolic triangle with angle sum x.

In hyperbolic geometry, we define the *defect* of a triangle to be the amount by which its angle sum differs from 180°. The larger triangles are in the hyperbolic plane, the greater their defect, and conversely, the smaller triangles are in the hyperbolic plane, the smaller their defect. This means large triangles have angle sums near 0° and small triangles have angle sums near 180°. This demonstrates the property of hyperbolic geometry that very small portions of the hyperbolic plane behave almost like the Euclidean plane.

We now look at some ways to model the hyperbolic plane.

Models of Hyperbolic Geometry

Henry Poincare (1854-1912) is credited in name with two models of hyperbolic geometry: the Poincare half-plane and the Poincare disk models of hyperbolic geometry. We will briefly discuss the two models, with the disclaimer that a thorough understanding of them requires a more rigorous study than is presented here. The Poincare half-plane model is the upper half-plane of the Euclidean plane (all points (x, y) such that $y > 0$) together with a hyperbolic metric, which is a formula for measuring distance in hyperbolic geometry. Lines in this model are represented by the arcs of circles in the upper half-plane whose center lies on the x-axis, and by straight Euclidean lines which are perpendicular to the x-axis. One might think of these straight Euclidean lines as being arcs of circles of infinite radii. Arcs in the upper-half plane that intersect on the x-axis represent horoparallel lines; arcs which do not intersect in the upper-half plane represent hyper-parallel lines, and arcs which intersect at right angles represent perpendicular lines. The angle between two intersecting arcs is determined by the measure of the angle between the tangent rays to the arcs. Figure 1-17 shows some lines in the Poincare half-plane. In Figure 1-17, observe that lines ℓ and k are horoparallel, lines n and k are hyper-parallel with common perpendicular p, and lines l, m, and q determine a hyperbolic triangle.

Now suppose one could pick-up the "ends" of the x-axis which lie at infinity and glue them together at a point. The result is something like a disk. What happens to our hyperbolic lines in the half-plane when we mold the half-plane into a disk? The lines perpendicular to the x-axis become diameters of the disk, and the arcs of circles with centers on the x-axis become arcs of circles that intersect the disk's boundary perpendicularly. This is basically the Poincare disk model.

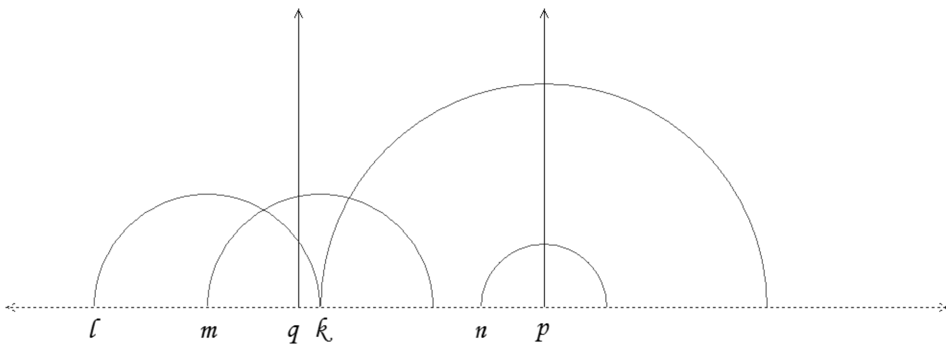


Figure 1-17

The Poincare hyperbolic disk is defined to be the interior of the disk of radius 1 about the origin, together with the hyperbolic metric. A hyperbolic point is thus a point inside the disk. A hyperbolic line is represented by an arc of a circle whose endpoints intersect the boundary of the disk at right angles. Diameters of the disk are also considered hyperbolic lines in this model (again, one might think of them as arcs of a circle of infinite radius). The reader should think of the boundary of the disk as representing infinity. Arcs that intersect on the boundary of the disk represent horoparallel lines which are asymptotic and "meet" at infinity. Arcs that do not intersect on the interior of the disk or on the boundary represent hyper-parallel lines. Arcs that intersect at right angles in the interior of the disc represent perpendicular lines. As before, the angle between two intersecting arcs is determined by the measure of the angle between the tangent rays to the arcs.

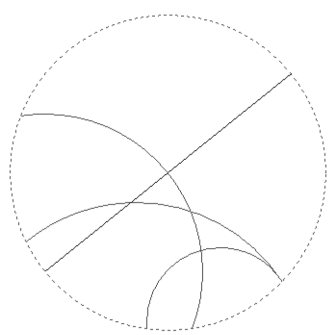


Figure 1-18

Examine Figure 1-18, which represents the Poincare disk, and locate a pair of horoparallel lines, a pair of hyper-parallel lines, a pair of perpendicular lines, and a hyperbolic triangle.

An interesting construction problem is how one might construct a line in the Poincare disk through two points. If the points lie on a diameter, the construction is obvious. If the points do not determine a diameter of the disk, however, how might one do this construction? This is explored in the problem set.

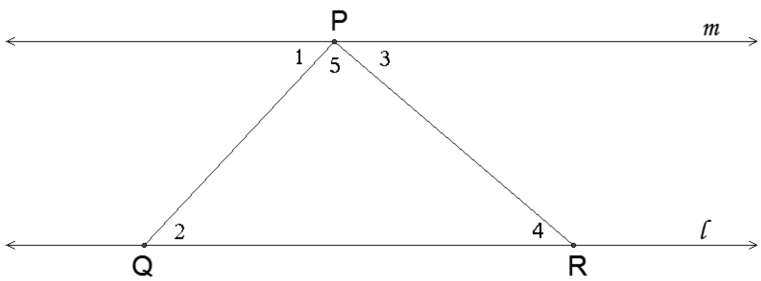
Problem Set 1

1. List the axioms and assumptions we used in our development of hyperbolic geometry. You may reference the appendix.

2. Find the flaw in the proof of the following claim:

   Claim: There is a triangle in hyperbolic geometry whose angle sum is 180°.

   Proof: Let ℓ be a line, and let m be a parallel to ℓ through point P not on ℓ. Let Q and R be distinct points on ℓ. Then lines $\overleftrightarrow{QP}$ and $\overleftrightarrow{RP}$ are transversals that cut the parallel lines ℓ and m.

   Then as in the figure, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Thus, the angle sum of $\triangle QRP$ is given by $\angle 2 + \angle 5 + \angle 4 = \angle 1 + \angle 5 + \angle 3 = 180°$.

3. Complete the following sentences:

   a. The angle sum of a quadrilateral in neutral geometry is ______________ 360°.
   b. The angle sum of a quadrilateral in hyperbolic geometry is ______________ 360°.
   c. Write a proof for your statement in part b.

4. Write a proof for Theorem 1-6.

5. In hyperbolic geometry, why can there be no squares or rectangles?

6. Show that in hyperbolic geometry there exist rhombi with equal angles. Do rhombi with equal angles exist in Euclidean geometry other than the square?

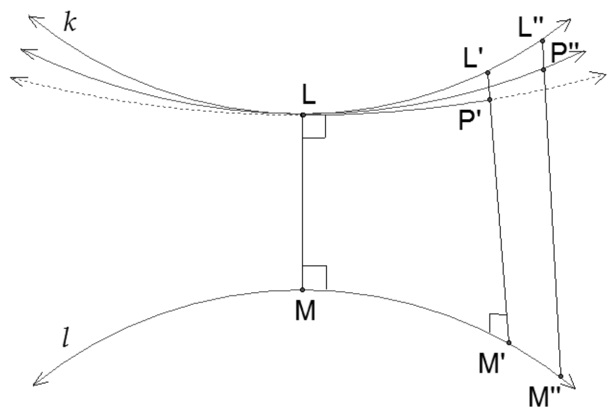
7. Proof of Theorem 1-9: Let ℓ and m be lines with a common perpendicular $\overleftrightarrow{AB}$ for A on ℓ and B on m. Theorem 1-8 tells us that $\ell \| m$.

   a. Show that there exists a transversal cutting ℓ and m that makes equal alternate interior angles and equal corresponding angles. *Hint:* If P is the midpoint of AB construct a transversal through P.

b. Where, if anywhere, did you use the Plane-Separation Axiom in part a.?

c. Show that if there exists a transversal of ℓ and m which makes equal alternate interior angles, and which does not intersect $\overline{AB}$ at the point midway between ℓ and m, them one can construct a second common perpendicular to ℓ and m. *Hint:* You may need to consider two cases.

8. Completing the proof of Theorem 1-11. Using the construction from the proof of Theorem 1-11 as depicted in figure below, choose a point L'' to the right of L' on k. Let M'' and P'' be constructed in the same way as M' and P' in the proof of Theorem 1-11.



Show that the lines $\overleftrightarrow{LP'}$ and $\overleftrightarrow{LP''}$ are not the same. Explain why this means there are infinitely many lines parallel to a given line through a point not on it.

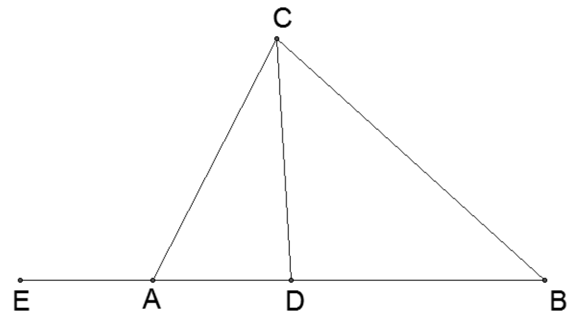
9. Prove that an angle inscribed in a semicircle in hyperbolic geometry is always acute.

10. Exhibit a pair of parallel lines and a transversal so that on one side of the transversal, the sum of the interior angles is less than 180°.

11. Show that no two hyperbolic lines are equidistant. *Hint:* Show that the distance from one line to another cannot be the same at more than two points. Recall that distance from a point to a line is given by the line segment perpendicular to the line through the point.

12. Consider triangle a $\triangle ABC$ and let D be a point between A and B. Extend $\overline{AB}$ slightly to create exterior angle $\angle EAC$.

Describe what happens to angles $\angle EDC$ and $\angle ACD$ as D approaches A along segment $\overline{AB}$. What happens to the angle sum of $\triangle ADC$? How might you use this construction to demonstrate a triangle with angle measure arbitrarily close to 180°?



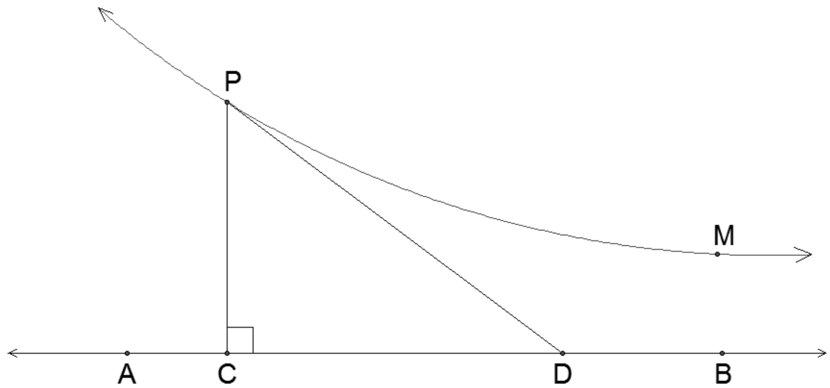
13. Finish the proof of Theorem 1-13. *Hint:* Use what you proved in problem 2 along with what you know about the angle sum of a hyperbolic triangle.

14. Recall that the definition of the *defect of a hyperbolic triangle* is the amount by which the angle sum of the triangle differs from $180°$, what do you think is the definition of the *defect of an n-sided polygon*? Give an upper bound for the defect of an n-sided hyperbolic polygon.

15. It can be shown that the area of a hyperbolic triangle T_d with defect d is given by the formula $A(T_d) = k \cdot d$ for some constant k which relates the units of length to the units of area. This means that the area of a triangle is determined by its defect.

    a. Give an upper-bound for the area of a triangle in hyperbolic geometry. This means that no matter how "big" we make our triangles, the area is always less than this upper-bound. Thus, unlike in Euclidean geometry, we cannot construct triangles of arbitrarily large area.
    b. Give a formula for the area of an n-sided hyperbolic polygon.
    c. Is there an upper-bound for the area of an n-sided hyperbolic polygon?
    d. Can we construct hyperbolic polygons of arbitrarily large area?

16. Let $\overline{AB}$ be a hyperbolic line and P a point not on it. Let $\overleftrightarrow{PM}$ be a line horoparallel to $\overline{AB}$ as in the figure below. Let $\overline{CP}$ be the segment perpendicular to $\overline{AB}$ through P, and let D be any point on $\overline{AB}$ between C and B. What happens to $\angle CPD$ as D moves to the right along $\overline{AB}$? What angle does it approach? This angle is called the *angle of parallelism* for segment $\overline{CP}$.



17. On your own paper, draw in the Poincare half-plane:

    a. A pair of hyper-parallel lines.

    b. The common perpendicular of your hyper-parallel lines.

    c. A pair of horoparallel lines.

    d. A hyperbolic triangle.

    e. A Saccheri quadrilateral.

18. To construct a line through two points in the Poincare disk. Using a compass and ruler:

    a. Draw a circle and label the center O.

    b. Pick two points inside the circle that do not lie on a diameter. Label them P and Q. Draw the Euclidean line $\overleftrightarrow{OP}$.

    c. Draw the Euclidean line perpendicular to $\overline{OP}$ through point P. Label the points of intersection of this perpendicular with the circle S and T.

    d. Draw the tangent lines to the circle at points S and T. These tangents intersect at a point. Label the point P'.

    e. Draw the circle through the points Q, P, and P'.

    f. At what angle does the circle you just constructed intersect the boundary of your original circle?

19. Now choose a point X in the Poincare disk other than your P and Q from problem 16. Using a compass and ruler, accurately draw the two lines through X horoparallel to the hyperbolic line $\overleftrightarrow{PQ}$ in problem 16. *Hint:* To construct a horoparallel, use the

same construction as in problem 16 to draw the hyperbolic line through X and Y, where Y is an appropriately chosen point.

20. Can you draw any other lines in the Poincare disk, besides those you drew in problem 16 which, are horoparallel to $\overleftrightarrow{PQ}$ through X?

Chapter 2: Elliptic Geometry

Hyperbolic geometry was the result of Saccheri investigating the hypothesis of the acute angle. Now we look at the results garnered from the hypothesis of the obtuse angle. From the introduction we know that Saccheri discovered that the hypothesis of the obtuse angle implied that the upper base is shorter than the lower base in the Saccheri quadrilateral, and that the angles of a triangle sum to more than 180°. We also mentioned that Saccheri reached a contradiction in this case by assuming that lines have infinite length, so we know that lines in *elliptic geometry* (for that is what this geometry came to be known as) have finite length. These lines must then turn inward on themselves, much like circles, in order to preserve the truth of Euclid's second postulate. We further assume that all lines have the same finite length. Since lines behave like finite circles, the notion of betweenness of points on a line no longer holds. Euclid's third postulate claims the existence of arbitrarily large circles, but since our lines are now finite in length, we may describe circles only with limited radius (less than or equal to half the length of a line), so we should modify this postulate appropriately. Furthermore, we are allowed the use of the first 15 of Euclid's propositions along with select other propositions such as the ASA congruence condition for triangles (Proposition 26) and the ability to construct an angle at a point which is congruent to a given angle (Proposition 23), as they do not depend on any of the changes we have made to his original assumptions. To all of this, we add one more axiom for use in later Theorems:

Axiom 2-1 (Pasch's Axiom)**.** *If a line intersects a side of a triangle, and does not intersect any of the vertices, it also intersects another side of the triangle.*

Pasch's Axiom, named for the German mathematician Moritz Pasch (1843-1930), can be derived from the Axiom of Separation mentioned in the chapter on hyperbolic geometry. In Elliptic geometry, however, the Axiom of Separation may not hold. We will discuss this further when we look at models of elliptic geometry.

Under Euclid's original assumptions, stated and unstated, the existence of parallel lines was certain. Now that we have altered some of Euclid's fundamental assumptions, however, the existence of parallels is by no means certain. In fact, the contrary turns out to be true: that in elliptic geometry there are no parallel lines! Some other interesting properties include:

1. The area of a triangle is determined by its angle sum.
2. All lines that are perpendicular to a given line meet at a point. This point is called the *pole* of the line.

3. If two triangles have three angles congruent, then the triangles are congruent.

Finally, we give our substitution for the parallel postulate.

Axiom 2-2 (Elliptic Parallel Postulate)**.** *The upper base angles of the Saccheri quadrilateral are obtuse.*

Theorem 2-1. *Let $ABCD$ be a Lambert quadrilateral with right angles at A, B, and C. Extend $\overline{AB}$ to $\overline{AE}$ so that $\overline{AB} \cong \overline{BE}$. Extend $\overline{DC}$ to $\overline{DF}$ so that $\overline{DC} \cong \overline{CF}$. Then $AEFD$ is a Saccheri quadrilateral.*

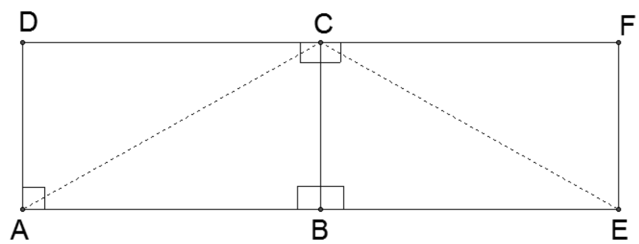


Figure 2-1

Proof. By SAS (Proposition 4), $\triangle ABC \cong \triangle EBC$, so $\overline{EC} \cong \overline{AC}$. Also, $\angle ACB \cong \angle ECB$, and since $\angle ACB$ is complementary with $\angle DCA$, this implies $\angle DCA \cong \angle FCE$. Then by SAS, $\triangle ADC \cong \triangle EFC$. Thus $\overline{EF} \cong \overline{AD}$. Also, we have $\angle FEC + \angle BEC = \angle DAC + \angle BAC = \angle DAB = 90°$. Therefore $AEFD$ is a Saccheri quadrilateral. □

Corollary 2-1. *The third angle of a Lambert quadrilateral is obtuse.*

Theorem 2-2. *In the Lambert quadrilateral $ABCD$ with right angles at A, B, and C, $\overline{AD} < \overline{BC}$ and $\overline{DC} < \overline{AB}$.*

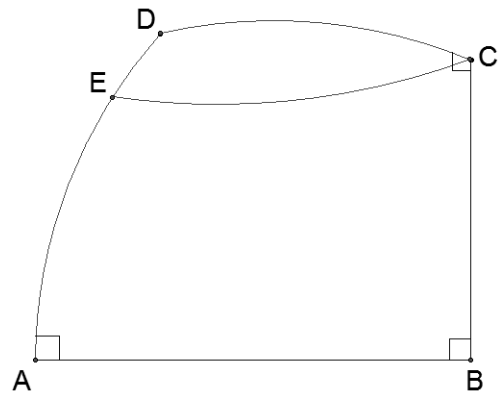


Figure 2-2

Proof. By way of contradiction, suppose that $\overline{AD} > \overline{BC}$. Then let E be the point on $\overline{AD}$ such that $\overline{AE} \cong \overline{BC}$. Then $\overline{EC}$ cuts the angle at C on its interior, and $ABCE$ is a Saccheri quadrilateral. This implies that $\angle BCE$ is obtuse, but since $\angle BCE$ is on the interior of $\angle C = 90°$, it must be less than or equal to $90°$, which is a contradiction. Thus, $\overline{AD} < \overline{BC}$. Similarly, $\overline{DC} < \overline{AB}$. □

Theorem 2-3. *In elliptic geometry, any two lines intersect.*

Proof. By way of contradiction, let ℓ and m be two lines that do not intersect. These lines are of finite length, so there exists a point on ℓ which is of least distance to m. Call this point A. This implies that the line segment perpendicular to m through A is shorter than any other perpendicular to m through any point on ℓ. Let B be the point of intersection of this shortest segment with m. We will show:

1. $\overline{BA}$ meets ℓ at right angles.

2. There is a point E on ℓ which is closer to m than A.

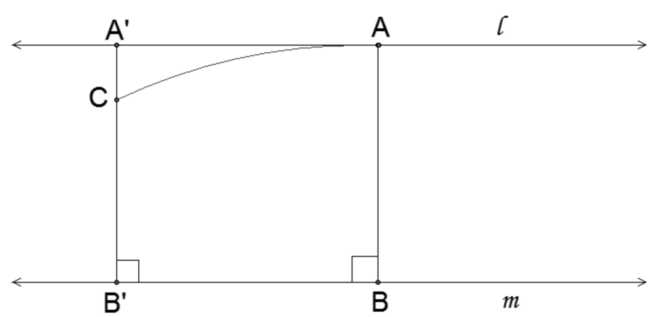


Figure 2-3

In order to prove 1, suppose $\overline{BA}$ does not meet ℓ at right angles. Then one of the angles at A must be acute. Let A' be a point on ℓ on the "side" of the acute angle. In elliptic geometry, there are no "sides" of a line, because "betweenness" no longer applies, but we use the term here to help visualize the argument. Let B' be the point on m such that $\overleftrightarrow{B'A'}$ is the perpendicular to m through A'. Since $\overline{BA}$ is shorter than $\overline{B'A'}$, let C be the point on $\overline{B'A'}$ so that $\overline{B'C} \cong \overline{BA}$. Then $BACB'$ is a Saccheri quadrilateral, so $\angle CAB$ is obtuse. However, $\angle CAB$ lies interior to $\angle A'AB$, which is acute, so we have a contradiction. Therefore, $\overline{BA}$ meets ℓ at right angles.

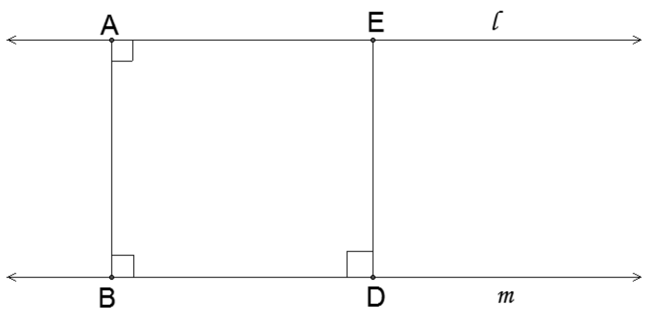


Figure 2-4

For 2, let D be any point on m apart from B, and construct the perpendicular to m at D. Let E be the point of intersection of this line with ℓ. Then $BDEA$ is a Lambert quadrilateral, so by Theorem 2-2, $\overline{DE} < \overline{BA}$, but this contradicts our assumption that $\overline{BA}$ was the shortest line segment through ℓ perpendicular to m. Therefore our original assumption that ℓ and m do not intersect must be at fault. □

Theorem 2-4. *The upper base of the Saccheri quadrilateral is shorter than the lower base, and the altitude is greater than both the arms.*

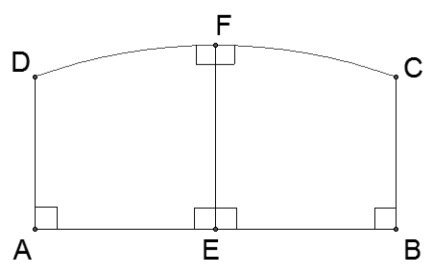


Figure 2-5

Proof. Let $ABCD$ be a Saccheri quadrilateral, and let E and F be the points on the lower base and upper base respectively that determine the altitude. Then $AEFD$ and $EBCF$ are both Lambert quadrilaterals, so $\overline{DF} < \overline{AE}$ and $\overline{FC} < \overline{EB}$ by Theorem 2-2. Therefore, $\overline{DC} < \overline{AB}$. Also by Theorem 2-2 we have $\overline{AD} < \overline{EF}$ and $\overline{BC} < \overline{EF}$. □

An interesting property of both hyperbolic and elliptic geometries is that they behave almost like Euclidean geometry on small, restricted areas. For example, the smaller one makes a triangle in either elliptic or hyperbolic geometry, the closer the angle sum is to 180°. This also means that many of the propositions and results of regular Euclidean plane geometry do hold in small portions of the elliptic plane. Recall that we could not use Euclid's propositions past Proposition 15, because the other propositions depend on the assumption that lines may be extended infinitely. If we restrict the area we are working in, however, and make our geometric figures sufficiently small, we are then able to extend lines "enough" to be able to use, at least locally, some of these propositions we originally discarded. In the next theorem, we restrict the area we are allowed to work on and borrow Euclid's Proposition 25 to prove that locally, the angle sum of a triangle is greater than 180°.

Theorem 2-5. *Locally, the angle sum of an elliptic triangle is greater than* 180°*.*

Proof. Let ABC be a right triangle with right angle at B. Let D be the point so that $\overline{AD}$ is perpendicular to $\overline{AB}$ and $\overline{AD} \cong \overline{BC}$.

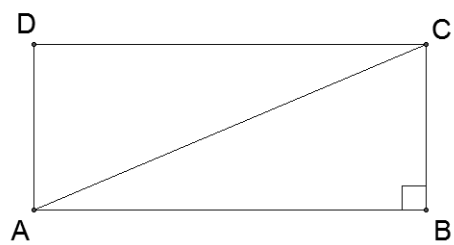


Figure 2-6

Then $ABCD$ is a Saccheri quadrilateral, so $\overline{DC} < \overline{AB}$. Then we have for triangles $\triangle ADC$ and $\triangle CBA$ that $\overline{AD} = \overline{BC}$ with shared edge $\overline{AC}$ and $\overline{DC} < \overline{AB}$. So by Euclid's Proposition 25, $\angle DAC < \angle BCA$. Therefore, $90° = \angle DAC + \angle BAC < \angle BCA + \angle BAC$, and it follows that the angle sum of right triangle $\triangle ABC$ is greater than 180°. Since we may subdivide any triangle into two right triangles by drawing the line segment perpendicular to the side subtending the largest angle and through the vertex of that angle, we then have that in any triangle in our restricted area, the angle sum is greater than 180°. □

Theorem 2-6. *In general, the area of an elliptic triangle is greater than* $180°$.

Idea of Proof. The proof of this theorem is done by breaking any triangle up into smaller and smaller triangles until we are able to apply Theorem 2-5. Then we repeatedly apply the theorem until we are able to show that the large triangle has angle sum greater than $180°$. □

Theorem 2-7. *The angles of a quadrilateral sum to more than* $360°$.

Proof. Divide a quadrilateral into two triangles and apply Theorem 2-7. □

We define the *excess of a triangle* in elliptic geometry to be the amount by which the triangle differs from $180°$ and the *excess of a quadrilateral* to be the amount by which the quadrilateral differs from $360°$.

Theorem 2-8. *If a line cuts a triangle so as to form one quadrilateral and one triangle, or two triangles, then the excess of the original triangle is the sum of the excesses of the smaller triangle and quadrilateral, or of the two smaller triangles.*

The proof is left as an exercise in the problem set.

Theorem 2-9. *[AAA] If two triangles have the three angles of one congruent to the three angles of the other respectively, then the triangles are congruent.*

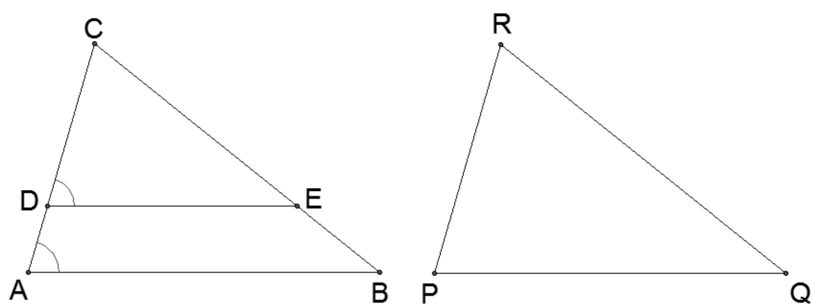


Figure 2-7

Proof. Let ABC and PQR be triangles with $\angle A \cong \angle P$, $\angle B \cong \angle Q$, and $\angle C \cong \angle R$. It follows that the excess of $\triangle ABC$ is equal to the excess of $\triangle PQR$. By way of contradiction, suppose that the two triangles are not congruent, and suppose (without loss of generality) that $\overline{AC} > \overline{PR}$. Let D be the point on $\overline{AC}$ such that $\overline{CD} \cong \overline{PR}$. Let E be the point on BC such that $\angle CDE \cong \angle A$ (Proposition 23). Then by ASA, we have $\triangle DEC \cong \triangle PQR$, and so $\angle CDE \cong \angle P$, $\angle DEC \cong \angle Q$, and $\angle C \cong \angle R$. Therefore, the excess of $\triangle DEC$ is equal to the excess of $\triangle PQR$. $\overline{DE}$ partitions $\triangle ABC$ into two triangles or into a triangle and a quadrilateral (here we invoke Pasch's Axiom). We will not lose anything by assuming that $\overline{DE}$ partitions $\triangle ABC$ into a triangle and a quadrilateral, because the other case plays out similarly. Let EX() denote the excess of a figure. Then applying Theorem 2-8, we have $\mathrm{EX}(\triangle ABC) = \mathrm{EX}(ABED) + \mathrm{EX}(\triangle DEC) = \mathrm{EX}(ABED) + \mathrm{EX}(\triangle PQR) = \mathrm{EX}(ABED) + \mathrm{EX}(\triangle ABC)$. Thus $\mathrm{EX}(ABED) = 0$, which contradicts Theorem 2-7. Thus, $\triangle ABC \cong \triangle PQR$. $\square$

Corollary 2-2. *In elliptic geometry, similar triangles are always congruent.*

Models of Elliptic Geometry

It may be helpful to visualize elliptic geometry as being somewhat like geometry on the surface of the sphere, where lines are represented by diameters called *great circles*. Great circles may be described geometrically as the intersections of the sphere with planes through the sphere's center. This by itself is not a true model of elliptic geometry, however, because it violates Euclid's first postulate, namely that two points determine a unique line. Can you find two points on the sphere that lie on more than one great circle? Such points are called *antipodal* points on the sphere, and they are those points which are diametrically opposite one another.

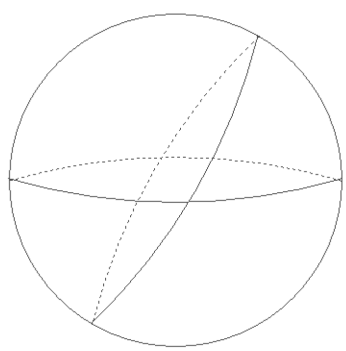


Figure 2-8

Riemann changed Euclid's first postulate to read "there is *at least* one line through any two distinct points." If this postulate replaces Postulate 1, many results are the same. This is sometimes called *double elliptic geometry*. Double elliptic geometry can then be modeled by the sphere with great circles as lines. Elliptic geometry can be modeled by the sphere as well, with one additional condition. Since we require the uniqueness of the line determined by two points, we *identify* antipodal points, that is, we consider two points which are opposite one another on the sphere to be the same point. Then this model satisfies all the axioms of elliptic geometry and is called the *Real projective plane*. When we take the sphere of radius 1, called the *unit sphere* as our base sphere for the Real projective plane, we have the following theorem by the French mathematician Albert Girard (1595-1632), which illustrates the result in elliptic geometry that the area of a triangle is determined by its angle sum. The theorem's proof is explored in the problem set.

Theorem 2-10. *The area of a triangle on the Real projective plane is equal to its angle sum minus* π*. That is, the area of such a triangle is equal to its excess. (Note that we use radians here since we are working on the sphere.)*

Since the Real projective plane is a sphere with antipodal points identified, we might think of a portion of the projective plane as being like a hemisphere without the equator at its boundary. Can you tell why the Plane-Separation Axiom fails to hold for this model? Thus we are able to easily visualize how lines and figures behave in elliptic geometry, at least on a portion of the elliptic plane. Results such as those in Theorems 2-3 and 2-6 no longer seem so strange, and one might even get an intuitive feel for how this geometry behaves.

One finds applications of the results of elliptic geometry in areas such as navigation (the shape of the earth is, after all, roughly spherical), and cosmology. Scientists believe that the shape of the universe may determine its future. If the universe is shaped so as to exhibit elliptic geometry, it is theorized that our universe at some point may cease expansion and begin to implode!

Problem Set 2

1. List the axioms and assumptions we used in our development of elliptic geometry. You may reference the appendix.

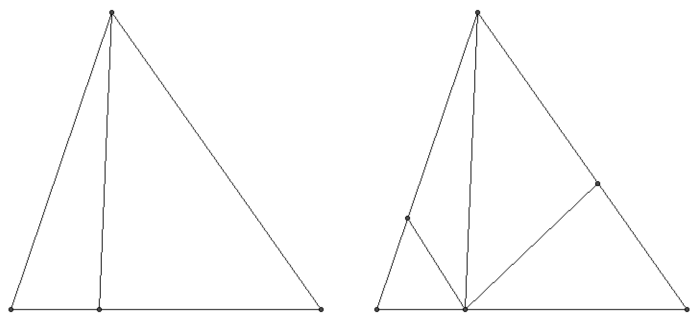
2. Explain why Corollary 2-1 follows from Theorem 2-1.

3. Prove that all lines which are perpendicular to a given line intersect at the same point. This point is called the *pole* of the line. (*Hint:* Euclid's Proposition 6 states that if two angles of a triangle are equal, the sides which these angles subtend are also equal. Use this, and a theorem about the Saccheri quadrilateral to do a proof by contradiction.)

4. Discussion of Theorem 2-8:

    a. Write a proof of Theorem 2-8.

b. Consider the case in Theorem 2-8 where we partition the original triangle into two smaller triangles. Now cut the two smaller triangles into smaller triangles. What is the relationship of the excesses of the smaller triangles to the excess of the original triangle? In general, if we proceed in this manner of cutting up the triangle, what is the relationship of the smaller triangles to the larger ones?



5. Can you give an upper-bound for the excess of an elliptic triangle?

6. Similarly to hyperbolic geometry, the area of an elliptic triangle is proportional to its excess. If an elliptic triangle T_e of excess e has its area defined by $A(T_e) = k \cdot e$ for a constant k, what is an upper bound for the area of a triangle?

7. Given an elliptic triangle, construct a Saccheri quadrilateral of equal area by following the steps in a. and b. below. Use great circles of the sphere as lines.

   a. Let $\triangle ABC$ be a triangle in the elliptic plane so that $\overline{AB}$ is the longest side. Construct the midpoints of $\overline{AC}$ and $\overline{BC}$. Label them D and E respectively. Draw line $\overleftrightarrow{DE}$.

   b. Construct the perpendicular to $\overline{DE}$ through A, the perpendicular to $\overline{DE}$ through B, and the perpendicular to $\overline{DE}$ through C. Label the points of intersections of these perpendiculars with $\overline{DE}$ as F, G, and H respectively.

   c. Prove: $ABGF$ is a Saccheri quadrilateral, and that the area of $ABGF$is equal to the area of $\triangle ABC$.

8. Given Saccheri quadrilateral in the elliptic plane, describe a procedure for constructing an elliptic triangle of equal area.

9. Prove that an angle inscribed in a semicircle is always obtuse in elliptic geometry.

10. Use a ball (orange, or other round object) to represent the sphere. Use great circles to draw on the ball a Saccheri quadrilateral. Approximate the measure of the upper

base angles with a protractor. Are they acute, obtuse, or equal? How does the upper base compare to the lower base?

11. Prove that in the double-elliptic geometry of the sphere, no two vertices of a triangle are antipodal points.

12. In the double-elliptic geometry of the sphere, two distinct lines form a lune in a hemisphere. Prove that the two angles of a lune formed in this way are congruent.

13. Recall that in the model of elliptic geometry on the unit sphere one way of ensuring the uniqueness of a line determined by two points is to identify antipodal points.

    a. Draw a sphere and demonstrate three pairs of antipodal points on it.

    b. When we identify antipodal points, the resulting surface is called the Real projective plane. Imagine what such a surface must look like. Try to draw a picture of how you imagine it.

    c. Explain why identifying antipodal points ensures the uniqueness of the line determined by two distinct points on the sphere.

14. Understanding radians:

    a. Consider the unit sphere. What is the radius of a great circle on the unit sphere?

    b. One radian is defined to be $180/\pi$ degrees. Let A and B be two opposite points on a great circle of the unit sphere. Let C be the circle's center. In radians, what is the measure of $\angle ACB$? ($\angle ACB$ is an example of a *central angle* because it has its vertex at the circle's center.)

    c. Draw two points on a great circle such that the central angle whose edges pass through those points has a measure of $\pi/2$ radians.

    d. The length of a circle is π times the circle's diameter. What is the length of a segment on a great circle of the unit sphere which subtends a central angle of measure $\pi/2$ radians, $\pi/4$ radians, and π radians? What is the relationship between the central angle and the arc it subtends?

    e. Repeat part d., but this time with a circle of arbitrary radius r. (Note: The relationship you discover between the central angle and the corresponding arc is one reason why it is nice to use radians when working with circles or spheres.)

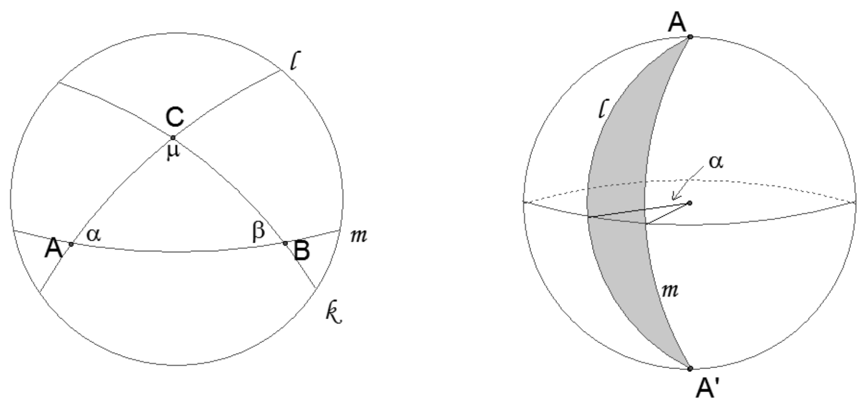
15. Consider the Real projective plane on the unit sphere as a model of elliptic geometry.

    a. The length of a line on the unit sphere is 2π. What is the length of a line on the Real projective plane?

b. The area of the unit sphere is 4π. What is the area of the Real projective plane?

16. Exploring the proof of Theorem 2-10:

    a. Let $\triangle ABC$ be a triangle on the Real projective plane, which for our purposes will look like a hemisphere of the unit sphere. Let $\triangle ABC$ be determined by lines ℓ, m, and k as in the figure below. Then lines ℓ and m meet at antipodal points A and A' and determine a lune. What is the area of the lune in terms of the angle α?



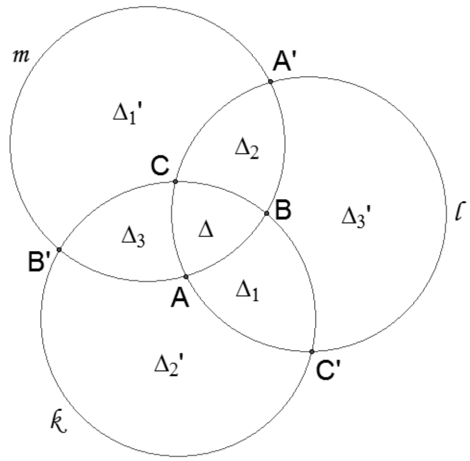
    i. *Hint 1:* α is the angle of the triangle at point A, but it is also the central angle formed by the equator that equally divides the lune as seen in the figure above.

    ii. *Hint 2:* Since $360° = 2\pi$ radians, the area of the lune times $2\pi/\alpha$ equals the area of the sphere. Convince yourself that this is true, and then use it to determine the area of the lune.

    b. Similarly to part a., what is the area of the lune determined by lines ℓ and k? By lines m and k?

    c. Since great circles of the unit sphere meet at antipodal points and equally divide the sphere into two hemispheres, for every triangle in one hemisphere there is a corresponding *antipodal triangle* in the opposite hemisphere. Show that the area of a triangle is equal to the area of its antipodal triangle.

    d. Imagine we could break apart and flatten the sphere and put it on a table to look at. Then it might look something like the figure below, where Δ' is the antipodal triangle of Δ, C' is the antipodal point of C, and so on.



Then the area of Δ equals the area of Δ', the area of Δ_1 equals the area of Δ_1', and so on. Suppose that Δ is the same triangle as in part a. What is the area of $\Delta + \Delta_1$, $\Delta + \Delta_2$ and $\Delta + \Delta_3$?

e. The area of the unit sphere is 4π, so $2\Delta + 2\Delta_1 + 2\Delta_2 + 2\Delta_3 = 4\pi$. Use this, and your results from part d., to show that $\Delta = \alpha + \beta + \mu - \pi$ as required by Theorem 2-10.

Chapter 3: Taxicab Geometry

We have thus far made a cursory examination of the two most notable forms of non-Euclidean geometry: hyperbolic and elliptic. We now give a brief introduction to taxicab geometry. Taxicab geometry involves working in the Euclidean coordinate plane, and satisfies all of Euclid's postulates and assumptions except the assumption that all straight lines may be made to coincide. This forfeits us the general use of congruence conditions for triangles in taxicab geometry. The reader might wonder at such an odd name for this geometry, seemingly un-mathematical when compared to the titles "elliptic" and "hyperbolic." The name arises intuitively from the way of measuring distance in this geometry. The usual way of measuring distance in Euclidean geometry is through use of the Pythagorean Theorem.

This gives us that if $S = (x_1, y_1)$ and $T = (x_2, y_2)$ are two points in the Euclidean coordinate plane, the distance between them is given by

$$d_E(S,T) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where we denote the Euclidean distance function by d_E. One begins a discussion of taxicab geometry by redefining this distance function. We still work in the Euclidean coordinate plane, and lines, angles, and points are all the same, but by changing the way we measure distance, we are brought to an entirely different geometry.

Definition 3.1. Let S and T be as in the previous paragraph, and let d_T denote the taxicab distance function. Then d_T is called the taxicab metric, and $d_T(S,T) = |x_1 - y_1| + |x_2 - y_2|$.

Imagine you live in apartment downtown in the imaginary City X, and you need to get to the opera hall. You live reasonably far away, so you decide to take a taxi. As with many cities, the map of the streets of City X looks like a grid. In Figure 3-1, the solid thin line represents the distance from your apartment to the opera hall as given by the Euclidean metric d_E, while the solid thick line represents the distance as given by the taxicab metric d_T; so $d_E(\text{apartment}, \text{opera hall}) = 5$, while $d_T(\text{apartment}, \text{opera hall}) = 7$.

Notice that the distance given by d_T is precisely the way a taxicab might travel along the streets of City X to get to the opera hall. This is where taxicab geometry derives its name. Thus we must think of distance in this geometry as though we had to travel through the infinite streets of City X (or parallel to them through city blocks) to get from point to point.

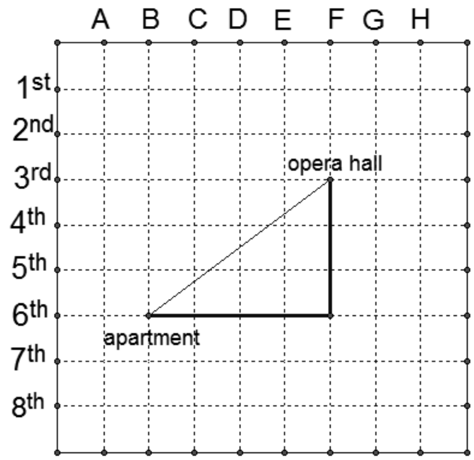


Figure 3-1

The taxicab metric alters our notion of certain geometric shapes. For example, a *circle of radius r around point P* (we will write $C(r, P)$) is defined to be the set of all points of distance r from p. We will denote $C(r, P)$ under the usual Euclidean metric by $C_{d_E}(r, P)$, and $C(r, P)$ under the taxicab metric by $C_{d_T}(r, P)$. We all know what a circle in regular Euclidean coordinate geometry looks like. Think for a moment how our new notion of distance alters the way we draw a circle in the plane. Figure 3-2 shows $C_{d_E}(2, P)$ on the left, and $C_{d_T}(2, P)$ on the right.

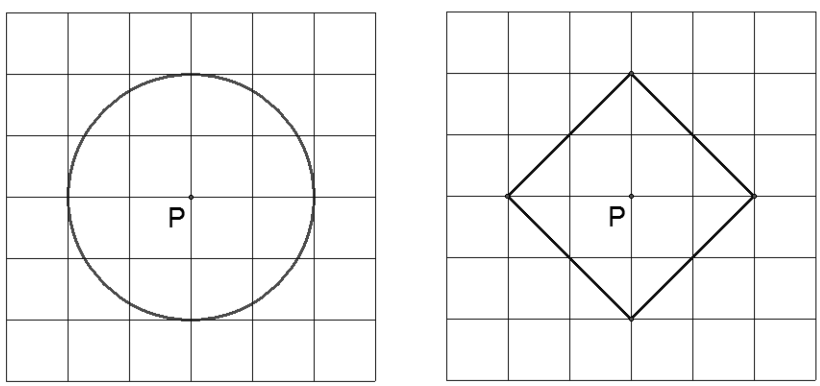


Figure 3-2

So circles in taxicab geometry are squares! Another consequence of the taxicab metric is that it destroys our ability to apply the congruence conditions for triangles so often used in other geometries.

Theorem 3-1. *In taxicab geometry, the SAS, ASA, SAA, and SSS congruence conditions for triangles to not hold in general.*

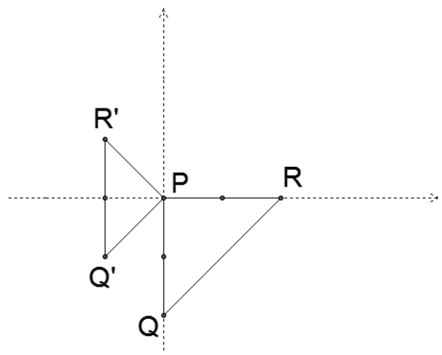


Figure 3-3

Proof. We exhibit a counter example to the SAS congruence condition. The reader is asked to demonstrate counterexamples for the other congruence conditions in the problem set. In Figure 3-3 we see two clearly incongruent triangles $\triangle PQR$ and $\triangle PQ'R'$ by Euclidean standards. Note that we measure the length of a segment by the distance between its endpoints. Then, since $d_T(P,Q) = d_T(P,Q')$, and $d_T(P,R) = d_T(P,R')$, and also $\angle RPQ \cong \angle R'PQ'$, if the SAS congruence condition were to hold, then we would have $\triangle PQR \cong \triangle PQ'R'$. However, $d_T(Q,R) \neq d_T(Q',R')$, so $\triangle PQR \ncong \triangle PQ'R'$. $\square$

One concept we have made continual use of in all the geometries we have discussed so far is the distance from a point to a line. The reader should already be familiar with the usual definition of the distance from a point P to a line ℓ. The distance is determined by the segment $\overline{QP}$, where Q is the projection of P on ℓ. This also applies in hyperbolic and elliptic geometries as well as Euclidean; however, the definition is quite different in taxicab geometry.

Definition 3.2. Let ℓ be a line and P a point. Then the distance from P to ℓ is denoted $d_T(P,\ell)$, and $d_T(P,\ell) = \min d_T(P,Q)$, where Q is a point on ℓ.

This may seem a bit analytical, and the reader may prefer a more intuitive description of how to determine the distance from a point to a line. One way to think about the distance in Euclidean geometry from a point P to a line ℓ is to think of inflating a circle centered at P until it just intersects ℓ. The radius of the inflated circle is then the distance from P to ℓ, denoted $d_E(P, \ell)$ since we refer to the Euclidean metric. Recall what circles look like in taxicab geometry, and apply the same procedure. We are left with the following result, the proof of which follows from the definition of distance and is left to the reader to explore:

Theorem 3-2. *Let ℓ be a line in the Euclidean coordinate plane with slope a*[1]*. Let P be a point.*

i. If $|a| < 1$, then $d_T(P, \ell)$ is the vertical distance from P to ℓ.

ii. If $|a| > 1$, then $d_T(P, \ell)$ is the horizontal distance from P to ℓ.

iii. If $|a| = 1$, then $d_T(P, \ell)$ is either the vertical or the horizontal distance from P to ℓ, for they are the same.

Now that we have some notion of how distance works in taxicab geometry, we may start to talk about objects being *equidistant*.

Let P and Q be points in taxicab geometry. Then the set of all points equidistant from P and Q is called the *taxicab midset of P and Q*. Let ℓ and m be lines in taxicab geometry. Then the set of all points equidistant from ℓ and m is called *taxicab midset of ℓ and m*.

In Euclidean geometry, the midset of two points P and Q is the perpendicular bisector of the segment $\overline{PQ}$. In order to find the center of the circle that inscribes a triangle $\triangle ABC$ in Euclidean geometry, one calculates the intersection point of the three Euclidean midsets of the vertices A, B, and C, that is, the intersection point of the perpendicular bisectors of the triangle's edges. How might we use a similar procedure to find the taxicab circle that circumscribes a triangle? As it turns out, we are not always able to find the circle's center.

Theorem 3-3. *It is not always possible in taxicab geometry to circumscribe a triangle in a circle.*

Proof. The reader is asked to think about the proof of this theorem in the problem set. □

Thus far we have explored a few properties of taxicab geometry which may seem interesting in themselves. The question may arise, however, whether there are any significant applications to this geometry. Taxicab geometry is a better model of urban geography than Euclidean geometry, and for that reason it has applications to urban construction problems. Some exercises in the problem set hint at these applications, which include finding optimal locations to build businesses and drawing district boundaries to fit certain

[1] Recall from algebra that if (x_1, y_1) and (x_2, y_2) are two points on ℓ, then $a = \frac{y_2 - y_1}{x_2 - x_1}$.

criteria. In this way, taxicab geometry proves to be as "useful" as other non-Euclidean geometries, and is perhaps more accessible to the reader than either hyperbolic or elliptic geometry.

Problem Set 3

For any problem that references City X, refer to Figure 3-1. Use the taxicab metric unless otherwise stated.

1. When we say that lines can not be made to "coincide" in taxicab geometry, we mean that we cannot move lines around in the plane without changing the distance between points on the line. Let $P = (1,0)$, $Q = (5,3)$, $P' = (-1,0)$, and $Q' = (4,0)$.

   a. Calculate $d_E(P,Q)$ and $d_E(P',Q')$. This means $\overline{PQ} \cong \overline{P'Q'}$ in Euclidean geometry.

   b. Calculate $d_T(P,Q)$ and $d_T(P',Q')$.

   c. Why does this show that lines in taxicab geometry cannot always be made to coincide?

2. In the scenario presented in Figure 3-1, the taxi driver travels 7 blocks to get you to the opera house. If the taxi driver has to pick up a second fare at the corner of 3^{rd} and C on his way to the opera hall, will he travel more, less, or the same number of blocks he would have traveled had he gone straight to the opera hall? (Presume all streets are two-way, and that he takes the shortest route.)

3. Suppose you work at a store on the corner of 4^{th} and D streets, and that the nearest restaurant is at 1^{st} and D in City X. You only have a very limited time for lunch. Supposing you walk the streets of City X at a constant speed no matter where you go, would it be faster to eat at the restaurant or go home for lunch? Now suppose you can walk "as the crow flies" to your apartment, so you do not have to stay on the streets. Should you eat at the restaurant or go home?

4. Referencing Figure 3-1 when necessary, graph the points and calculate the following:

   a. $d_T((1^{\text{st}} \text{ and } A), (5^{\text{th}} \text{ and } C))$

   b. $d_T((7^{\text{th}} \text{ and } G), (2^{\text{nd}} \text{ and } G))$

   c. $d_E((1^{\text{st}} \text{ and } A), (5^{\text{th}} \text{ and } C))$

   d. $d_E((7^{\text{th}} \text{ and } G), (2^{\text{nd}} \text{ and } G))$

   e. $d_T((3,-2), (-2,4))$

f. $d_T((4,-3),(-2,5))$
g. $d_E((3,-2),(-2,4))$
h. $d_E((4,-3),(-2,5))$
i. $d_T((1/2,1/2),(-2,3/4))$
j. $d_T((3.5,2.3),(-2.3,3.5))$

5. For each set of points P and Q, find the set of points in taxicab geometry that are equidistant from P and Q. This set is called the *taxicab midset* of the points P and Q:

   a. $P=(-1,0)$ and $Q=(3,2)$
   b. $P=(0,1)$ and $Q=(2,5)$
   c. $P=(0,0)$ and $Q=(3,3)$

6. City X has two high schools: one at 1^{st} and B, and one at 7^{th} and D. Draw a school district boundary so that each student in City X attends the high school nearest their home.

7. Redraw the school district boundary to accommodate a new high schools being built at 4^{th} and G.

8. There are three major corporate office buildings in City X: one at 1^{st} and C, one at 7^{th} and F, and one at 5^{th} and A. Local Daycare wants to open a new daycare facility at equal distance from each of the three office buildings. Where should it open this facility?

9. Repeat exercises 4, 5, and 6, but this time use the usual Euclidean metric d_E.

10. Exhibit a counterexample in taxicab geometry for the following congruence conditions:

    a. ASA
    b. SAA
    c. SSS

11. Find an isosceles triangle in taxicab geometry with incongruent base angles.

12. Find a triangle in taxicab geometry with two congruent angles which is not isosceles.

13. Find in taxicab geometry a right triangle which is equilateral.

14. Let ℓ be the line through $(0,2)$ and $(1,3)$. Calculate:

a. $d_T((0,0),\ell)$
b. $d_T((2,-4),\ell)$
c. $d_T((-2,4),\ell)$
d. $d_E((0,0),\ell)$
e. $d_E((2,-4),\ell)$
f. $d_E((-2,4),\ell)$

15. For which lines in the coordinate plane does the taxicab distance from a point to a line equal the Euclidean distance from a point to a line.

16. Graph the points $(1,2)$ and $(-4,3)$, and let ℓ be the line through these points.

    a. Graph the set of all points P such that $d_T(P,\ell) = 1$.
    b. Graph the set of all points P such that $d_E(P,\ell) = 1$.
    c. Graph the set of all points P such that $d_T(P,\ell) = 3$.
    d. Graph the set of all points P such that $d_E(P,\ell) = 3$.

17. Can you find a line ℓ such that the set of all points P such that $d_T(P,\ell) = 1$ is the same as set of all points P such that $d_E(P,\ell) = 1$?

18. Let $A = (3,-1)$, $B = (0,7)$, and $C = (-7,-3)$.

    a. Circumscribe an Euclidean circle about $\triangle ABC$.
    b. Circumscribe a taxicab circle about $\triangle ABC$.

19. Let $A = (-3,1)$, $B = (0,2)$, and $C = (5,6)$. Try circumscribing a circle about $\triangle ABC$. What goes wrong? Can you find another triangle about which it is impossible to circumscribe a circle? Note that this proves Theorem 3-2.

20. Let $A = (3,-3)$, $B = (-3,6)$, and $C = (5,4)$. Recall from Euclidean geometry that the center of an inscribed circle of a triangle is the intersection of the angle bisectors of the triangle (which is the intersection of the midsets of the edges of the triangle).

    a. Inscribe a Euclidean circle in $\triangle ABC$.
    b. Think of a procedure for finding the center of the taxicab circle inscribed in a triangle.
    c. Inscribe a taxicab circle in $\triangle ABC$.

21. Do you think it is always possible to inscribe a taxicab circle in a triangle? Do you think the inscribed taxicab circle is unique?

22. City X has been experiencing smog problems. The mayor decides to build a new trolley station with three trolleys that each travel to a major drop-off point in hopes of reducing the smog in the city. The drop off point for Trolley 1 is to be located at 5$^{\text{th}}$ and A; the drop off point for Trolley 2 is to be located at 1$^{\text{st}}$ and G; and the drop off point for Trolley 3 is to be located at 7$^{\text{th}}$ and D. Due to lack of city funds, the mayor wants to build the trolley station at a location so that the sum of the distances traveled by the trolleys from the station to their drop off point is at a minimum. Where should the mayor have the trolley station built?

Chapter 4: Appendix

We list for the reader Euclid's postulates, common notions, unstated assumptions, and propositions from Book I of *The Elements*. These are referenced frequently throughout the text. Most of the translation is taken directly or paraphrased from [6].

Postulates:

1. There is a unique line through any two distinct points.
2. A line segment segment may be continued in a line beyond each of its endpoints.
3. A circle of any (positive, real) radius about any point may be constructed.
4. All right angles are equal to one another.
5. If a transversal cuts two lines so as to form acute interior angles on the same side, then the lines intersect somewhere on the side of the acute angles.

Common Notions:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Unstated Assumptions:

1. Lines are infinite in extent.
2. Lines and angles may be made to coincide, and therefore moved around in the plane.
3. **Plane-Separation Axiom:** Each line in a plane separates all the points of the plane that are not on the line into two nonempty half-planes with the following properties:

(a) The half planes are disjoint convex sets.

(b) If P is in one half plane and Q is in the other half plane, the segment $\overline{PQ}$ intersects the line that separates the plane.

4. **Betweenness:** If A, B, and C are distinct colinear points, then exactly one of the following holds: A is between C and D, B is between A and C, or C is between A and B. Moreover, given distinct points A and B, there is always a point C such that C lies between A and B on the line $\overleftrightarrow{AB}$.

Propositions:

1. On a given finite straight line to construct an equilateral triangle.
2. To place at a given point (as an extremity) a straight line equal to a given straight line.
3. Given two unequal straight lines, to cut off from the greater a straight line equal to the less.
4. (SAS) If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base,the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.
5. In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.
6. If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.
7. Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.
8. (SSS) If two triangles have the two sides equal to two sides, respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.
9. To bisect a given rectilinear angle.

10. To bisect a given finite straight line.
11. To draw a straight line at right angles to a given straight line from a given point on it.
12. To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.
13. If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.
14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.
15. If two straight lines cut one another, they make the vertical angles equal to one another.
16. (Exterior Angle Theorem) In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.
17. In any triangle two angles taken together in any manner are less than two right angles.
18. In any triangle the greater side subtends the greater angle.
19. In any triangle the greater angle is subtended by the greater side.
20. (Triangle Inequality) In any triangle two sides taken together in any manner are greater then the remaining one.
21. If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.
22. Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.
23. On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.
24. If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

25. If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.

26. (AAS, ASA) If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

28. If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

29. A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

30. Straight lines parallel to the same straight line are also parallel to one another.

31. To draw a straight line through a given point parallel to a given straight line.

32. In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

33. Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

34. In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.

35. Parallelograms which are on the same base and in the same parallels equal one another.

36. Parallelograms which are on equal bases and in the same parallels equal one another.

37. Triangles which are on the same base and in the same parallels equal one another.

38. Triangles which are on equal bases and in the same parallels equal one another.

39. Equal triangles which are on the same base and on the same side are also in the same parallels.

40. Equal triangles which are on equal bases and on the same side are also in the same parallels.

41. If a parallelogram has the same base with a triangle and is in the same parallels,then the parallelogram is double the triangle.

42. To construct a parallelogram equal to a given triangle in a given rectilinear angle.

43. In any parallelogram the complements of the parallelograms about the diameter equal one another.

44. To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

45. To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

46. To describe a square on a given straight line.

47. In right-angled triangles the square of the side opposite the right angle equals the sum of the squares of the sides containing the right angle.

48. If in a triangle the square of one of the sides equals the sum of the squares of the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

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