

FACTORS INFLUENCING THE PROBLEM SOLVING OF COLLEGE STUDENTS SOLVING A
MATHEMATICS PROBLEM IN A SMALL GROUP: TWO CASE STUDIES

by

LINDA BAILEY CRAWFORD

(Under the direction of JAMES W. WILSON)

ABSTRACT

This study investigated the problem solving of college students as they worked together in a small group to solve a mathematics problem. Eight students participated in the study and were divided into two groups of four. All eight students completed a semester of precalculus mathematics (MATH 1113) the spring semester of 2002. The researcher was the teacher for the two precalculus classes from which the participants were selected. The students were asked to participate in the study after the MATH 1113 class was completed. Data collection for the study occurred between May 15, 2002 and August 14, 2002.

Data for the study included individual interviews, observations and videotapes of problem-solving sessions, and written reflections. The final interviews used video clips from the problem sessions to stimulate participant recall. Group 1 required four sessions of approximately one hour each to complete the problem whereas Group 2 solved the problem in one session of about one hour. The approaches used by the two groups were different and the dynamics within the two groups were different.

The study identified the factors influencing the group problem solving and described how these factors influenced the problem solving. Schoenfeld's (1992) definition of problem solving as learning to think mathematically was the definition of problem solving used in the study. The factors influencing collaborative problem solving identified by Watson and Chick (2001) were used as starting points to identify the factors influencing the problem solving in the two groups.

Social, cognitive, and external factors were identified and found to interact. Social factors influencing the problem solving included leadership factors, egocentrism, and social collaboration. Cognitive factors included cognitive ability, prior experience, a sense-making perspective, communication factors, the big picture, and goal focus. External factors included task factors, outsider, and logistical factors.

The processes the groups used to solve the problem were described through the identification of these influencing factors. The problem solving in the two groups differed as a result of how the participants allowed these factors to influence them personally. A significant observation from the study was the number and type of misconceptions the students possessed.

INDEX WORDS: Small-Group Learning, Collaborative Learning,
Problem Solving, Precalculus Mathematics

FACTORS INFLUENCING THE PROBLEM SOLVING OF COLLEGE STUDENTS SOLVING A
MATHEMATICS PROBLEM IN A SMALL GROUP: TWO CASE STUDIES

by

LINDA BAILEY CRAWFORD

B.S., Georgia College, 1975

M.Ed., Augusta College, 1988

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial
Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2004

© 2004

Linda Bailey Crawford

All Rights Reserved

FACTORS INFLUENCING THE PROBLEM SOLVING OF COLLEGE STUDENTS SOLVING A
MATHEMATICS PROBLEM IN A SMALL GROUP: TWO CASE STUDIES

by

LINDA BAILEY CRAWFORD

Major Professor: James W. Wilson

Committee: Jeremy Kilpatrick
Paul R. Wenston
Patricia S. Wilson

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
August 2004

To my husband, Don,
for always being my best cheerleader!

ACKNOWLEDGEMENTS

Writing this dissertation was in many ways a solitary endeavor. The hours spent reading, summarizing, transcribing, analyzing, writing, and editing were quite lonely. Completing this dissertation, however, was not a solitary endeavor.

Thanks must first go to Abby, Eve, Jane, Sara, Helen, Lisa, Sandy, and Tess. These students gave up a part of their summer to participate in this study. I am indebted to them for the insights I gained about how students learn and think as they interact with each other in a small group.

I give my sincere thanks to my major professor, Dr. Jim Wilson. I am sure he thought I would never finish this project! Thanks for believing in me and providing the feedback that I needed to finish.

I also wish to thank my doctoral committee: Dr. Jeremy Kilpatrick, Dr. Paul Wenston, and Dr. Pat Wilson. Thanks also to Dr. Henry Edwards.

My parents, Otis and Helen Bailey, have always encouraged me and let me know they are proud of me. It was perhaps my growing up as the eldest of five children where I first learned the importance of working together!

Family, friends, and colleagues have supported me with their prayers, their encouraging words, and their willingness to listen. Thank you for caring about me.

I am especially grateful to my husband, Don, and my daughter, Emily. They saw me when I was most discouraged and frustrated, but never did they suggest I quit. Thanks for encouraging me, for listening to me, for loving me.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	v
CHAPTER	
1 INTRODUCTION	1
Purpose and Rationale	4
Research Questions	5
Theoretical Framework	6
Definition of Terms	7
2 REVIEW OF RELATED LITERATURE	8
Small-Group Learning	8
Problem Solving in Mathematics	14
Problem Solving in Small Groups	16
Concluding Remarks	25
3 METHODOLOGY	26
Design of the Study	26
Description of MATH 1113 Course	29
Selection of Participants	31
The Task	31
Limitations of the Study	32
Data Collection	32
Timeline for Data Collection	35
Organization of the Case Studies	36

4	CASE STUDY OF GROUP 1	37
	Participants	37
	Investigation of the Buried Treasure Problem.....	50
	Reflections and Final Interviews	151
	Synthesis	186
5	CASE STUDY OF GROUP 2.....	200
	Participants	200
	Investigation of the Buried Treasure Problem.....	209
	Reflections and Final Interviews	245
	Synthesis	277
6	ANALYSIS OF THE CASE STUDIES.....	284
	Social or Interpersonal Factors	285
	Cognitive Factors.....	287
	External Factors.....	294
	Influencing Factors from the Participants' Perspectives	297
	Factors Influencing the Group.....	297
7	SUMMARY, IMPLICATIONS, AND RECOMMENDATIONS	301
	Purpose and Rationale Revisited.....	301
	Framework Revisited	303
	Conclusions	305
	Implications.....	309
	Limitations of the Study.....	310
	Recommendations for Future Research	311
	Concluding Remarks.....	312
	REFERENCES	313

APPENDICES.....	317
A REFLECTIONS QUESTIONNAIRE.....	317
B EMAIL AUTOBIOGRAPHY ASSIGNMENT.....	319
C NOTEBOOK GUIDELINES	321
D NOTEBOOK CHECK	323
E INITIAL INTERVIEW GUIDE	325
F FINAL INTERVIEW GUIDE	330
G EXAMPLE OF GUIDE USED FOR STIMULATED RECALL.....	332

CHAPTER 1

INTRODUCTION

Numerous documents have been written in recent years advocating reform in mathematics education (American Mathematical Association of Two-Year Colleges, 1995; National Council of Teachers of Mathematics, 1989, 2000; National Research Council, 1989). Both curriculum and pedagogical changes have been recommended. One pedagogical strategy that is being recommended in both precollege and college classrooms is the use of small groups. This pedagogical tool is often identified by terms such as small-group learning, small-group teaching, cooperative learning, collaborative learning, work groups, and group work. Although the terms are different and are often used interchangeably in the literature, the goal is the same. That basic goal is stated by Johnson, Johnson, and Smith (1991) in their definition of cooperative learning as the instructional use of small groups so that students can work together in order to maximize their own as well as each other's learning. Although an integral component, cooperative learning involves more than just putting students together in small groups and giving them some task to do. It also involves giving deliberate thought and attention to various aspects of the group process (Davidson, 1990).

One way I have used small-group learning in my college precalculus mathematics class is through an out-of-class problem-solving activity that I refer to as a "project." Students, in groups of four or five, are given about two weeks to work on the project. They submit their findings in essay form, called a write-up. The write-up is a narrative in which they provide their work as well as an explanation of their thinking. For some projects each individual student submits his own write-up; for other projects a single write-up is submitted from the group. An

example of a project is in Figure 1 below. This particular project was adapted from a problem by Philipp, Martin, and Richgels (1993).

Your agency represents Joe Starr, the hottest movie star of the 1990's. He has been approached by three separate motion picture studios, each of which would like to hire him to star in one of their upcoming films. All three of the studios plan to shoot their pictures in July, placing him in the position of having to select one studio. All three of the studios have assured him that their movies will require somewhere between 2 weeks (approximately 14 days) and 3 weeks (approximately 21 days) for the shoot. He is pleased with all three of the parts, and he wants to accept the most lucrative job. The motion picture studios have been experimenting with some rather unusual salary contracts. He has been offered the following contracts:

Universal: A flat rate of \$100,000 for each day of work.

Orion: \$10 for the first day of work, with the daily salary doubling for each additional day.

Touchstone: One penny on the first day of work, with the daily salary tripling for each additional day of filming.

Each studio has guaranteed Joe that he will be paid for between 14 and 21 days of work. Joe has asked you and your team to study the contracts and determine which contract he should accept. Will your answer depend upon the number of days Joe will work? If so, then under what conditions (how many working days) would it be most lucrative for Joe to work for Universal? for Orion? for Touchstone? Joe wants a full report with the team's recommendations. Because this is such an important decision, he has asked that you give him your recommendations, but he also wants you to provide documentation for your recommendations. He has promised the team a bonus if the report is well presented with supporting documentation that he can understand.

Figure 1. An example of a project.

Augusta State University is a commuter school with many nontraditional students enrolled. Many of these students have jobs, families, and other commitments, so scheduling group time to work on these projects is quite difficult. Although students have trouble finding time to get together, I have always believed the effort is well worth it. Some of the benefits I have seen are the following:

1. More involved problems can be assigned with greater likelihood of a solution as students work and learn together.
2. Students learn different strategies from the other members of the group. They also learn about different forms of technology; we use the graphing calculator in class, but often students will suggest their group use a spreadsheet or a computer algebra system to solve a problem.

3. Students gain confidence in their ability to solve problems as they have to explain their thoughts to others. Their problem-solving ability is also shaped as they ask questions and have group members clear up their misconceptions.

4. Students get to know other members of the class and appear to be more comfortable with each other and with me. They ask more questions during class and are more willing to respond to my questions and the questions of their classmates. The class becomes more like a community of learners.

Prior to being assigned the projects, students are given opportunities to work with a partner during class time. On the first day of the semester students are told to find a partner for that class period. On subsequent days students are to choose a new partner so that they begin to know the members of the class. I use two cooperative learning strategies frequently in class. The first strategy, "talk to your neighbor," is implemented during a whole-class discussion or lecture; students are instructed to discuss in dyads a question or problem posed to the class. This is an informal way to get students to share ideas and engage in conversation. Fogarty (1990) notes it is almost impossible to turn to your neighbor and not say anything. In the second strategy, "think-pair-share", a question is posed to the class. Students silently think of a response individually for a given period of time, then pair with their partners to discuss the question and reach consensus. Students are then asked to share their agreed-upon answers with the rest of the class.

Three weeks into the semester groups of four or five are selected. I have used various techniques for determining the groups. Some techniques I have used are as follows:

1. Students select their own groups but must make sure the group can meet at least twice during the week.
2. Students list the names of one to three people whom they would like to work with on a piece of paper. I try to assign each student to a group with at least one of his choices.

3. I completely determine the groups using such criteria as SAT and ACT scores to group students.

Once the groups are formed the assignment for the project is given. Some groups function well, with much learning taking place. Other groups see the project as just a task to be completed. Some students have reported excitement and enjoyment as they work together, but other students have complained. Even though I believed these group activities were beneficial, I also believed I needed to understand the group processes that influence the learning during these small-group, problem-solving sessions. To gain that understanding, I needed a closer look at a small group of students engaged in an activity similar to those I might assign as a project. This study was designed to facilitate my understanding by giving me that closer look.

Purpose and Rationale

In a review of the research on using small groups in teaching Noddings (1989) identifies two primary reasons as to why researchers recommend the use of small groups: (a) to improve learning outcomes and (b) to promote development. Outcome-focused researchers claim small groups strengthen learning outcomes usually associated with direct instruction, particularly in basic skills; these researchers focus on how small groups can be used to help students learn the content of the traditional curriculum. Researchers concerned with development claim small-group processes contribute to development (or developmental learning) and to higher-order thinking skills required for problem formulation and solution; these researchers focus on cognitive, social, and moral development.

More research in the area of small-group learning needs to be done. In particular several researchers have indicated more research is needed to determine how the use of small groups affects the development of students. For example, Davidson (1985) claims additional research is needed to determine the extent to which small-group instruction affects problem-solving ability and higher-order thinking skills. Davidson and Kroll (1991) state research is

needed to study the interactions that take place during cooperative group work to determine how academic, social, or psychological effects are produced.

In Bossert's review of the literature on cooperative learning (1988-1989), he concluded little is known about what actually goes on in small, heterogeneous, cooperative learning groups that stimulates the use of appropriate cognitive processes. Even less is known about how students in cooperative groups actually conceptualize their task. He claims an important area for research is to study students' cognitive processing in various cooperative learning techniques.

Good, McCaslin, and Reys (1992b) state that most research on student learning in small groups has not focused on how students learn to think and learn while interacting with peers on a cooperative task. They believe, however, that a potential benefit of having students learn in small groups is to help them develop problem-solving skills and dispositions.

According to Schoenfeld (1992), to understand how students develop their mathematical perspectives, researchers must consider the mathematical communities in which they live and the practices that underlie those communities. He further states understanding the role of interactions is the key to understanding learning.

Research Questions

The research community provided me with the rationale to study the interactions of students as they work in small groups. My own interest in what transpires as I place students in small groups to work on a project gave me a problem to solve. This study was the result. My purpose for the study was to examine the factors that influence the problem solving of a small group of students as they work on a mathematics problem together. The research questions which guided the study are the following:

1. What factors influence the problem solving of a small group of students as they work together to solve a mathematics problem?
2. How do these factors influence the problem solving in the group?

Theoretical Framework

The participants do not enter a study as blank slates. Their experiences, their beliefs, and their personalities are only some of the factors that influence their participation in a study. When a small group of students are engaged in solving a mathematics problem together, additional influencing factors are present. In developing my theoretical framework, Watson and Chick's (2001) study was of interest to me. Their study was concerned not with the overall success of collaborative group work in mathematics but rather with the identification of factors associated with outcomes that could be characterized as productive or not productive while the students worked together. To describe the factors that influenced the outcomes of the collaborative problem solving of the two groups I observed, I used Watson and Chick's factors as a beginning point. They grouped their seventeen influencing factors into the three categories of social or interpersonal, cognitive, and external. Some of their factors were influences in my study. They also identified some factors which I did not find in my study. I also found it helpful to consolidate several of their factors into one which I termed *sense-making perspective*.

Schoenfeld (1992) suggested any research in problem solving should include the researcher's operational definition of the term *problem solving*. My view of problem solving for this study coincides with the definition Schoenfeld (1994) himself has adopted. That is, problem solving is the means by which one learns to think mathematically where

learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure—mathematical sense-making.

(p. 60)

Definition of terms

Several terms used throughout the study need clarification. These terms are as follows.

1. *Lifting*. This term is taken from the study by Watson and Chick (2001) and is used when the cognitive level of the members of a group has been raised.
2. *Vandalism*. The term comes from research by Goos (1998) and is described as “taking inappropriate action to deal with an impasse” (p. 226). For example, a student may choose to perform some totally illogical procedure just to get an answer.

CHAPTER 2

REVIEW OF RELATED LITERATURE

As I considered the literature related to my study, three areas became apparent. The first was literature associated with small-group learning. The second area was that of mathematical problem solving. And the third area was the literature that addressed the connections between these two areas.

Small-Group Learning

The use of small groups is a pedagogical strategy that is being recommended in both precollege and college classrooms. I reviewed the literature to find out what researchers mean by small-group learning and what they claim are the advantages and disadvantages of this strategy.

What is Small-Group Learning?

Using small groups is a pedagogical tool identified by various terms such as small-group learning, small-group teaching, collaborative learning, cooperative learning, work groups, and group work. Davidson (1985) describes the basic characteristics of this pedagogical tool as follows: The class is divided into groups of two to six students apiece and each group is given its own working space. Each group is expected to discuss mathematical concepts and principles, to practice mathematical techniques, and to solve problems. As the students work, the teacher moves from group to group, checking the students' work, and providing help in various ways. The groups may also meet outside of class to work on projects.

The different terms used to identify this pedagogical tool are often used interchangeably in the literature although there are different models for structuring learning in small groups. Although I could find no differences in small-group learning, small-group teaching, work groups and group work, the terms collaborative learning and cooperative learning appeared to be used

differently. As I tried to discern the differences between the two, I was glad to see others have also struggled with the differences. Panitz (1996) believes his quest for an understanding of the distinction between the two concepts has gotten him somewhat closer to an understanding. The distinction he makes is that collaborative learning is a personal philosophy, not just a classroom technique. It can be used as people interact with others in the classroom, at committee meetings, in community groups, within families—virtually in any environment where people deal with others. Collaborative learning suggests a way to deal with people when they come together in a group that will respect and highlight the abilities and contributions of the individuals in that group. On the other hand, cooperative learning is defined by a set of processes. To learn cooperatively, the people in the group are expected to interact according to these processes to accomplish a specific goal or develop an end product. Panitz points out cooperative learning is more teacher-centered since the teacher defines the processes students will use as they interact. In contrast, Panitz claims collaborative learning is more student-centered. His contrast of cooperative learning and collaborative learning as teacher-centered versus student-centered has to do with the way the learning situations are set up.

Johnson (n.d.) explained the difference between collaborative and cooperative learning with the following answer to a question submitted to the website on cooperative learning:

The terms collaborative and cooperative are used interchangeably much of the time. Some like to think of collaborative as a broader term including all kinds of “group work” some of which is not structured cooperatively. Cooperative implies that students have a group goal, are individually accountable to contribute and learn, are working as skillfully with each other as they can, are working Knee to Knee (usually in a group of two or three), and take time to process how well they functioned as a team and what they can do to be even better. Much different than some unstructured “group work”.

My understanding of the literature leads me to conclude collaborative learning is the umbrella term under which all other forms of small-group learning, cooperative learning included, live.

Johnson, Johnson, and Smith (1991) define cooperative learning as the instructional use of small groups so that students can work together in order to maximize their own as well as each other's learning. Researchers (R. T. Johnson & Johnson, n.d.; Smith & Waller, 1997) involved in the model of cooperative learning stress five components are necessary in order for small-group learning to be cooperative. I understand these five components to be representative of the processes Panitz referred to as those the teacher defines in a cooperative situation. These five components are: the group members must be committed to a common goal; they must face-to-face promote each other's learning and success; they must hold each other personally and individually accountable to do an equitable share of the work; they must be skilled in the use of interpersonal and small-group skills; and they must as a group process how effectively the members are working together. Cooperative learning groups can be either informal or formal and a number of different models have been developed for implementing these types of groups.

Bean (1996) describes a method of small-group learning that he characterizes as a "goal-oriented use of small groups aimed at giving students supervised practice in disciplinary thinking under the tutelage of the teacher as coach" (p. 150). This method has somewhat of a structured quality in the sense that it includes the following components: (a) the teacher presents a problem that requires critical thinking; (b) the students work together in small groups to find a solution the group can agree upon; and (c) the teacher acts as a coach by observing the students as they work and critiquing their solutions. Bean has observed several advantages, including several which are social, of using small groups in the classroom. An advantage he cites which may lead to cognitive gains for students is that learning is more active as students are given the opportunity to "practice disciplinary inquiry and argumentation under the tutelage of a teacher as coach" (p. 167).

Although Bean's goal-oriented use of small groups appears less structured than that recommended by the proponents of cooperative learning groups, his use of small groups

nevertheless places responsibility on both the teacher and the students in order for the experience to be successful. Davidson (1990) also observed small-group learning involves more than just putting students together in small groups and giving them some task to do. It involves giving deliberate thought and attention to various aspects of the group process. A successful small-group experience will not just happen but will occur only as teacher and students assume their obligations.

What can happen when students learn in small groups?

The 1989 report entitled Everybody Counts: A Report to the Nation on the Nature of Mathematics Education (National Research Council, 1989) recommended small-group learning as one of the tools to help students learn mathematics well. The report pointed out that educational research offers strong evidence that students must *construct* their own mathematical understanding in order to learn mathematics well. Opportunities to examine, represent, transform, solve, apply, prove, and communicate within the mathematics curriculum are needed for students to understand the mathematics they are learning. These opportunities will occur most readily when students work in groups, engage in discussion, make presentations, and in other ways take charge of their own learning.

Small-group work has been found to promote the use of higher-order thought processes, foster the use of academic problem-solving skills and perspective-taking skills, increase opportunities for oral rehearsal of information, and encourage and involve peers in learning which can increase friendship, acceptance, and cognitive processing skills (Bossert, 1988-1989; D. W. Johnson & Johnson, 1985). In addition to these reasons Good, McCaslin, and Reys (1992a) cite the following reasons as to why cooperative groups may lead to improved achievement and social relations:

1. Subject-matter knowledge is increased. A student working alone may not know how to approach a problem but when students work together in a group they can pool their

understanding, content knowledge, and problem-solving skills to increase the likelihood of solving the problem.

2. Students value shared academic work. If the cooperative task promotes more student understanding of mathematical ideas, students are more likely to value this shared work.
3. Students can regulate their own resources. The group situation allows students to be more flexible about the time and energy they expend.
4. Students learn to manage others' resources. As students work in groups they learn how to obtain help from others.
5. Students develop appropriate attitudes toward challenging work on shared tasks. Students are more likely to see challenging tasks as do-able when they have the shared expertise of their peers available to them.
6. School tasks are similar to those outside school. Tasks worked in a cooperative setting are more like those which occur at home and in the workplace where everyone contributes (or at least does his or her part).
7. Group members serve as models for one another. Students can learn how to learn from the members of the group.
8. Students develop an expanded understanding of self and others. Students learn to appreciate individual strengths and weaknesses in themselves and others.

Cohen's (1994) review of research on small-group learning looked at the interaction occurring in small groups to identify the conditions which make for productive interactions. A proposition central to her review is that the "relationship of the total amount of interaction within a group to achievement differs according to the nature of the task" (p. 3). An ill-structured task is more likely to generate the open exchange and elaborated discussion (that is, interaction) necessary for conceptual learning to occur within small-group work. Interaction can be constrained in situations where conceptual learning is the goal when tasks are too structured in

design, teachers are unwilling or unable to delegate authority to the groups, and students are unprepared for small-group work. The review furthermore suggests status issues in a small-group situation are minimized when ill-structured problems are assigned.

A meta-analysis of within-class grouping (Lou, Abrami, Spence, Poulsen, Chambers, & d'Apollonia, 1996) looked primarily at the effects of within-class grouping on student achievement at the elementary, secondary, and postsecondary levels. For the meta-analysis the term *within-class grouping* referred to small-group instruction, loosely defined as “the physical placement of students into groups for the purposes of learning” (p. 423-424). The researchers concluded that student learning appears to be facilitated by within-class grouping particularly in large classes and especially in math and science courses. Furthermore, low-ability students benefit most when placed in mixed-ability groups but overall they found students benefit more from participating in a group of students having similar abilities. The analysis also revealed the best with-in class grouping practices went beyond the physical placement of students into groups by also adapting instructional methods and materials for small-group learning.

Small-group work, however, is not a panacea. As with any instructional method, there are problems associated with its use. Good, McCaslin and Reys (1992a) identify the following problems that may occur during small-group work:

1. Students' misconceptions are reinforced. If the group members hold common misconceptions, these may be reinforced as the members interact.
2. Students shift dependency from teacher to peer. Rather than the group's functioning collaboratively, one of the group members may become the expert or authority figure.
3. Students value the product more than the process. In some groups the effort is on getting the task completed rather than understanding the mathematical concepts.

4. Students value group processes more than the academic product. When too much emphasis is placed on acquiring the skills needed to learn to cooperate with others, subject matter learning may be de-emphasized.
5. Students receive differential attention and status. The different abilities of students in heterogeneous groups contribute to different experiences within the group.
6. Some students believe they are not able to contribute.
7. Some students may learn they do not need to contribute. Students may learn to engage in social loafing; this can occur when they either do not value the group work or when they try to protect their self-esteem because they are not strong students.
8. Group accountability may mediate failure-avoiding and success-enhancing behavior. Learning may be sacrificed as students withhold information to avoid being labeled as a “know-it-all” or to encourage other students to participate more.

Problem Solving in Mathematics

A search of the literature reveals conceptions of *problem solving* differ. Stanic and Kilpatrick (1988) observed that even though the National Council of Teachers of Mathematics asked in their 1980 document *Agenda for Action* for the focus of school mathematics to be problem solving, “there is no adequate clarification of what problem solving is, why we should teach it, or how the position taken fits into a historical context” (p. 1). Schoenfeld (1994) cites a survey he conducted in 1983 in which he found there was no universal kind of problem-solving courses. Instead he found five rather different courses, all of which were referred to as a problem-solving course. The five types of courses he identified were:

1. Seminars to prepare students for competitions such as the Putnam.
2. Courses designed to “provide my students with an introduction to what it means to think mathematically.”
3. Courses for future teachers of mathematics, with an emphasis on learning to solve problems, so that one could then teach students to do so.

4. Courses in mathematical modeling.
5. Remedial courses, in which slightly nonstandard problems were used as a means to help students “develop basic thinking skills.” (p. 42)

Because of this lack of uniformity in the concept of problem solving, Schoenfeld (1992) says that every study or discussion of problem solving should be accompanied by an operational definition of the term *problem solving* and examples of what the author means. Schoenfeld complies with his request by describing his problem-solving course as the second type listed above, designed to help students learn to think mathematically. For Schoenfeld (1994), problem solving is the means by which one learns to think mathematically where learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure—mathematical sense-making. (p.60)

When NCTM identified problem solving as one of the five process standards in its document, Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000), the organization explained its position on problem solving.

Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking. (p. 52)

The identification of problem solving as a process standard elevates the role of problem solving in the classroom. It is not an isolated unit of study but is instead an essential component to all mathematics learning that is woven throughout all content areas.

This concept of problem solving implies students are not blank slates, and the problem solving they engage in should provide them with the opportunity to connect ideas, to strengthen their understanding of ideas, and to confront their misconceptions. Schoenfeld (1983) recognized the role of the teacher when he claimed it is important for teachers overtly to help students make connections because students may not see the connections otherwise. Furthermore, the teacher must search for misconceptions because evidence of these misconceptions will likely surface only if students are required “to show us what they “know” “(p. 27). In settings where the teacher presents material to the students or where the students regurgitate material to the teacher (such as is true in formal mathematics classes), evidence of misconceptions are not likely to be seen. Herein lies an advantage of problem solving in small groups because such a pedagogical approach has the potential for students to show us what they know so their misconceptions can be confronted.

Problem Solving in Small Groups

For me, the next body of literature to consider was that of problem solving in small groups. I first looked at how researchers connect problem solving to small-group work. Then I looked specifically at several studies that have been conducted in this area.

What is the connection between problem solving and small-group work?

The literature supports having students work in small groups, and the literature supports having students engage in problem solving. Having students solve problems in small groups can be the best of both worlds. Unquestionably, when students engage in mathematics in small groups, the opportunity for interactions to occur among the students is much more likely than in a large group setting. The nature of those interactions will vary among students and among groups. Nevertheless, Schoenfeld (1992) points out it is through interactions with others that

students develop their sense of mathematics. He advocates students working in small groups on mathematical problems when he acknowledges that students develop their mathematical point of view as they work in a “community of mathematical practice” (p. 344).

Other researchers have also pointed out how the interactions of students during small-group work can help them become better problem solvers. Kilpatrick (1987) observed that when students work together, they engage in a dialogue with others that resembles the kind of internal dialogue that good problem formulators appear to have with themselves. When students talk to others, they have the opportunity to reflect on their ideas and to develop the language skills needed to express those ideas. The writers of Everybody Counts: A Report to the Nation on the Nature of Mathematics Education (National Research Council, 1989) noted that not only should students work cooperatively in small groups to solve problems, but as they work these problems they should learn to justify their approaches by presenting convincing arguments when confronted with conflicting ideas and strategies.

Of concern for Lester (1985) is his observation that mathematics instruction “is training [students] to be rigid in their thinking, not *flexible* and *adaptable*, is teaching them how to perform procedures but not *when* and under what conditions to perform them, and is showing them what to do but not *why* to do it” (p. 43). Resnick (1988) reiterates Lester’s concern by also noting students are more likely to become flexible and inventive mathematical problem solvers if they understand mathematics as a domain that welcomes interpretation and the construction of meaning. Solving problems in small groups is certainly no panacea but it could be a pedagogical approach that can contribute to the development of a flexible and adaptable attitude toward mathematics. For some researchers (Good et al., 1992a; Good et al., 1992b) this progression of development appears to be that as students solve problems in small groups, they become adaptive learners which can then help them develop problem-solving skills and dispositions. To see why this progression occurs, it was helpful to look at the conditions that Resnick (1988) claims socially shared problem solving apparently sets up that are important in

developing problem-solving skills. The social setting provides occasions for the following conditions:

1. Effective thinking strategies are modeled so that when a student observes how others attack a problem, he can become aware of the mental processes that might otherwise remain entirely implicit.
2. Thought processes can be critiqued and shaped as student thinking is shared with the group.
3. The interaction with the group provides a kind of scaffolding whereby an individual gains the ability to solve a problem that he was unable to solve alone.
4. The social setting can provide motivation and support as students are encouraged to try new, more active approaches.
5. A disposition to meaning construction is shaped as students learn “they have the ability, the permission, and even the obligation to engage in a kind of independent interpretation that does not automatically accept problem formulations as presented.” (p. 40)

Structuring classroom work so that students can engage in small-group, problem-solving activities is a time-consuming venture, especially if the goal is to help students learn to think mathematically. This challenge to help students learn to think mathematically is a long-term commitment as pointed out by Resnick (1988) when he argues for the long-term engagement needed to treat mathematics as an ill-structured discipline.

If we want students to treat mathematics as an ill-structured discipline—making sense of it, arguing about it, and creating it, rather than merely doing it according to prescribed rules—we will have to socialize them as much as to instruct them. This means that we cannot expect any brief or encapsulated program on problem solving to do the job. Instead we must seek the kind of long-term engagement in mathematical thinking that the concept of socialization implies. (p. 58)

Schoenfeld (1983) cites four reasons to justify the class time he allows his students to engage in small-group problem solving. The first reason is simply that the format allows for teacher intervention in the process before the product is completed. Secondly, solving problems with one's peers provides the opportunity for discussion as the group weighs the merits of plausible approaches to make a decision on the approach to take. The discussion within the group is the kind of discussion students should be having internally when engaging in problem solving. On the other hand, when a student works a problem alone, generally the first plausible approach is the one used. The third reason is that problem solving is not always a solitary endeavor and by having students work in groups they are reminded it is not always solitary. The fourth reason is the reassurance students gain as they realize others also struggle to solve problems.

Later Schoenfeld (1994) observed that "when mathematics is taught as received knowledge rather than as something that (a) should fit together meaningfully, and (b) should be shared, students neither try to use it for sense-making nor develop a means of communicating with it" (p. 57). This observation lends support to structuring activities for students to work together to solve problems. For it can be through such activities that students learn to make sense of the mathematics as they share their ideas with their peers. They can learn to communicate mathematically as others make requests for further elaboration and explanation.

What are some studies related to mathematical problem solving in small groups?

Watson and Chick (2001) studied eight groups of three students: two groups from Grade 3, four from Grade 6, and two from Grade 9. They identified a total of 17 factors that influenced the outcomes of collaboration on two open-ended tasks. Although these factors were grouped into the three categories of cognitive, social or interpersonal, and external, they found many of the factors interacted with each other. The extracts in the study showed these factors contributed to positive cognitive changes (lifting), negative cognitive changes (falling), and no change (hovering).

Assuming students will communicate when asked to solve problems in small groups is perhaps a fallacy. It is at least a fallacy to assume they will communicate in ways teachers expect. Sfard and Kieran (2001) analyzed the interactions of two 13-year-old boys learning algebra. They began their experiment believing student collaboration and mathematical conversation are the best ways to learn mathematics. Examination of their data, however, forced them to realize “the merits of learning-by–talking cannot be taken for granted” (p. 70). Their study provides a framework for analyzing communication. They first define communication “as an attempt to make other people act or feel according to one’s intentions” (p.56). Communication is considered effective as long as there is no evidence to suggest otherwise. They concluded from their analysis that if “conversation is to be effective and conducive for learning, the art of communicating *has to be taught*” (p. 71). This conclusion has significant implications for students expected to work in small groups.

The learning opportunities that arise as students work together on a mathematical task was the focus of a study of pairs of second grade students (Yackel, Cobb, & Wood, 1991). The researchers identified three types of learning opportunities:

first, the opportunity to use aspects of another’s solution activity as prompts in developing one’s own solution; second, the opportunity to reconceptualize a problem for the purpose of analyzing an erroneous solution method; third, the opportunity to extend one’s own conceptual framework in an attempt to make sense of another’s solution activity for the purpose of reaching consensus (p. 406).

The learning opportunities they identified do not typically occur in traditional classrooms. Furthermore, because these opportunities arose in the course of collaborative dialogue, “the rationale for the developmental approach to collaborative learning is not restricted to the resolution of conflicting points of view” (p. 406).

Cobb (1995) conducted four case studies which focused on the small-group interactions of pairs of second-grade students as they develop their conceptions of place-value numeration

and construct computational algorithms. Cognitive constructs and sociological constructs emerged in the development of the case studies. Of particular interest for my study were the two sociological constructs Cobb identified. The first sociological construct was the type of interaction. He observed two types of interaction, which he called univocal and multivocal. An interaction was classified as univocal if the perspective of one child dominated the interaction. In contrast, a multivocal interaction often occurred when an obvious conflict existed and both children insisted their own reasoning was valid. Multivocal interactions occurred as one child challenged his partner's assumptions by explicitly questioning his explanation. The case studies indicated multivocal interactions were typically productive provided the students established a level of mathematical communication that worked for them. Univocal interactions, however, generally did not produce learning opportunities for either student. Cobb also observed an interaction might involve either direct or indirect collaboration. Direct collaboration occurred when the students explicitly coordinated their attempts to solve a task. In contrast, indirect collaboration was defined as that situation whereby a student was not obliged to listen to another and yet capitalized on the other's comments indicating he was monitoring what his partner was saying and doing to some extent. Interestingly, unexpected learning opportunities often resulted from indirect collaboration whereas direct collaboration generally did not result in learning opportunities.

The second sociological construct identified by Cobb dealt with a power imbalance. A power imbalance occurred when the partner was perceived to be the mathematical authority of the group. In such a situation, a child deemed his effectiveness in the group was dependent upon his allowing his partner to dictate the mathematical activity of the group. A second type of power imbalance arose as one child was allowed to regulate the way the group interacted as they did and talked about the mathematics. The child controlling the way the interactions emerged was viewed as the social authority of the group.

Cobb concluded that in a classroom in which an inquiry-mathematics microculture has been established, two aspects of children's social relationships are vital. These vital aspects are:

1. The development of a taken-as-shared basis for mathematical communication.
2. The routine engagement in interactions in which neither child is the mathematical authority, namely those involving multivocal explanation. (p. 124)

Several studies (Goos, 1998, 2000; Goos & Galbraith, 1996; Goos, Galbraith, & Renshaw, 2002; Goos, Renshaw, & Galbraith, 1998) of collaborative problem solving in secondary mathematics have resulted from the work of researchers in Australia. These studies were part of a larger study that in part investigated patterns of classroom social interactions associated with metacognitive activity. Goos (1998) looked at the factors that can lead to metacognitive success or failure. She identified three "red flags"--lack of progress, detection of an error, and anomalous result—that must be recognized and acted upon appropriately if metacognitive success is to occur. Less successful outcomes are more likely when students are guilty of metacognitive blindness (failing to notice that something is amiss), commit metacognitive vandalism (taking inappropriate action to deal with an impasse), or see a metacognitive mirage (seeing difficulties which do not exist and mistakenly react when no action is necessary). Later analysis (Goos, 2000; Goos et al., 2002) found that a small-group, problem-solving session led to unsuccessful outcomes when the students made poor metacognitive decisions resulting from a lack of critical engagement in each other's thinking. In contrast, successful outcomes occurred as students challenged, questioned, and abandoned ineffective strategies and actively adopted and applied effective strategies.

A report by Lumpe (1995) addressed the question "Can group learning impact concept development and problem solving in science?" In the report he categorized and defined group learning strategies, examined the theoretical issues surrounding group work, and surveyed the

research conducted in this particular realm. His research resulted in the following practical suggestions for science teachers:

[I]f teachers desire to use peer interaction to foster concept development and problem solving, they should (a) use peer collaboration as opposed to other peer interaction strategies, (b) assign cognitive roles alongside managerial roles or embed cognitive cues in the lesson, (c) build social skills over time, and (d) use heterogeneous grouping. (p. 308)

According to Lumpe, in order for group work to enhance concept development and problem-solving ability, peer group lessons must be designed to maximize sociocognitive functioning so that beneficial conflict can occur.

Some research on problem solving in small groups has been conducted at the college-level. Kroll (Kroll, 1988; Lambdin, 1993) identified and analyzed the monitoring moves and roles of three pairs of preservice elementary teachers as they participated in a cooperative mathematical problem-solving task and formed six conclusions about the nature of metacognition and cooperative problem solving. These six conclusions are as follows:

1. *Ignoring*. Knowing when and how to ignore the suggestions and comments of others is important.
2. *Awareness and regulation*. Knowing what and when to monitor (awareness) is important but equally important is knowing how (regulation) to monitor.
3. *Sudden insights*. Insights appear as an individual steps back and examines the situation from a new perspective.
4. *Complementary roles*. Assuming differentiated roles can be an effective problem-solving strategy.
5. *Reflection vs. commitment*. Going with the first suggested approach rather than reflecting on the effectiveness of various approaches often leads to unsuccessful problem solving.

6. *Persistence*. Persistence with an approach is not necessarily beneficial in problem solving. It may be wiser to abandon an approach and try something else.

A focus of Kroll's study was to examine the way a partner's questions and statements can direct and change the course of a problem solution.

Dees (1991) studied students in a college remedial mathematics course. In two lab sections (the treatment group) students were encouraged to work together throughout the semester. The students in the other two lab sections (the control group) were not discouraged from working together but were not actively encouraged to do so. She concluded a minimal level of teacher intervention can induce cooperative behaviors among students. Furthermore, she found the students in the treatment group generally performed better than students in the control group on measures of higher cognitive skills.

Schoenfeld (1985; 1987; 1992) has done extensive work with college-age students in the problem-solving courses he teaches. One aspect of his work includes the observation of students as they work in pairs to solve problems. A framework of mathematical thinking has resulted from his work (Schoenfeld, 1992). This framework of mathematical thinking consists of five aspects of cognition: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices. A thumbnail description of each concept follows but does not do justice to the research Schoenfeld has conducted. The knowledge base includes not only what students know but how they organize and access what they know. Problem-solving strategies include the heuristics students use in their approaches to solving problems. The concept of monitoring and control is under the umbrella term of metacognition and influences the decisions students make as they engage in problem solving. In the area of beliefs and affects, Schoenfeld discusses student beliefs, teacher beliefs, and general societal beliefs about doing mathematics. Finally, the concept of practices considers the mathematical communities in which students operate.

Another study dealing with college-age students was conducted at North Carolina State University and explored the dynamics and their implications for engineering education (Haller, Gallagher, Weldon, & Felder, 2000). The work sessions of four groups of three to four students in a sophomore-level chemical engineering course were analyzed to understand how students taught and learned from one another. The researchers found the group members generally engaged in two types of teaching-learning interactions. These interactions were identified as transfer-of-knowledge sequences (TK) and collaborative sequences (CS). TKs were characterized by distinct teacher and pupil roles whereas students in CSs worked together with no clear role differentiation. Furthermore, interactional problems were primarily products of TKs and associated with students identified as either blockers or constant pupils. Students described as blockers habitually discouraged others' contributions, and those described as constant pupils habitually assumed the pupil's role.

Concluding Remarks

Schoenfeld's (1994) definition of problem solving as the means by which one learns to think mathematically is the operational definition of problem solving I have used for this study. The small-group situation in this study is closely related to that described by Bean (Bean, 1996) as goal-oriented. Although the research by Watson and Chick (2001) used students in grades three, six, and nine, I decided to use their framework as a guide to help me identify the social, cognitive, and external factors that influenced the problem solving of the students as they worked on a mathematical task. Although some of their seventeen factors were present in my study, other factors also surfaced. In particular, by comparing the two groups in the study, several group processes that influenced the problem solving became evident.

CHAPTER 3

METHODOLOGY

The instructor of every research course I have taken has adamantly stressed the research questions determine the type of study. My research questions directed me toward a qualitative study. The sections that follow include a description of the design of the study, an overview of the MATH 1113 course, how the participants were selected, a description of the task used in the problem sessions, the limitations of the study, how the data were collected, the timeline for data collection, and an explanation of the organization of the case studies.

Design of the Study

A qualitative research design allows for description and interpretation. This design was appropriate for my study because I was interested in describing the factors that influenced the outcomes of collaborative problem solving and also interpreting how those factors influenced these outcomes. The study considers the factors from the researcher's perspective as well as from the participant's perspective. The study necessitated description and interpretation of the observed interactions of the students within a group and the reported individual thoughts of the students. According to Merriam (1998) qualitative research is conducted to understand the meaning people have constructed, how they make sense of their world, and the experiences they have in the world. I was interested in the factors influencing problem solving of a group as they worked collaboratively on a mathematical task. As the researcher, I tried to make sense of the problem-solving behavior and the mathematical thinking of the students in each group; my personal experiences certainly affected my interpretation of the students' behavior and thinking. The students' identification of the factors that influenced the outcomes were determined by their own understanding of themselves, of the task, and of the group.

It was my goal to investigate the factors influencing the problem solving of a group of students as they worked on a shared task. Of the various types of qualitative research, I believed a case study design was most suitable for this study and expected each of the two groups to constitute a case. According to Merriam (1998) “a case study design is employed to gain an in-depth understanding of the situation and meaning for those involved. The interest is in process rather than outcomes, in context rather than a specific variable, in discovery rather than confirmation” (p. 19). The case study design made sense for my study. My focus was on the processes the students used as they worked on the task rather than on a correct solution to the problem. I believed the unit of analysis, the case, would be the group, and the study would consist of two cases because I looked at two different groups. Each case was bound by the group of students as they worked on the Buried Treasure Problem.

I found, however, I could not think about the group without thinking about the individuals within each group. After all, their personal experiences contributed to the factors influencing the problem solving for the group. I received some help with my dilemma of how to handle the group versus the individuals by looking at the comments of Patton (2002) whereby he acknowledges the complexity of defining the case. It is his observation that “when more than one object of study or unit of analysis is included in fieldwork, case studies may be layered and nested within the overall, primary case approach” (p. 298). Although each group of four students exploring the Buried Treasure Problem constituted a bounded system, I found it necessary to look at the individuals nested within each group as cases as well. To gain the in-depth understanding of the factors influencing the problem solving of the group, I collected data that also helped me understand the factors influencing the individuals as they worked on the task with their peers.

The description and interpretation of the factors influencing the problem solving had to be couched in the interactions of the students. Thick, rich description was used to describe

these interactions. Important parts of the description are the thoughts and understanding the students have of themselves, the task, and the group.

Each student participated in at least two individual interviews. The first interview was conducted prior to the first problem-solving session. The purpose was to gain background information and insight into how each participant saw himself or herself as a problem-solver. The final interview was conducted after the last problem-solving session. This interview was used to gain a better understanding of the participant's mathematical thinking and to determine the participant's perspective of the sessions. All interviews were semi-structured. Interview guides are in Appendices B and C.

The problem-solving sessions took place during June and July of 2002. The sessions were audio- and videotaped. The data from the problem-solving sessions used in this study came from the work each group did as they solved the Buried Treasure Problem. Hence, the number of problem-solving sessions reported is different for the two groups because Group 1 needed four sessions whereas Group 2 solved the problem in one session. Group 2 did investigate a second problem in two additional sessions although this data are not part of this study.

A characteristic of qualitative research is that the researcher is the primary instrument for collecting and analyzing data. This human instrument, unlike an inanimate instrument such as a questionnaire, can respond to the situation by maximizing opportunities for collecting and producing meaningful information (Merriam, 1998). As researcher, I observed the problem-solving activity and took notes on what I observed. Because it was important for me to understand the interactions from the participants' perspective, however, I asked questions to seek clarification as to what the students were thinking and doing. At the conclusion of the problem-solving session, I collected the work and notes generated during the session. This information was used as data. Following the activity, the students reflected on the problem-solving activity by responding to a questionnaire (see Appendix A). The responses obtained on

the questionnaire were used to identify interactions to observe in subsequent problem-solving sessions.

Data collection and data analysis were ongoing. An advantage to videotaping the sessions was that clips from the video could be shown to a participant for clarification. The clips were used to stimulate the participant's recall and allowed greater insight into the thoughts and actions occurring at the time.

Description of MATH 1113 Course

Each participant had been enrolled in one of my two MATH 1113 (Precalculus Mathematics) classes at Augusta State University during spring semester 2002.

Augusta State University is a regional university in the University of Georgia system. The school, a commuter school, has no dormitory facilities. The enrollment is about 6,000 students with a 32% minority enrollment of which 24% are African American. The average age is 25 years and 35 years for undergraduate and graduate students, respectively.

Each group of four students consisted of four females. Three of the students in each group were Caucasian, and the fourth was African American. All of the students in Group 1 were enrolled in the MATH 1113E class which met on Tuesdays and Thursdays from 10:00-11:15 AM. Group 2 consisted of two students from the MATH 1113E class and two students from the MATH 1113B class which met from 8:30-9:45 AM on Tuesdays and Thursdays. The classes ran from January 2002 until May 2002.

MATH 1113, precalculus mathematics, is a study of functions—linear, quadratic, polynomial, trigonometric, exponential and logarithmic functions. The format of the spring 2002 class consisted of lecture, teacher-led discussions, and opportunities for students to pair up.

Several requirements for the course need to be pointed out. The first is the email assignment that provided a way for the students to introduce themselves to me by responding to several questions concerning their interests, responsibilities, and perception of themselves as mathematics students (see Appendix B). Secondly, students were required to maintain a

notebook for the course according to the criteria given in Appendix C. The notebooks were checked on test days using the criteria listed in Appendix D. Finally, ten percent of the course grade was “In-class work” determined by the quality of participation with others and in whole-class discussions. On the first day of class, the class and I discussed the subjectivity associated with this portion of their grade. I also shared my belief that working with others is a worthwhile and valuable experience. The “In-class work” portion of the grade was instituted to encourage participation and collaboration with others during class.

The students began working in self-selected pairs on the first day. They were asked to select a partner for each class period, and I suggested they work with different people for the first couple of weeks to help them select a long-term partner. Even though I reminded them to work with different people, they were somewhat resistant to move around and select different partners. Thus, some of the partner pairs became established early in the semester once a pair of students found they were comfortable with each other. Other students never really formed an established partner pair and when asked to pair up during class, their interaction with another person was minimal. There were other students who always chose to work alone, and eventually I relented. If a partner was absent, a student could join a neighboring pair of students to form a group of three for that class period.

The TI-83 graphing calculator was a required tool. The textbook, *Precalculus: Mathematics for Calculus*, 3rd edition by Stewart, Redlin, and Watson (1998) was not linearly followed and was used primarily as one of the sources for homework problems. Students were given numerous handouts consisting of activities I developed. The ideas for the activities came from various sources including some of the reform precalculus material.

Assessment for the course included two major tests, a final exam, quizzes and homework assignments. After the first major test, students submitted corrections for which they could earn an additional point for each question missed. In order to receive the point, they had

to explain why the problem was missed in addition to submitting the correction. No group projects were assigned during this semester.

Selection of Participants

Participants were selected based on my observation of them during the MATH 1113 course. There were frequent opportunities for them during this course to “Talk to your neighbor” or to “Think-Pair-Share.” As I watched them, I noted those students whom I would approach after the class ended for participation in the study. I wanted students who did not isolate themselves from other students but were willing to discuss mathematics when asked to do so. It was only after the MATH 1113 course ended and grades were turned in that I asked students to participate in the study. Data collection would be during the summer term of 2002 so student availability as well as willingness would be considerations in participant selection. Not all students take classes during the summer and many have summer jobs, making participation in a study difficult. Nevertheless the first four students from MATH 1113E I asked to participate agreed to do so, and Group 1 was formed. I began data collection with them before Group 2 was formed. In forming Group 2, one of the four students I asked was not available for the summer so I asked a fifth student who agreed to participate.

The Task

The task used in this study is the Buried Treasure Problem taken from *Connecting Mathematics* (Froelich, 1991), one of the books in the Addenda Series. This book is a resource book for students in grades 9-12. The original problem has grades 9-12 written at the bottom so I purposely deleted this information from the problem before giving it to the participants in this study. I pilot-tested a variation of the Buried Treasure Problem in a study I conducted during Summer 1998 with graduate students and knew the problem could be solved in several different ways. Furthermore, it was unlike any problem the students in the current study experienced in the MATH 1113 class.

A second problem, the Oil Tank Problem, was taken from the course material of a problem-solving class taught by Dr. James W. Wilson at the University of Georgia. The Oil Tank Problem was used only in Sessions 2 and 3 of Group 2, and the data from these sessions have not been used in this study.

Limitations of the Study

No attempt was made to balance the groups according to gender or race.

Not all students in Group 1 were present for all four problem sessions. Although this nonattendance was frustrating for me at the time, their absences were typical of what occurs when students work in a group over an extended period of time.

The participants did not always respond to the questions for reflection in a timely manner.

There were some inconsistencies in the data collection for the two groups. Group 1 worked on the Buried Treasure Problem only, whereas Group 2 worked on this problem as well as the Oil Tank Problem. Although both groups discussed the Buried Treasure Problem with me in the final interview, Group 2 also individually wrote up the solution for the Oil Tank Problem. Furthermore the final interview for each member of Group 2 consisted of two parts, conducted on two different days. The final interview for each member of Group 1 was conducted on a single day.

Data Collection

The initial interviews were conducted in my office and were semi-structured. The questions used as a guideline are in Appendix E . These interviews were recorded with a digital audio recorder, saved to my computer, and transcribed using transcription software.

The problem-solving sessions were held in the psychology laboratory at Augusta State University. The students sat around a circular table. I was in the room as the students worked and interacted with the students to have a better understanding of their mathematical thinking. The problem sessions were videotaped using two different cameras to capture the session from

two different angles. One video camera was mounted on the wall, and the second camera was on a tripod. A digital audio recorder placed on the table where the students worked was used to capture the audio in the sessions. These recordings were saved to my computer. As the students worked on the problem collaboratively, I observed the students and took field notes and asked questions to help me understand their thinking.

Group 1 participated in four problem-solving sessions and worked on the Buried Treasure Problem only. Group 2 participated in three problem-solving sessions. In their first session, they worked on the Buried Treasure Problem, and the group solved the problem in that session. Group 2 worked on a second problem (the Oil Tank Problem) for two additional sessions, but the data collected during these sessions were not considered in this study. The four problem sessions for Group 1 were transcribed. Sessions one and two for Group 2 were transcribed. I transcribed the problem sessions by watching the videotapes and listening to the audio of the sessions saved to my computer. The transcription of a problem session included not only verbatim conversation but also my perception of the group processes occurring during the session.

A reflections questionnaire (see Appendix A) was given out at the end of the first problem session. The participants were asked to respond to these questions in narrative form although some of the students merely answered the questions with short answers. The responses were faxed or emailed to me. The same questionnaire was used again after the second session for Group 1.

Each member of the group was interviewed individually after the group completed its problem sessions. The final interviews (see Appendix F for the Final Interview Guide) were held in the psychology laboratory in a room adjacent to the one used for the problem-solving sessions. The final interview for each member of Group 1 was held within one week of the final session and was 60 to 90 minutes in length. During the first part of the interview, the participant shared with me her understanding of the problem. I asked specific questions about the solution

to the problem. The participant and I looked at the written work she generated during the sessions, and I questioned her about this work as well as her general understanding of how the problem was solved. During the remaining time, she and I watched videoclips from the sessions that I had identified and about which I had questions. The videoclips and questions for each participant were noted during my viewing of the tape prior to the interview. (See Appendix G as an example of the guide used for stimulated recall.) As the participant and I watched the clips together, I asked questions to understand the nature of her mathematical thinking and what processes she perceived as factors influencing her mathematical thinking.

The final interview for Group 2 consisted of two parts, occurring on two different days. Each part lasted 60 to 90 minutes. The first part took place within the first week of the final problem-solving session. During this part, the participant and I watched the entire tape of the first problem-solving session in which the group solved the Buried Treasure Problem. I asked questions about the decisions made during the session and questions about her understanding of the problem and its solution.

The following week each Group 2 participant was interviewed again for 60 to 90 minutes. Video clips from the second and third problem-solving sessions during which the students had worked on the Oil Tank Problem were used to trigger recall of the student's thinking. As we watched I asked questions about her understanding of the problem and its solution. I also tried to understand what group processes she perceived as factors influencing her mathematical thinking.

The study generated a large amount of data. I found it useful to record in a computer file the observations and insights I gained as I transcribed the interviews and problem sessions. These recordings were helpful in identifying emerging themes and patterns. During transcription the Comment feature of Word was used to keep track of the influencing factors I saw.

Timeline for Data Collection

Each member of Group 1 was initially interviewed during the last two weeks of May 2002 after the MATH 1113 course ended. Group 1 participated in four problem sessions. The first of these sessions was on June 17 and the final one was July 16.

Each member of Group 2 was initially interviewed during the last week of June 2002. This group participated in 3 problem sessions, beginning on July 8 and concluding on July 22. The schedule for the interviews and the problem sessions is given in Table 1. At least one week separated the problem sessions for each group.

Table 1
Timeline for data collection

May 15	Abby initial interview
May 15	Eve initial interview
May 23	Sara initial interview
May 28	Jane initial interview
June 17	Group 1 Session 1
	Reflection
June 27	Group 1 Session 2
	Reflection
July 9	Group1 Session 3
July 16	Group 1 Session 4
July 17	Abby final interview
July 17	Jane final interview
July 18	Sara final interview
July 19	Eve final interview
June 26	Helen initial interview
June 27	Lisa initial interview
June 27	Sandy initial interview
June 24	Tess initial interview
July 8	Group 2 Session 1
	Reflection
July 15	Group 2 Session 2
July 22	Group 2 Session 3
	Explanation of Oil Tank Problem
August 2	Helen 2 nd interview
August 12	Helen 3 rd interview
August 2	Lisa 2 nd interview
August 14	Lisa 3 rd interview
August 5	Sandy 2 nd interview
August 14	Sandy 3 rd interview
August 7	Tess 2 nd interview
August 14	Tess 3 rd interview

Organization of the Case Studies

Each case study consists of four parts. The first is a description of the participants with data garnered primarily from the initial interviews. The second part is a description of the investigation of the Buried Treasure Problem. The data for this part came from my on-site observations of the sessions and repeated viewings of the videos of the sessions. The third part is the examination of each participant's understanding of the problem solving that took place. The data for this third part came primarily from the final interviews and the responses to the reflections questionnaire. The fourth and final part is my synthesis of the factors that influenced the problem solving of the group as they investigated the Buried Treasure Problem.

The case studies consist of numerous quotes. Each participant is identified by her pseudonym, and I have identified myself as LBC.

CHAPTER 4
CASE STUDY OF GROUP 1

In the following sections the case study of Group 1 is presented. The description of the participants in the first section capitalized on data from the initial interviews. The second section is a scrutiny of the four sessions in which the group investigated the Buried Treasure Problem. In the third section data from the reflections of the sessions and the final interviews have been used to present each participant's perspective and understanding of the experience of working together to solve the Buried Treasure Problem. The fourth session is my synthesis.

Participants

Group 1 consisted of Abby, Eve, Jane, and Sara. Eve is African American, and the other three are Caucasian. Some demographics for the students are in Table 2.

Table 2
Demographics for Group 1

Student	Age	Major	Highest SAT MATH Score	Highest SAT Verbal Score	MATH 1113 Final Exam Grade
Abby	20	Early Childhood Education	470	510	48
Eve	19	Computer Science	500	470	96
Jane	18	Biology	490	450	69
Sara	20	Art	530	560	74

Jane

Jane, the oldest of three children, graduated from high school in 2001 and had just completed her second semester at ASU. She plans to major in microbiology and wants to get her Ph.D. in pathology and study infectious diseases. Both parents are college graduates. Her mother has a degree in finance and has recently gotten her MBA. Her father is an engineer, and

when she needs help with a math problem, she asks her father. In the interview I was trying to understand how she approaches problem solving so I asked her what she does when she sees a problem that is totally unlike any problem seen before. Immediately and tersely she responded with, "Ask Dad. Ask a parent."

Jane said that in high school she did well in math, getting B's. She said that she is very good with algebra and especially enjoyed doing proofs in geometry but thought that was "kind of a funny thing." She took a trigonometry course during her junior year but because she made 68 for the second semester, she had to take the course again during the senior year. She did not admit that her failure was due to a lack of understanding, and when she made 93 for both semesters of her senior year, she cited this achievement as evidence that she did understand what she was doing.

Jane admitted she gets nervous on tests and has been diagnosed with test anxiety. She is currently taking medication and practicing breathing exercises to help control the anxiety.

Jane said she enjoyed proofs in geometry. This admission seems odd because she did not enjoy the writing and explaining the class was expected to do in MATH 1113. Jane said of the MATH 1113 class:

I had a lot of problems writing down on paper how I did this. And in math it normally was...you really don't explain, you know, write down why you do it...this is just how it works. You have these theorems, normally you don't know the name of the theorems...you just know this is how you do it...and a lot of times I could not explain, I can't write down what I mean...why I'm doing this.

Jane has a rule-based approach to mathematics. She said that "you read a word problem and you look for the word "is" cause you know that means equals." She believes she can handle the statistics class she has signed up to take in the fall because her mother told her statistics is mainly formulas. She had no trouble using trigonometry to solve velocity problems in her high school physics class but acknowledged that she had the formulas and "if you give

me an equation I can do it.” The math problems she claims to be best at are the algebra or the triangle trig problems because those are more formula-based. Mathematics, however, has not been enjoyable to her, and she identified it as her least favorite subject “just because it is numbers, number crunching.”

Jane’s frustration with the MATH 1113 course just completed was evident during her initial interview. She finished the course with a C. She expressed frustration with the requirement to give explanations for her work. She attributed this frustration with already having had trigonometry twice before and therefore knew what she was talking about but “having to write it down...words get all jumbled.” When asked if she believed she got better at explaining, she responded, “No....I actually had come out thinking I didn’t know anything after this semester. And that honestly is the truth. And I don’t even know why—you know, cause I knew—I actually did better on the trig identities because that was a killer my junior and senior year of high school.” Difficulty with trigonometric identities in high school seems to contradict her enjoyment of proofs in geometry and her “being good with algebra. “

During the MATH 1113 course the students were frequently asked to pair up and discuss some question or some problem. When Jane was asked during the initial interview whether or not this was effective for her she responded:

Well, the thing is if you do that what happens if your partner doesn’t know anything to begin with...they don’t understand the whole concept...they are not really learning anything...and in discussing it, discussing it can be a good thing, but if the person you are working with...for me it is more of looking at it on my own, independently doing it...it’s...I’ve always thought of math as being an independent, an individual class...you know, each person does their own thing...I don’t mind getting into groups and working problems...you know, we had done cut-out puzzles and you had to find...we would four problems on each block and put it together and make a puzzle out of it ...that wasn’t a big thing.

Jane's comments reveal that group work is not her preferred way for learning mathematics. Her preference is a lecture with the teacher "working the problems at first to show what it is and then giving a problem and working it and then going and doing the homework the night of and then coming back and going over the math homework and why people missed it." She said that in general "working in groups in math, I just never even really saw a lot of help from that. Most of the time I would just go home, read the book, do the problems, look at the examples and use the notes and go from there."

Abby

Abby, an only child, has just completed her second year at Augusta State. Abby likes children and is seeking a major in early childhood education. Not only does she babysit but she has also volunteered in one of the classes at the local elementary school. MATH 1113, a course Abby struggled through, is not required for her program of study but was taken simply because she enjoys mathematics. It is interesting that she claims to enjoy mathematics but gets mad when she gets stuck on a problem and will stop and put it away even if it is the first problem in homework. She believes she does not do that well in math because she gets mad and quits. Her ninth grade algebra class was a Learning Logic course in which students worked at their own pace to learn Algebra I. I asked Abby to tell me about the Learning Logic course, and she had negative comments:

It.....it was just awful. For math you need somebody talking to you, you need a person like in front of the classroom; you need to be able to ask questions. And with that it [the Learning Logic program] told you steps and it gave you all the steps that you needed but it couldn't...it wouldn't tell you why (*she emphasized the why*) something happened, it wouldn't tell you why a number was on this line and get down to step 4 and it wasn't there any more, you know where it went or why it happened and you couldn't ask it why it happened and the teacher was in there but....it wasn't the same....

Although Abby disliked the class, she did not blame the teacher. She saw the teacher as willing to help, but also believed the teacher was limited in the help she could provide. She referred to the teacher's role in the classroom when she said, "it wasn't that she [the teacher] didn't want you to ask, she wasn't...it wasn't the same as if she was teaching it because a computer was teaching it its way and it wasn't the same as if she was teaching it. And she would tell you that it wasn't the same..."

Abby learned algebra as a series of steps to follow. Her assessment of the approach used in the Learning Logic class is revealed in the following comments:

A computer turns it ...and especially math, anything really, but especially math the computer turns into something not the same as someone trying to tell you and you can't, if it does something you can't ask it why it did it and sometimes I really don't think she knew why. I mean you know it would just show the next line and it wouldn't show you the work, it wouldn't show you OK you cross this out and get x and it would just show you the next line and stuff but...and sometimes I don't think she really...she knew why it did what it did. You know she was like...just follow the steps and I was like where did the number go? You know? And I hated that. It was the awfulest [*sic*], it was awful.

Her experience with Learning Logic gave her a rule-based approach to learning algebra.

Because she talked about how the computer constrained her from understanding why, I believed Abby realized there is more to mathematics than that which she experienced. Abby seemed to understand that knowing why is important whereas Jane had less of an appreciation of the importance of why. Abby said that she would often miss a problem because she would make "up a rule for something that there's not" [an appropriate rule]. She attributed this tendency to make up rules to "not knowing it enough or not knowing it well enough to know that you can't, you know...it's like the ones that we had [problems in MATH 1113], you know...just because we had, if it looked this way you can do that but if it looked another way they are not

the same, not realizing that and trying to make it work and it really won't do that, it really won't work that way."

Abby "did better in geometry" than in her other high school mathematics courses. When asked why she believed she did better in geometry, she said, "the way she [the teacher] explained it cause you know it had the pictures--the like the triangles--and it just made more sense than the other stuff, like the formulas and stuff." No technology was used in the geometry class and when asked if the students used hands-on materials, she replied, "I think we did. I know she [the teacher] had stuff like that she showed us and another thing, too, I think she was very enthusiastic about that class."

Abby's rule-based learning of algebra has limited her problem-solving ability. She said that she is best at math problems that can be worked with "a formula or that just comes out and says this is, you know, what I want you to do." She gets confused when she solves a word problem or a problem that has two or three parts. She describes her difficulty with solving problems with many parts:

I try to jump in and figure out everything instead of....I won't, I don't pick a point and start there and try to go....I just want to figure it all out and trying, you know, I've got this and I'm going to get this, I don't work it out in steps—I try to figure it out too quickly, I won't take the time to read it two or three times, I want to read it and just do it and you know try to do everything all at once and I don't look and see, well, do I really need this...part that's in here, do I really need it?

Abby described herself as a shy person. Although she wanted to attend the University of Georgia, she "was scared [she] would go and get lost there [at UGA] because [she] was more shy and...couldn't just go up and talk to someone." She instead opted for Augusta State because it is a smaller school. Although she is attending a smaller school, she credits her college experience with helping her overcome some of her shyness. She sees herself as having changed and describes herself as "a lot different than when I graduated from high school

and I talk to people more and I made some more friends here and I can just talk to people.” When I asked her what contributed to her change since attending college, she said “being away from like high school because those people I had seen my whole life and whatever and it was almost like they expected me to be a certain way.”

Abby grew up in a small town where she says everybody knows everybody and as a result is “sort of, kind of friends.” Although she described herself as shy, a better description may be insecure. She said that she likes to learn, likes “to know many forms about stuff and to know what’s going on to know if somebody’s talking about something I like to be able to know what they are trying to talk about...I don’t want to feel like I don’t know what’s going on.” Abby is very concerned about what other people think of her. I think she feels inadequate around others and yet tries very hard to hide this from others. On the first day of the semester in MATH 1113 students are asked work on a math problem in pairs. I questioned Abby about how she felt that first day when told to work with another student, and her response reveals her feelings of inadequacy and insecurity:

I think just because of having to try to talk to somebody else and feeling like...and feeling like they are going to know something that I’m...Gosh, I’m so dumb, you know,...but any time I have to work with...it’s just I feel like they are going to think I’m so dumb because I don’t know....but after I work with somebody I don’t feel like I....but it is just after that first time, gosh I’ve got to talk to this person or whatever.

Abby rarely asked for outside help during the semester that she took MATH 1113. She never came by my office to get help and never did she go to the Math Lab (a free tutoring service provided by the Mathematics and Computer Science Department). She said she tried to meet with her partner a couple of times but apparently these meetings were not very productive. During these meetings Abby said, “we would try to discuss stuff. But most of the time we had the same stuff, and she didn’t quite understand what I didn’t quite understand.” I asked her why she did not come to see me or try to get help from someone else and she answered, “I don’t

know...this was totally wrong, I thought that maybe it would click, you know in the next class or whatever...I always thought I was going to get it in the next class—I would think, you know, it will come in the next class.” Abby’s thinking “it would click” indicates a view of mathematics that is limited to isolated topics to learn. She said that occasionally she would get help before class from another girl in the class. However, she asked for help “mostly if we were going to turn something in.” She was seeking help not for understanding but for getting a better grade on an assignment.

Eve

Eve is the youngest of four children. She lives at home with her parents and describes her family as very close. She chose ASU because it is close to home. She said that she has always enjoyed school to the point of hating to miss even when sick. A leader in high school, Eve served on the student council, was in the club for Future Business Leaders of America, and was the senior class president. She was offered numerous scholarships to attend college and realized the importance of a college education in getting a good job. Her major is computer science.

Eve likes mathematics and says that although it is challenging “most of the time I can get it to where I can understand it and it’s always a subject that I have always done good in.” She describes her family as having a strong math background. Although she says the members of her family have not studied mathematics to the extent that Eve has, Eve will often get help from them when she has “questions about Precalc.”

Eve described her College Algebra class as “simple.” The precalculus course, however, “was different because in high school we just did things and we didn’t find out the reason why certain things were like...formulas, how we derived it the things we did...we were just looking for the end to it. But precalculus in college we had to go into graphs and how you come to a graph, how was it formed and the basis behind the things of it and just not like well that’s it and you just go on with that, that’s the way it is.” Eve’s experience with mathematics prior to the precalculus

course was more rule-based. She saw a different view of mathematics in the precalculus course which obviously impacted her because she told her high school calculus teacher what she had learned about “the background things.” Her high school teachers asked her to help them with some of their classes and she described her experience as “interesting because the kids were like ‘Oh I’ve seen that,’ you know, that is how you come to a graph, you know, and that’s why it shifts over and it was good to know the meaning behind it instead of just saying that’s it and take it, to know the purpose for it.” Because her teachers requested her help, perhaps they were acknowledging their own rule-based approach to mathematics.

In the first interview I asked Eve to describe the kind of math problems that she is best at. Eve said that she is best at problems “when you have something where you can just solve for it...I just like if you give me a problem and I have to solve for like x or something, I like those problems.” Although I’m not sure what she would do with the problem, she said, “if you have like $x = 5xy + 6$, you know, I could... I just like those problems to solve.” Problems requiring algebraic manipulation to solve for a variable appeal to her.

After describing the kind of problems she is best at, she quickly said that problem solving is “alright but I don’t care for it.” I asked her what did she mean by problem solving and she answered, “The wording and....I always had trouble with putting you know the things together. When the teacher does it, I would like “that’s right” but when I do it, it seems like I could never get it together like....what does it mean or what it meant in the wording.”

I asked her what she does when she gets stuck on a problem. Looking back in the book for a related problem, reworking the problem again, or seeing if it could be worked another way were strategies she gave. She said that often she gets stuck “cause I probably didn’t add correctly or did something that I probably shouldn’t have done—like something that wasn’t mathematically correct and that’s how I got stuck. Cause if I rework it, a lot of times, you know, the corrections are made and I catch it.” I was getting a sense that she believed most of her

getting stuck was due to careless errors so I then asked, “Do you ever just not understand the concept?” She responded with “Sometimes. On one problem I just could not get it...I just did not know where to go so I just skipped and went on to the rest of the problems.” It appears that she has had few problems that she has had to grapple with.

When Eve had math homework to do, she preferred to work by herself rather than working with her classmates outside of class.

It seems like I just have to do problems by myself in order to get it cause it seems like other people will confuse me but I have gotten together with other people...well basically what I do I have already worked the problem and then we just come back and converse and see how we worked it cause it seems like I can't work well...with other people. I mean I will work a problem but I just have to do it myself and know its [unclear] and then we'll see if I...we came to the same answer.

When she paired up with other students in class, little collaboration and discussion occurred. Generally each person worked the problem and then they compared answers. She described the situation as

Well, I looked at the problem and I asked my partner, “What do you want to do?” And then meanwhile I was already writing or you know strategizing stuff and then most of the time I was like “Well, I came up with this and how did you solve it?” That's how...basically how...we both worked it and then I would like, “How did you solve it ? or did you come up with this answer” and see if it was right or not.

The purpose of working with other people during class for Eve was primarily to check her answer to a problem. If she knew how to solve the problem, she worked alone. She felt little responsibility in helping her partners understand what she was doing.

I would ask well, you know, well what do you think, or how would you go about this, or what was better...like a problem that you know wasn't...I wasn't like getting through

but...if I pretty much knew how to do it and stuff I would work it and then ask them and then if they didn't understand or I didn't understand then we would talk about it.

Sara

Sara has an older brother and a younger sister. Her father was in the military and the family spent three years in Germany. When Sara was eight years old, her father was stationed at Fort Gordon so the family moved to Augusta. Because Sara is from an Hispanic family, she says it is only natural that she enjoys dancing. She named modern dance, African dance, ballet, hip hop, and “out and about like going out type dancing” as particular interests. She also has a strong interest in art and attended a local magnet school for students interested in the arts. In high school she studied photography, drawing, and ceramics and at the time of the initial interview was majoring in art at Augusta State. Recently, though, she has begun to consider a major in graphic design that would allow her to combine her love of art with mathematics.

Sara has well-developed visualization skills and is a very good writer as well. I recall how beautifully she was able to express herself when asked to explain the thinking she used to solve problems in MATH 1113. Asked if translating her thinking into words has always been easy for her, she responded with the following comments:

I think so. I think I've always been as far as like...not necessarily teaching somebody or in telling somebody certain things I have always been able to break it down real simplistically, give it to you in a matter that is not hard to understand—I've always tried to break down things as simply as possible, make them as, you know, no fancy nothing added to it just....like I'm a trainer at Carraba's (*a local Italian restaurant*), a server trainer and I feel like because of my patience and because of the way I explain things to people that it is easy for them to understand, most of the time.

Sara's favorite subjects are art and math. Although she took AP Calculus in high school, her first mathematics course at ASU was College Algebra. In high school she took mathematics courses because she “could do it but ... didn't really like it and ... didn't really pay attention to it.”

Her attitude toward mathematics, however, has changed since attending ASU. The following comment reveals that her experiences in College Algebra and Precalculus were helping her develop more of a conceptual understanding of mathematics.

...in like Dr. Robinson's class [College Algebra] and your class [Precalculus] when you actually start to understand it and you can see it...especially if you have like graphs incorporated to a real life...this depends on this therefore this is what you get. It is just much easier for me to be able to understand things if I can see them or if I can work them out so I can produce something so somebody else can see and say... this is how I explain what I am thinking."

Sara performed very well on the first test given in Precalculus but did a poor job on the second test. When I asked her about the difference in her performance, she explained that

probably in the back of my head I was thinking... I've been doing good so far, it'll probably be the same, I'll probably do good again—I probably got a little behind on the homework and then also like the identities I didn't know that everything was going to be so---like it was more you had to draw from your knowledge of maybe a certain other aspect of something we learned in order to be able to answer this question. I don't think I was expecting that and for that test if you didn't know the identities then there was a lot that you couldn't come to. But there were certain things on the test, too, that as I was looking at them like physically looking at them I realized that oh you have to...this graph is...you know, I don't know, I can't remember any problems, but I figured stuff out as I was going along that you know I was relating this to something I knew from beforehand and then it made sense to me then.

Sara was beginning to make connections among different concepts. It was unfortunate that she had not allowed herself time prior to the test to begin to make those connections. A time span of one hour and fifteen minutes, however, is not sufficient time to pull ideas together.

Nevertheless, her attempts at making connections during the test indicate a realization that mathematics is more than rules to be learned.

Sara admitted that College Algebra had been primarily a review. I then questioned her about the Precalculus (MATH 1113) course and asked if most of the topics studied were review since she had taken AP Calculus in high school.

...[A] lot of them were review but this time versus in high school it was I wanted to learn more and the way it was presented to me it was more the why along with the how instead of just how to and so now that it has been presented to me like that I know it and I know why it works the way that it does and that will stay with me now instead of just learning how to and forgetting it later on.

I asked Sara what kinds of math problems she thinks she is best at. Rather than telling me which she is best at, she responded with “I hate word problems.” She tempered this initial response with “I like trigonometry-like word problems, like figuring out distances and stuff like that cause you get to use trig.” I encouraged her to explain what she meant when she said that she hates word problems. She explained:

Like the one where there is a river and he is making a fence around the river—I guess more algebra-based word problems because I can’t put them, it’s hard for me to...you know...there is not already a picture there, there’s not already a written problem there—you have to come up with a picture, you have to come up with a problem and sometimes it’s hard for me to read into it and I can’t necessarily see what they are asking for.

Remembering that Sara is a visual learner, I asked her if she will draw a picture when given one of these problems. She explained that she will draw a picture “but sometimes I won’t know how to set the equation up based on the picture, so...it just depends.” This explanation leads me to believe she has difficulty representing situations algebraically.

Sara said that she hated having to pair up with a student in the MATH 1113 class. The two girls on her left were friends from high school and generally worked together; Jane was on

her right so generally the two of them would pair up. The work, however, with Jane was not very collaborative because “a lot of times that it happens in the back of our minds we both knew that you know we would kind of be like OK this is the answer, Cool, we are done. But generally, I don’t know, sometimes it was very quick and you would just discuss something for a minute or two....” When asked what could have been done in the MATH 1113 class to make the group effort more effective, Sara suggested

if maybe...there would have been an actual, something that you actually had to show for it...you know maybe have to have like if there is 10 groups maybe 10 different problems and somebody has to get up and explain why...So that there is an actual, you actually have to show something at the end of it instead of just saying OK we did it.

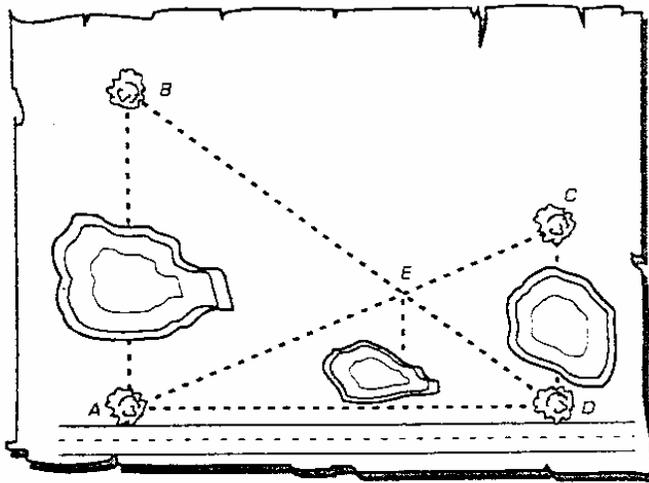
When I suggested, “So you didn’t feel very committed to the effort, perhaps?”, Sara responded, “Right. Probably because there was nothing to show for it. We could just look at someone and say, “Hey, what are you having for dinner tonight?” and you know it looks like you are talking about what you are supposed to be talking about.”

Investigation of the Buried Treasure Problem

Group 1 spent four sessions of approximately 1 hour each to solve the Buried Treasure Problem (see Figure 2). All four students were present for the first and second sessions but Sara missed the third session and Eve missed the fourth session. They did not let me know they would be absent prior to the sessions.

ACTIVITY 9
BURIED TREASURE

You are on a trip to one of the Cartesian Islands. In the cellar of your cabin you have found an old map that claims to hold the secret location of a buried treasure. The map shows four trees at locations *A*, *B*, *C*, and *D*. The trees at *A* and *D* are along a road that runs through the island. The tree at *B* is directly north of the tree at *A*, and the tree at *C* is directly north of the tree at *D*.



The map's directions say to trace a path from the tree at *B* to the tree at *D*, and to do the same for the trees at *A* and *C*. The treasure is buried halfway between the intersection of these paths and the road. Because three small lakes dot the island, you are only able to measure the distances between *B* and *D*, *A* and *C*, and *A* and *D*:

B to *D* 300 paces

A to *C* 260 paces

A to *D* 240 paces

To find the location of the treasure, you must determine the distance from *E* to the road and then walk half that distance from *E*. How far is *E* from the road?

Figure 2. The Buried Treasure Problem.

The students were not adequately prepared to work collaboratively. Sara did not want anyone to be left out and frequently asked for suggestions. There were times when she sacrificed her own ideas so that the ideas of others could be explored. Jane and Eve were very focused on getting the answer but their means were quite different; whereas Eve tried to make

sense of the problem and the mathematics, Jane was quite willing to vandalize the mathematics in her pursuit of “the answer.” As the constant pupil, Abby’s notion of working in the group was so that she could learn from the others. Because “solving this problem together as a group” meant different things to the individual group members, their work was not very collaborative.

These students had been in the same section of MATH 1113 during the Spring Semester 2002. They did not, however, know each other well. Their collaborative work in MATH 1113 the previous semester was no more than informal “talk to your neighbor” sessions. Jane, Sara, and Eve had occasionally worked together in class. Abby had not worked with any of the girls in class and said (2nd interview, p10) that she did not even remember Eve’s being in the class. I told the students I was studying how students work problems in groups. Their previous experience in mathematics classes did not adequately prepare them to work collaboratively. Thus, when asked to work collaboratively on this problem, they approached the problem with different expectations. Their interpreted expectations to the task may have constrained their ability to solve the problem in a timely manner. My analysis of the data revealed the following task expectations for each participant.

Eve was so quiet during the first session that I believed she really did not want to participate. I mistook her silence for arrogance. As the sessions continued, I realized Eve felt an obligation to be a team player and yet found working problems as a group uncomfortable for her. During the final interview she admitted there were questions she did not ask during the first session “because the atmosphere was different so I was just trying to feel around...I mean cause we were in, we were working all together...and you know we couldn’t just run off and do our own thing...you know we work as a team...so I was just trying to feel how everybody else was feeling so I just laid low....” Eve, accustomed to working alone, felt constrained by the task of working collaboratively. Eve not only felt the obligation to be a team player but she was also obligated to herself to make sense of the problem. Frequently she would cognitively withdraw from the group and work alone.

Sara, too, was obligated to work the problem as a team. Of the four, Sara was the one who made the most overt efforts at involving the others in the group work. These efforts included questions such as, “Does anybody have any suggestions?” or “Anybody have any ideas?” Even though Sara solicited feedback from the others, her efforts were not very effective. Most of the time, Sara got no feedback other than a giggle from Abby or a cryptic remark from Jane. For example, in the second session, Jane was entering data into the calculator and Sara asked, “What are you doing?” Jane replied, “Right now? I’m just messin’ around.” Later in this same session, to Sara’s request, “Does anyone think it is worth trying or should we try something else?”, Jane obscurely replied, “Anything will work—hopefully something’s gonna work.” Another illustration that Sara’s requests for comments fell on deaf ears occurred during the fourth session; Eve was absent for this session and Jane and Abby were giving Sara no help. Sara frustratingly asked for some feedback.

Sara: Does anybody agree or disagree that what I’m doing is going to help at all or any other ideas?

[10 seconds of no response]

LBC: What do you need for them to do? What would you like for them to do, Sara?

Sara: Just respond.

[Abby giggled but still no other response—20 seconds of silence.]

LBC: What do you think?

[still no response so I questioned again]

LBC: Do ya’ll trust her arithmetic?

Sara: It’s been a while.

[Abby giggled.]

Abby: That makes sense to do that, though.

Jane: I trust her a heck of a lot more than I trust myself right now.

This exchange illustrates how Abby would respond with a giggle and Jane would offer a cryptic comment. Even after this exchange there were 75 seconds during which Abby and Jane watched as Sara worked quietly.

Sara said during the first interview that she is basically a shy person; furthermore, she did not want to be a leader although I believe the others saw her as the leader of the group. I think one of the reasons this group took so long to solve the problem was because Sara was unwilling to be the leader the others perceived her to be. As a result, when a suggestion was made, it was often implemented without exploring its potential and weighing it against other options. Sara's refusal to be the leader and her desire to validate the suggestions of the others caused the group to "chase a few rabbits."

Whereas Eve and Sara felt obligated to work the problem as a group, Jane's and Abby's obligations to the experience were more egotistical. Jane's goal was to get an answer but she was not interested in making sense of the problem. Jane's approach to mathematics was very procedural, and she often committed vandalism to the mathematics as she searched for an answer. Subsequent episodes will also illustrate her attempts to save face.

Abby was the constant pupil in the group. For her, the task provided her with the opportunity to be taught. She rarely offered a suggestion or asked a question because of her fear of being wrong. In the final interview I asked her about her reluctance to ask questions.

Abby: I didn't question so much because...I was thinking that maybe that...I might have thought that wrong and that they were right...I didn't ask many questions.

LBC: And why would that have been a problem with being wrong?

Abby: I don't know...sigh...it's just this...me and the...I don't know...I don't like to be wrong...not because I don't like to be wrong but because I don't want people to know I am wrong.

Not only did she not make suggestions or ask questions, she would also immediately look at whomever began talking as if fearful of missing something. She admitted during the final

interview that she “was trying to pay more attention than to write stuff...if I had started writing, I was scared that I would just kind of go off...” She would discreetly look at the work of others and even occasionally copy from their papers. She wanted to understand the problem but was unwilling to ask the others for help.

One of Watson and Chick’s (Watson & Chick, 2001) cognitive factors that influenced the outcome of collaborative mathematical problem solving was “the big picture.” Their description of this factor underscores the importance of keeping the overall task in perspective as the group works on subsidiary tasks that contribute to the final goal. Initially, however, it is important that the group have some sense of what the overall task is. According to Schoenfeld (1983), generating or considering plausible approaches is important in problem solving. They did not ascertain that everyone understood the problem. They did not discuss the options that existed for solving the problem. Their disregard for planning was evident after the initial reading of the problem. Even before Eve and Abby finished reading, Jane began talking, revealing her tendency to be egocentric. Aware Eve and Abby were perhaps not finished, Sara asked, “Are you folks done?” Sara’s question again shows her obligation to the group experience. Abby, not unexpectedly, replied, “Uh huh,” but Eve continued reading and ignored Sara’s question. Although all four had ample time to READ the problem, Eve’s continued focus on the problem indicated she was trying to understand the problem; this was an example of withdrawing cognitively. Eve shared during the final interview that she was uncomfortable on the first day because she did not understand the problem. Having time to think about the problem between sessions 1 and 2 helped her understand what the problem was about. As a result, Eve asked questions and made suggestions during the subsequent sessions. Following the first session Abby discussed the problem with her colleagues at work; this discussion perhaps helped her understand the problem and resulted in the contributions she made during Session 2.

When Eve finally looked up, Sara, tried to initiate a discussion of the problem by saying, “I’m still trying to figure out exactly where the treasure is located...right?” Sara’s subtle attempt

was ineffective. Although Jane responded with “Right” and then read “in order to find the location of the treasure, you must determine the distance from E to the road—and walk the half distance from E,” nothing else was said. Sara, reluctant to be a leader, did not explicitly suggest they discuss the problem.

Related to the factor of “the big picture,” was the group’s failure to use the conclusions made in one session in a subsequent session. For example, at the end of Session 1 I asked them to write a reminder to themselves of where to begin the second session. When Session 2 began, no one referred to the reminder or to the work they did during Session 1. Later during Session 2, I suggested Sara read her reminder aloud to the group. Eventually they worked on ratios between the triangles, and Sara’s paper provided a good record of this work. Sara was absent during Session 3, but her work was available. It was late in Session 3 before Abby looked at Sara’s work from Session 2 dealing with the ratios. Finally, at the beginning of Session 4 Sara looked at her own work from Session 2 rather than asking the group about any progress they made during the third session. Although Abby’s and Eve’s work from Session 3 had the proportions that Sara ultimately used to solve the problem, Abby (Eve was absent) was hesitant to share with Sara what they did in Session 3. In fact, Abby ignored my requests that she share the work with Sara. I had to insist that Abby show Sara the work. Moreover, Sara showed no eagerness to see the work they did in the third session. She asked a number of times if anyone had any suggestions but never did she specifically ask about the work they did. This lack of interest on Sara’s part was perhaps reflective of her desire to figure the problem out herself and revealed egocentrism. (See Figure 3 for a schematic drawing of the problem.)

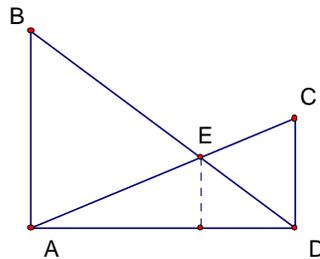


Figure 3. Schematic drawing of the Buried Treasure Problem.

Other factors also influenced the collaborative problem solving of the group. Many of these factors interacted as indicated in the twists and turns the group took as they solved the problem.

Session 1

With no discussion of the problem or of possible approaches, Jane suggested they use the Pythagorean Theorem to find the lengths of AB and CD. Abby and Sara agreed, but Eve said nothing and continued to read the problem.

Finding the lengths of AB and CD, however, was especially problematic for Jane and Eve and yet neither asked for help from the group. I was surprised at the difficulty they had finding these lengths.

Sara asked if the others got 180, prompting a puzzled look from Jane.

Jane: For [unclear]. Oh, total?

Sara: For AB.

Jane: Oh, I haven't even done that yet.

Jane's work indicates that she first treated AB as the hypotenuse by writing $AB^2 = (300)^2 + (240)^2$. Then, when her value $\sqrt{147600}$ did not give Sara's result of 180,

egocentrism was a factor as she saved face by claiming that she had not “done that yet.” Jane did not ask for help but finally fixed her problem by changing the problem from addition to subtraction. She carelessly wrote, however, $\sqrt{82400}$ instead of $\sqrt{32400}$. Her work is shown in Figure 4.

$$\begin{array}{l}
 AB^2 \Rightarrow (300^2) - (240^2) \\
 AB^2 = 90000 - 57600 \\
 \sqrt{AB^2} = \sqrt{\cancel{90000} - 57600} = \sqrt{82400} \\
 AB = 180
 \end{array}$$

Figure 4. Jane's work to find the length of \overline{AB} .

After getting the value of AB, Jane was still slow to obtain the value of 100 for CD. She eventually confirmed her answer agreed with Abby's. Though Jane was tenacious in getting the answers reported by the others, I am not certain she understood why her initial answers were incorrect. As subsequent episodes will show, Jane had a tendency to commit vandalism with the mathematics whenever a procedure did not work. The correct answers of 180 and 100 may have been obtained but without understanding. As a result, a learning opportunity for Jane was missed.

Eve's calculations of AB are shown in Figure 5. She clearly made an incorrect statement of $AB = 300 - 240$. This error was not made in her calculations for the length of CD. Furthermore, when I questioned her in the final interview about the statement $AB = 300 - 240$, she saw

$$\begin{array}{l}
 (AB)^2 + (240)^2 = (300)^2 \\
 AB = 300 - 240 \\
 \textcircled{AB = 180}
 \end{array}$$

Figure 5. Eve's solution of AB.

immediately that 300 and 240 should have been squared. Eve admitted during the final interview that she had a hard time understanding the problem on this first day. When the others began talking about finding the lengths of the sides, Eve continued to read the problem—apparently trying to make sense of what was asked. Feeling compelled to keep up with the group but frustrated because she did not understand the problem, she made errors in the procedure. Furthermore, she was the only one of the four without a calculator so she wrote the answer of 180 after hearing Sara's result. Not asking for a calculator was further evidence of her unease during this session. There was a fourth calculator that Sara brought on the desk adjacent to their table. Eve neither asked if another calculator was available nor if she could share one. I identified Eve's making the procedural error and not asking for a calculator as evidence of her frustration with not being able to make sense of the problem. Egocentrism also came in to play as she did not make her needs known to the group.

Although Sara asked, "Is everybody OK with those two measurements—so far?" no one responded. Clearly neither Jane nor Eve was OK when Sara asked, and yet neither of them asked for help. Because the four students worked on their own paper with no common workspace, the errors in Jane's and Eve's work went unobserved. Furthermore, Jane and Eve could not see Sara's and Abby's work to realize the errors they themselves were making. Because the work was not shared, the opportunity to deal with the errors made while finding the lengths of AB and CD was missed.

Earlier in the session Jane claimed to see at least three right triangles and now saw another.

Jane: Well, that's another...looks like another right triangle and we just have to break the other triangles....

Eve: Yeah.

Jane: ...but are they even the equal length...the hypotenuse [*sic*]? [Jane always mispronounced *hypotenuse*.]

Sara: Probably not.

Jane: ...if BD is 300, that doesn't mean it's the same length on each [inaudible].

Sara: Yeah, I don't think we can assume that...the picture...it's probably not drawn to scale but even if we look at the picture here, it doesn't look like it is half....

Having no common workspace was troublesome because no one could be certain as to what Jane and Sara were referring. Sara could only assume she knew to what Jane was referring which showed her lack of regard in *making sense* of another's thinking.

Sara admitted during the final interview that she did not want to be a leader and yet she did not want to leave anybody behind. I think these two positions were in conflict with each other. Sara did not want to leave anybody behind but her attempts at encouraging collaboration were too superficial to be effective. For example, after the above dialog between Jane and Sara, Sara asked after 30 seconds of reading, "Anybody have any ideas?" Neither Jane nor Eve acknowledged the question and Abby simply looked at Sara and giggled. Sara, apparently frustrated with the lack of progress and with the lack of participation, waited only briefly before responding to her own question.

Sara: I know that we have to break up the triangles somehow...probably...throw an x...an unknown length in there and then solve for the unknown length...then probably plug it back in to like two separate equations...and solve for an unknown side.....

Sara's suggestion of breaking up the triangles and using two equations may have been triggered by Jane's earlier statement indicating that Sara was lifted by Jane's comment. This situation where Jane lifted Sara is an example of what Cobb (1995) referred to as indirect collaboration. Sara's suggestion of breaking up the triangles and using two equations, however, was not acknowledged by the others. Although Jane may have lifted Sara, later I find out Sara's trivial comment of "throw an x" was a precursor to a major misconception that was a thread throughout these four sessions. Getting no response from the others, Sara slid her paper a short distance on the table and said, "...cause there's right triangles right here." The opportunity

for making sense of Sara's suggestion was lost, however, when Jane suggested finding the angle measures.

Jane: What would happen if we tried to solve for an angle, as stupid as this sounds? If we went and tried to find the angle...of D....

Sara: We can try it.

Jane: ...the triangle.....and then we could do.....cause we'd ultimately find the angle there....but....

Abby: Yeah...yeah ...cause if you found that...see if I am saying that correctly ...then you could do something to find that...some kind of way.

Jane: Right...but the problem is 240...

Abby: Yeah....

Jane: So...if we did...if we found this side... [pointing to the drawing on Abby's paper]

Sara: We could start with this angle right here...cause if...these two angles are the same size....

Jane: Cause of the congruency?

Sara: See what I am saying? And then if we know this angle, we know that angle.

The above episode has several factors that influenced the problem solving. First of all, the group did not discuss the problem, and Sara appeared frustrated with their unwillingness to offer suggestions. Then egocentrism was displayed when Jane ignored Sara and suggested finding the angle measures. The group did not consider the suggestion made by Sara but instead proceeded with Jane's suggestion. Sara, perhaps feeling an obligation to validate the suggestions of others (social collaboration), agreed to try Jane's suggestion. The communication was ineffective because no one tried to make sense of statements made by another. The participants were really talking past each other. Sara claimed two angles were the same size but she did not identify the two angles. No one questioned her which showed their disregard for making sense of her thinking. (Later in the session I asked Sara to clarify

which angles she meant.) Although Jane probably could not tell which angles Sara was talking about, she asked “Cause of the congruency?” This question is one of several illogical and irrelevant comments made by Jane during the course of the four sessions. I believe these comments were her attempts to indicate/signify/suggest her understanding. Jane may have believed these contributions added to the discussion when they really showed her lack of understanding of the problem solving. These contributions were evidence of not only Jane’s egocentric nature but also showed her procedural approach to mathematics. These illogical comments made by Jane were usually ignored.

All in all, the communication in the above exchange was not effective because it added nothing to Jane’s earlier suggestion of finding the angle measures. Furthermore, Eve listened to their exchange but was still dubious about how the angle measures would be useful. Her question showed she was thinking about the big picture.

Eve: Ok, now how is that going to help us in the long run?

Jane: Well, if we do the tangent or if we do the sine or cosine, we might possibly be able to find this side...the length of this side.

Abby: Yeah...if you find that and then...cause you know that’s like a right triangle....you know...Ok! [Abby frequently had trouble expressing her thoughts clearly.]

Jane: Does that make any sense?

Sara: Well, we might not know yet but we can try it...just to fill ‘em in.

Eve: It won’t hurt to try it.....

Eve doubted how the angle measures would be useful. Her doubt could have prompted the group to consider other possible approaches. Her lack of leadership skills was evident as she conceded to finding the measures without sufficient justification. At the beginning of the second session, I asked if they wondered why they found angle measures. Sara’s and Eve’s responses revealed how comforting implementing a procedure can be even though knowing how the information would fit into the big picture was not readily apparent.

Sara: ...cause it's the one thing I think we all knew how to do...for sure.

Eve: Cause we would know which way **not** [Eve emphasized] to go, if it didn't help.

The group needed a strong leader to suggest they step back and explore their options. Lacking this leader, they decided to proceed with finding the angle measures. Although they claimed it was a procedure they could do, they had trouble finding the angle measures. No communal work space and poor communication made it difficult to understand each student's comments. Eve struggled to understand that an angle's measurement is not dependent upon the lengths of the sides of the angle. She asked the question in the following episode but it was not until Session 2 that understanding occurred for her. Eve's tenacity in making sense of this idea not only lifted her, but it will also be shown her tenacity helped Sara consider a relationship she did not at first see.

Eve:now which angle are we going to find...angle D?

Jane: Should we start with angle D or angle B?

Eve: It doesn't matter...won't it be the same?

Sara: I think B would be easiest cause we know both of these lengths right here.....I mean....

[pause]

Eve: But how--which angle--I mean--for this right here or this whole AB uhm...

Jane: Yes, the whole ABD and this angle is going to be the same regardless if it's in this triangle or this little triangle--

[Jane pointed to her paper as she explained this to Eve.]

Eve: uh huh

[Eve did not look at what Jane was pointing to but looked at her own paper.]

Jane: cause it's at the same angle measurement.

Sara: Yeah...if we have both of these angles right here— [Eve glanced up as Sara began speaking.]

Jane: uh huh, and you know this is 90—[?] [Eve glanced toward Jane and shook her head as if she did not agree.]

Sara: can we figure out the length?--I don't know...can we do that? I don't think so.

Jane: I don't know.

Sara: so should we figure out these angles?

[No response.]

[pause]

Eve: So which angle are we going to figure out-----B?

Sara: Uh huh

Jane: [softly] Figure out B.

The lack of communication and failure to ask for elaboration and clarification is evident in this episode. First of all angle D is ambiguous because three angles have a vertex of D so the group really needed clarification about Jane's statement. Eve added to the confusion by claiming, "it doesn't matter...won't it be the same?" Did she mean the same amount of work, the same measure, the same approach, or something else? No one, however, asked her for clarification. Sara, I believe, suggested finding angle B because she knew the sides of triangle ABD; I believe she was initially looking at angle D as an angle of triangle RED and not of triangle ABD. (We agreed to label the point R where the perpendicular segment from E intersects AD.) Eve questioned whether the angle B they were finding was the one in triangle ABE or the one in triangle ABD (where E and D are collinear). Jane, pointed to her paper and explained to Eve, "...the whole ABD and this angle is going to be the same regardless if it's in this triangle or this little triangle." Jane's explanation apparently did not lift Eve because Eve asked a similar question in Session 2. In the final interview Eve explained that she believed the measure of angle B was dependent upon the triangle used; her question showed a conceptual misunderstanding of angle and angle measurement.

The next fifteen minutes were spent finding the measures of the angles in the right triangles. They faced many hurdles as they tried to find these angle measures. The difficulties they had were surprising to me because the calculations were procedures we had carried out in MATH 1113. Many mistakes were made with ineffective communication and the lack of a communal workspace contributing to their inability to identify these mistakes.

Jane suggested they use the tangent function to find the measure of angle B, but Sara was the only one of the four who actually expressed a correct relationship using the tangent function. Sara wrote

$$\tan(B) = \frac{O}{A} = \frac{240}{180} \text{ where } O \text{ and } A \text{ represent the opposite leg and adjacent leg, respectively.}$$

Jane wrote

$$\angle B \rightarrow \text{Tan} \frac{O}{A}$$

$$\text{Tan} \frac{240}{180}$$

and Abby wrote $\text{Tan} \frac{240}{180} = .$ Eve wrote $\tan 180^\circ$ and then later wrote $\tan x = \frac{180}{40}$. Jane

asked, "So it would be the tangent of 240 over 180?" Jane's question revealed a misconception about the role of the tangent function as she was interpreting the ratio of the opposite leg to the adjacent leg as the input of the function rather than the output. But Sara agreed with Jane by claiming, "That's what I have," although her equation was not at all as Jane stated. Sara was caught up in her own work and misunderstood Jane. Watson and Chick (2001) claim that egocentrism "is more passive, exemplified by a student being "wrapped up in his or her own little world" with little or no cognitive engagement" (p. 139). Sara was cognitively engaged but just not listening carefully to Jane. But not listening carefully, plus with Sara's tendency to validate the contributions of others (social collaboration), Sara verified Jane's statement was correct when it was not.

I believe Abby's work reflected what she heard Jane say rather than her own ideas about the relationship in the right triangle as Abby was unsure of her mathematics skills and frequently wrote what others said or would copy discreetly from their papers. Perhaps having a communal workspace would have allowed Sara to realize that what Jane wrote was incorrect. Had Sara listened more carefully to Jane, she may have also realized Jane's misconception. Finally, Abby could have initiated communication by questioning Sara and Jane about the tangent ratio rather than just writing down what she heard Jane say. Abby was so unsure of her mathematics ability that she willingly accepted the claims of the others rather than making sense of what they said. At this time Eve had also written an incorrect statement but no one was aware of this as she did not share her work with the others. The lack of communal workspace, ineffective communication, egocentrism, social collaboration, and sense-making factors influenced the work they did to find the angle measures. The following description shows their struggle.

Because Sara validated Jane's incorrect statement, Jane now believed $\tan \frac{240}{180}$ would give her the measure of angle B. She asked the group if the calculator should be set to degree mode and got an affirmative answer. But when she entered $\tan \frac{240}{180}$ in the calculator and obtained .023, an answer she did not expect, she assumed the mode was the problem and announced, "I think it's radian actually. Now that I think about it."

When Jane's answer was not what she expected, she assumed the mode was the critical issue rather than questioning her logic and reasoning. Jane was very procedural and when a problem did not work out as she expected, she looked for a "quick fix to a procedure" rather than acknowledging the existence of a conceptual misunderstanding. Sara, still not willing to be a leader and perhaps fearing she would be perceived as taking over, solicited support from Abby and Eve by asking, "What do you guys think [about the mode]?" Clearly

Sara did not agree with Jane, but she did not express her disagreement. No one tried to help Jane make sense of the mode so Jane justified her choice of radian mode by saying, “Cause if you do it in radian mode you get point zero two three [i.e., .023] but if you go and put it in degree mode, you get four point one three [i.e., 4.13].” She obtained these values by calculating

$\tan \frac{240}{180}$; this value is .023 in degree mode and 4.13 in radian mode. However, Jane reversed

the modes and incorrectly reported the value as .023 in radian mode and 4.13 in degree mode.

I did not understand why these values led Jane to conclude the mode should be radians but she, nevertheless, attributed her wrong answer to a procedural deficiency rather than a conceptual deficiency.

Sara, Abby, and Eve stared at Jane as she offered this explanation, indicating confusion about what she was saying. Sara has obtained neither of the values Jane mentioned but did not ask Jane for justification or further explanation. Abby, characteristically said nothing, but Eve asked, “And that’ll be the angle?” to which Jane answered “Uh huh.” There was a five second pause before Sara admitted, “Alright, I’m lost.” Jane perhaps sensed their doubt and tried to save face by saying “I don’t even know if that’s possible. “ Rather than questioning Jane, Sara accepted responsibility for not understanding her by saying she was lost. Here was an opportunity for communication that could have aided the problem solving; however, Jane, sensing their doubt and needing to save face, merely backed down thus eliminating the opportunity for lifting. Ignoring Jane’s reported values of .023 and 4.13, Sara returned to her dilemma of finding the angle measure when the value of the tangent is known. She has found the tangent of angle B to be 1.33 but has forgotten how to determine the measure of the angle from that information. Furthermore, she trivializes the issue of the mode.

Sara: As far as whether...regardless if it’s in radian or degree mode, once you get 240 over 180—

Jane: uh huh

Sara: I forgot how to...what's...how do you figure out the actual angle?

Sara's question caused Jane to revise her statement of $\tan \frac{240}{180}$ to $\tan x = \frac{240}{180}$, but she

did not share her revision with the others. Again, no common workspace precluded the need for explanation. As typical, the variable x was used without denoting its representation. To save face, Jane confidently suggested a procedure.

Jane: Oh, hah!...just kidding...cause you have to cross multiply!"

Sara: Ok. [social collaboration]

Jane: That's why it didn't work—[pause]--cause you do $\tan x$ --it'd be 180 tangent x equals 240...that's why it didn't work.

Jane wrote on her paper $\angle B \rightarrow$
 $\rightarrow 180 \tan x = 240$, and Sara similarly wrote

$\tan(B)180 = 240$. Neither shared her writing with the others. Subsequently Sara announced, "I got zero." Her work did not reveal how the value of zero was obtained although I assumed she entered into her calculator $\tan 180$ (in degree mode) to get a value of zero. Unfortunately, neither I nor the others asked Sara for clarification. Jane was still punching numbers into her calculator and mumbled an incoherent statement containing the phrase "degree mode." Abby looked at Sara and gave a nervous but supportive giggle, also a form of social collaboration.

Eve first wrote $\tan 180^\circ$ and then hearing and misunderstanding the comments of the others wrote $\tan x = \frac{180^\circ}{40}$ and $40 \tan x = 180^\circ$. When Sara announced the result of zero, Eve put on the brakes by saying, "Wait, wait, wait...uhm..." Eve realized the work was not making sense to her. As if trying to gather her thoughts, Eve waited fifteen seconds and then turned toward Sara for clarification. The following episode contains several factors that influenced their progress.

Eve: We are trying to find angle B...so we use $\tan x$ equal to 180 over 40...right? That's right?

Sara: I have...opposite being 240 and adjacent being 180.

Eve: So it's 240...

Sara: over 180

[Jane, who must have been listening more than I realized, leaned over, looked at Eve's paper, and pointed to Eve's incorrect ratio of 180 to 240.]

Eve: [noting her error] Oh, ok--240 over 180.

Sara: Right.

Jane: Yeah.

Sara: But then I get confused about how to solve the rest.

In the episode Eve tried to make sense of her work and asked for help. Sara merely stated what she had written rather than helping Eve understand why her statement was wrong; this again showed Sara's reluctance to lead. It also showed Sara's egocentrism; Sara would get caught up in her own work so that she did not listen carefully to the questions and comments of the others. As a result, she validated incorrect statements, and because she was viewed as the leader, her validation was detrimental. For example, hearing only "240," Eve inserted the digit 2 to make the denominator 240. It was Jane who pointed to Eve's paper and showed her she should have 240 over 180; Eve then revised her work to $\tan x = \frac{240^\circ}{180^\circ}$. It was unclear, however, why she used degrees as the units. She also used the variable x without denoting its representation.

Jane's suggestion of cross-multiplication was not helpful so Sara asked, "Is it possible to find this angle or not?" Eve pointed to the MATH 1113 book on the table and Jane, passing it to her, said, "I think it's chapter 6. [unclear]...trig." They did not try to make sense of the conclusions they had reached nor did they pull their resources together to see if anyone had ideas to suggest. Eve looked through the book, but this was a solitary endeavor other than Jane's suggestion to look in chapter 6.

The group had been using the tangent function to find angle B. In the following episode, Jane and Sara decide to find the measure of angle A using the sine function. Although Jane assumed Sara meant angle A, at no time has Sara talked about finding the measure of angle A. Again, I believe Sara responded affirmatively without listening to Jane's question.

Sara: I wonder if finding this angle will help any.

Jane: This A?

Sara: Uh huh.

Jane: I was just thinking that....It might because you have a ninety degree angle and if you find that you'll know what [unintelligible].

Sara: Yeah ...try to find that-----Ok—that's opposite and adjacent also. [On Sara's paper she has written $\tan(A) = \frac{100}{240}$ without realizing she is in the same dilemma as when she had

$$\tan(B) = \frac{240}{180}]$$

When Sara expressed doubt about finding the angle measure, Jane's first reaction was, "I was just thinking that..." This comment allowed her to save face since she was the one who had suggested finding the angle measures. Nevertheless, Jane offered a weak justification as to why the angle measure would be useful and Sara accepted it. They have, however, clearly forgotten how to solve for an angle. Abby and Eve were of no help as Abby kept glancing at Sara, and Eve ignored the others while looking through the book. Jane eventually suggested they forego using the tangent function.

Jane: For this one you could even--you don't even have to just do tangent—you could do it another way, too...cause you have all three sides.

Sara: Oh, that's right. Yeah, let's try something other than tangent...I like that...like sine, that's easy.

Jane: Opposite over hypotenuse [*sic*].

Jane and Sara have been unsuccessful finding angle B with the tangent function and decided finding angle A with the sine function will be easier. They did not discuss why they believed this approach would be easier so there was no opportunity for anyone to justify. They did not yet realize the same difficulties exist. Abby watched Sara work but would not lean over into Sara's space nor would she ask Sara to explain what she was doing. Sara incorrectly wrote

$\sin(A) = \frac{100}{300}$; although Abby looked over Sara's shoulder, she did not pick up on the error.

Later in the session, Jane realized Sara's mistake as Sara read aloud from her paper.

Jane asked Eve if she found what she was looking for in the book and even though Eve said, "Uh huh," she did not show or tell the group what she had found and no one asked. Instead she set the book in front of Jane who began searching the book, an endeavor that lasted throughout the rest of the sessions. Because Eve was looking through the book, she was only partially aware of Jane's and Sara's conversation.

Eve: And we are doing sine of 240?

Jane: We are finding A, so we are using---

Eve: [interrupting and sounding frustrated] Oh, so we are not trying to find B?

Sara: I think we all got—I got confused on that.

Jane: Got confused on that—we are going to see if we can do A.

Eve: Do A?

Eve apparently overheard the conversation about using the sine function because she did mention "sine of 240." However, she was unaware Sara and Jane were working on angle A. When Eve was looking through the book, I assumed she was looking for a way to find the angle given a trigonometric value of the angle; in other words, I expected her search to reveal a way to use the inverse trigonometric function. However, as I examined the tapes and noted what occurred later I now believe she had forgotten how to express a trigonometric ratio for an acute angle of a right triangle and was looking for this information. As evidence, she had trouble

expressing the tangent ratio of angle B, but after looking through the book she easily wrote $\sin A = \frac{100}{260}$. Perhaps she was reluctant to admit she had forgotten how to set up a trigonometric ratio which may explain why she did not share with the others her finding from the book and merely responded, “Uh huh,” when Sara asked if she found what she was looking for. Furthermore, when Jane later mentioned using the inverse function to find the angle measure, Eve reacted as if she had not considered this. Abby also wrote the ratio $\frac{100}{260}$ but did not indicate its representation leaving me to assume she got this ratio from one of the others.

The group was still at the same point they were 6 minutes ago with trying to determine the angle measure given a trigonometric value of the angle. Sara again says, “But once we get that, how—I mean—I think we will kinda have problems trying to figure out the actual angle.” Looking through the book, Jane responded, “That’s what I’m looking at now.” Jane had the book in the center of the table and leaned into the center of the table. This is the first time any common workspace has existed, and it really was productive. It took Jane one minute of searching to read aloud from the book, “In some of the problems we need to find the angle in a right triangle whose sides are given. In this example—to find the angle whose sine is not given in a table, we use the inverse sine or arcsine key on the calculator.” Jane realized the significance of this statement and looked at Sara as she asked, “Could that be our problem? Not using the inverse?”

Eve’s attention has been piqued; she looked at Jane and asked, “The inverse?” and then began silently reading the passage from the book. Jane, Eve, and Abby are all focused on this common workspace. Sara, who had been questioning how to find the angle, ignored Jane and continued working independently,; the others were leaning into this common workspace, but Sara was sitting very much apart from them. Sara had withdrawn cognitively and obviously heard none of what Jane read because she mumbled more to herself, “Is this possible, finding

the angles? I know it is.” Three minutes after Jane read the passage aloud, Sara looked up at Jane who slid the book toward Sara and said, “I think it’s supposed to be the inverse.” After reading, Sara said, “Ok, I see...that seems right...using the inverse function.” Jane shared with Sara only after Sara indicated some interest; furthermore, Sara was interested in what Jane was doing only after being unable to make sense of the situation herself. Sara had been looking for the very information Jane gave her and yet she was not tuned in to what the group was doing to realize the information was available. This was another instance of ineffective communication that hindered the problem solving—Jane had a good suggestion but failed to communicate it well, and Sara was so caught up in her own work that she was not listening. A help to the problem solving was having the one book that was shared with all the members of the group. Jane’s putting the book in the center of the table and her leaning into the middle of the table was beneficial in creating a common workspace that helped focus the attention of the group members on the common goal of finding the angle measure.

Each student was able to find an angle measure. Eve and Jane both reported values of 22.62 for the measure of angle CAD. Abby then shared, “I went back and did the other one and I got the same thing you did—the 53.13° .” Because I was not sure Abby fully understood how to find the angle measurement, I quizzed her about how she found 53.13. Her response is so indicative of one who is unsure of herself and of one who is accustomed to getting results from others. She was unable to offer a mathematical explanation of her thinking, and I don’t think she was bothered by this (i.e., she was not embarrassed to admit she did not fully understand how she got the answer); instead she acknowledged she obtained information from others.

Abby: Well, she [pointing to Jane] did it, but then looking in here, too—the first time we did it when we were trying to do the tangent or whatever it wadn’t coming out right, but if you do the inverse tan, tangent or whatever, and put it in there you get 53.13—which makes more sense than what we were gettin’ the first time.

Abby looked at Jane's paper but wrote nothing about the inverse function on her own paper. I believe she saw what Jane wrote (see Figure 6) and then entered $\tan^{-1} \frac{240}{180}$

LA $\sin A = \frac{100}{260}$
 22.62°

Figure 6. Jane's solution of angle A's measure.

into her calculator (she had previously written $\tan x = \frac{240}{180}$ with no mention of the representation of x). Thus, Abby used what Jane wrote to help her calculate the value 53.13. After Abby offered this explanation, I looked at Jane and asked if she had obtained the same result as Abby. When Jane said, "Uh huh," Abby continued, "You did it first and then I was just..." Jane then interrupted Abby with a most illogical statement.

Jane: Cause tangent is undefined at pi normally because one over zero is undefined and that makes sense because you already knew the lengths but you were trying to find the angles so you had to do the inverse to find the angle—I just remembered that.

Jane often gave illogical explanations as she tried to make sense of the mathematics. Earlier in the session, Jane gave a similar explanation:

Jane: Cause tangent and 180 at pi is 0...because on the unit circle it's one zero [i.e., the ordered pair (1,0)...and if you do the tangent, it's undefined."

Neither of Jane's illogical explanations were challenged by the others in the group. In fact, the explanations were completely ignored. During the final interview I asked Jane to watch the clip where she talked about the tangent being undefined at pi but as will be shown she was reluctant to make sense of her comments.

They found all the angle measures in the right triangles but did not discuss how they would use the information. In the next episode it was apparent Sara and Jane were trying to find lengths of segments, but their comments were not collaborative. In fact, their explanations were unclear and worsened by the lack of a communal workspace. It was during this episode that I became aware of the students' misconceptions of variable.

Sara: So now we can take the opposite and—oh, wait a minute—now we have to find the hypotenuse, though...

Jane: Here's the next problem—it's because the 240 cannot be the whole length.

Sara: Yeah...but we know...but we know 240 minus x ...

Jane: Would give us that side.

After watching the tapes repeatedly and carefully examining their work, I can only guess at Sara's references to opposite and hypotenuse. More than likely, no one in the group understood her references. Furthermore, I am not sure what Jane meant by "240 cannot be the whole length" although she dragged her finger over the drawing as if referring to segment AD. In all probability, she had written 240 adjacent to segment AD earlier in the session. When Sara said, "but we know 240 minus x ," Jane attached " $-x$ " to the 240 and announced that "would give us that side" as she labeled segment RD as $240 - x$. Later, however, Jane labeled ER as x while continuing to have $240 - x$ written adjacent to RD, evidencing her misconception about variable representation.

Although Sara and Jane were both talking, they were talking past each other. Sara was really thinking aloud and paid little attention to what Jane said. Furthermore, Sara did not help Jane and the others understand her perspective. Confirming her procedural approach to mathematics, Jane changed "240" to " $240 - x$ " without any justification other than hearing Sara's comment. Sara then expressed the ratio for the tangent of angle RED with a measure of 53.13° and evidence of the trouble with variable representation is apparent.

Sara: We can take the tangent of 53.13 degrees and we'd have to use opposite over adjacent—opposite is going to be the long leg which is 240 minus ... some length x ...and then adjacent would be... 300 minus x —cause there's extra stuff up here—does that seem right?

Jane: Yeah, but that brings in too many variables.

Sara: I'm just trying to...get ideas...of you know...

I did not understand what Jane meant by “that brings in too many variables.” Examining Jane's paper, I can see that when she drew the triangle ED[R] she first wrote E adjacent to the segment from point E to the road, perhaps indicating E as the length of this segment. Later she marked through E and wrote x next to the segment. Because she had E and $240-x$ as two of the sides of the triangle, this may explain her comment about too many variables. Unfortunately I can only guess at her intended meaning as no one asked Jane to explain. Sara often asked the others for suggestions and yet when Jane claimed “that brings in too many variables,” Sara became defensive and did not ask Jane to elaborate on what she meant. Had Sara been more open to other's suggestions and encouraged communication, the group may have been more successful in solving the problem.

Jane's observation of “too many variables” was not explored, and Jane and Sara both wrote $\tan 53.13 = \frac{240-x}{300-x}$. I believe Sara was looking at RD as the opposite side and calling it $240-x$; however, I believe she used ED as the adjacent side and represented this length as $300-x$. Thus, two significant errors were made: the first was in using the wrong sides of the right triangle to express the tangent ratio, and the second was in using variable representation incorrectly. Jane, however, accepted Sara's suggestions.

Our time was up; I asked each of them to write themselves a reminder of “anything in your mind that you think perhaps you may need to remember.” Although Abby had not participated in the conversation with Sara and Jane, she nevertheless wrote the same illogical statement Sara and Jane wrote (see Figure 7).

End Here
 $\tan 53.15 = \frac{240-x}{300-x}$

Figure 7._ Abby's reminder at the end of Session 1.

As was so often the case, Abby copied work from the others. Sara's reminder (see Figure 8) indicated a more qualitative approach for proceeding.

I know were going to have to set two equations together and solve for a variable

Figure 8._ Sara's reminder at the end of Session 1.

Although Sara mentioned this approach prior to Jane's suggestion of finding the angle measures, the group did not explore the approach during this first session.

Session 2

No one came to the second session with the problem solved. Jane, as another example of saving face, told me in the final interview that she chose not to work on the problem outside of the sessions "cause I didn't think it really would have been that fair to the group if I had figured it out." Abby said that she talked to her colleagues at work about the problem and as a result was able to make several contributions during Session 2. Eve took a more active role in

this second session. She appeared uncomfortable during the Session 1 and really was quite aloof. I believe part of Eve's aloofness during Session 1 resulted as she tried to make sense of the problem but could not do so in the group environment. Accustomed to working alone, Eve could have benefited from some quiet time. The time between the first and second sessions afforded her some time to make sense of the problem so that she could take a more active role during the second session.

In the Session 1 Jane claimed to remember a similar problem from MATH 1113 so at the beginning of the second session Jane started her search of this similar problem. Sara asked, "What are you looking for, Jane?"

Jane: In chapter 6—I remember we worked the problem where you had—you didn't know the length—you had like AD, you knew it was the whole length--was 240--but you had to find out x which happened to be in triangle uhm [pause] ED--and there is a way of working it with the variable—somehow or 'nother. [Point R was not labeled yet so Jane named the triangle with only the two labeled vertices.]

I believe Jane thought the Buried Treasure Problem was similar to a problem from the previous semester merely because the group talked about using a variable x in Session 1. She believed that if she found this problem, she would find a procedure that could be applied to the Buried Treasure Problem. Consequently, for most of this session she was not engaged in what the others were doing but instead conducted an independent search for this problem. Jane's prior experiences in mathematics contributed to her belief a template problem existed to help them solve the Buried Treasure Problem.

At the end of the Session 1, I observed misconceptions about the concept of variable representation. It was early in this Session 2 that their misconceptions about this concept began to be problematic.

Eve: Ok, we are using triangle ED, right?

Jane or Sara: Uh huh.

Eve: Cause I thought it was like 240 minus x for the bottom half and then we put x which one are we trying to find--E to the road? So wouldn't that like x , I believe--I'm not sure—in the problem, mark that as the unknown since we are looking for that—so we use the variable?

Sara: We use the variable x for which side?

Eve: For E to the road, was x ?

Sara: [ignoring Eve's question and looking at Abby] We don't know the length from A to C, either, do we?

Eve: From A to where?

Sara: From A to C.

Eve and Abby: Yes, 260.

The above dialogue shows Eve was more involved in the group. Her involvement was probably a result of a better understanding of the problem. The dialogue also shows the group struggled with representing relationships symbolically. Even though Eve labeled the distance from E to the road as x , she observed one of the pieces of segment AD was $240 - x$. Sara, thinking the length of AC was unknown, suggested x could also represent that length. I decided it was important for me to help them deal with this misconception so my role in this session was also more involved. My first question to deal with their misconception was, "I'm hearing x and I'm hearing 240 minus x —what is x ?" Interpreting my question as the value of x rather than the meaning of x , Eve and Sara respectively replied, " x equals....." and "Is....." Their incomplete replies showed they were thinking about a value. Abby, however, showed a better understanding of variable and pointed to her paper as she answered.

Abby: It would be that...wouldn't it—that part?

Eve: Yeah, that part.

Abby: Cause the whole thing is 240 but we just want to know that part so like we want to know that part and that part so we can figure out that part.

A factor that particularly influenced Abby's participation in the problem solving was getting verification of an idea. She was very timid about making a suggestion unless she was certain it was correct. In the above exchange, we see that Abby continued her explanation once she got verification from Eve. Furthermore, Abby's participation in this session was a result of the verification she got from her colleagues at work.

Eve could see Abby's paper as she pointed to "that part," but Sara did not look at where Abby was pointing and apparently was not listening.

Sara: So do we want to assign like this left hand side like $240 - x$ and this over here x ?..... I still don't know how that will let us find the smaller side.

Although Sara related x to a physical entity in the above comment, the idea of the unique representation of variable was not yet developed for her as evidenced by her later suggestions. Although no one challenged or supported Sara's suggestion, the expressions on the faces of Abby and Eve indicated some doubt about her suggestion. The following exchange shows Sara continued to misinterpret my questions concerning the representation of x ; she assumed I was asking about the value of x .

LBC: When you say, " x ," what is x ? Is it important to know what x is? "

Sara: Eventually.....I think right now x is just—we have to—we have to throw an x in there so that we can start dividing this up, I think into 2 separate sets of equations [referring to her suggestion at the end of Session 1] because with the information we have now, I don't think we can find any of the sides we are looking for unless we break it down into separate equations, solve for x , and then—[pause]

The above comments illustrate how poor communication made the problem solving difficult since they could not be sure they understood each other. In the above comments it was difficult to know which parts of the drawing the students referred to when they said "this part," "the left hand side," "this over here," and "the smaller side." Because there was no common workspace, the students could at best only assume they understood the imprecise references.

More troubling to me was my realization that they did not appreciate the need for a taken-as-shared basis for communication.

It was also disconcerting for me to realize how poorly I communicated my questions about the variable. Sara thought I was asking about “the value of x ” rather than “the representation of x .” Realizing this ambiguity was a learning opportunity for me and reminded me that I inadvertently contribute to misconceptions.

Sara again referred to using two equations, a suggestion she made in the first session. Her flippant remark of “throw an x in there” shows her misunderstanding of the role of a variable. I asked her what she meant by “throw and x in there.”

Sara: Uhm...like on the bottom side, like the road side, maybe separate the left—there’s two triangles, two right hand [sic] that this length that we are looking for divides—this part into and maybe give this like—I don’t know [sounding unsure]

Eve: [jumping in to help Sara] I think we want to put the x in there so we can like.....begin to solve somehow.

The comments above show that for Sara and Eve, x was an unknown value whose purpose was limited to symbolic manipulation so that an unknown value could be found. The idea of using variables and related algebraic expressions to represent physical entities and to state relationships among these entities was not understood. This lack of understanding can be seen in the following description.

The conversation about the variable x led Eve to write x adjacent to segment ER . (Earlier I suggested they name the point R where the segment from E intersected \overline{AD} .) Jane, however, had 240 minus x written adjacent to segment AD . These decisions were made with no communication as a group so I decided to share with the group how Eve and Jane represented segments \overline{ER} and \overline{AD} , respectively. Essentially my comment was ignored and certainly no one pointed out the inconsistency. The only effect of my statement was for Eve to

attach “- x ” to 240 on her drawing to mimic Jane’s representation. It was not incongruous for Eve to write $240 - x$ adjacent to segment AD while x was written next to segment ER so I believed she did not understand the concept of variable representation. I asked her why she made the change, but her explanation did not reveal any recognition of the incongruity.

Eve: Because you know the whole side is—this whole length—is 240 so—well, no, really it shouldn’t even be like that because we really need to find both sides in order to get this side. [Eve pointed to AR and RD as she explained and apparently claimed these lengths would be needed in order to get ER.]

Sara then decided to simplify the work by pointing out that triangles AER and RED were “basically the only ones they needed to deal with anymore.” Her suggestions to deal with these triangles reveal her misconceptions related to variable representation.

Sara: Should we maybe deal with the hypotenuse as a length—minus x —like 300 minus x over here and then 260 minus—well, I don’t know if that would help either.

[A pause of 18 seconds as they continued looking at their own work.]

Sara: Well, maybe if we took these lengths right here [Sara pointed to Eve’s paper.]—this hypotenuse and that hypotenuse—and assign them a length, one 260 minus x —we could then find the length of each of the bottom sides in terms of x —with the known angles—

Sara reached across the table and pointed to Eve’s paper as she spoke, but this action did not create a common workspace. At best the only outcome of Sara’s reaching into Eve’s space was for Eve to ask Sara to repeat her statement. Although she observed the lengths of segments AE and DE were parts of the lengths of segments AC and BD, her suggestion to represent segments AE and DE as $260 - x$ and $300 - x$, respectively, showed she did not recognize the significance of using the same variable x . The other students paid little attention to Sara’s comment. The lack of a common workspace was compounded by their inattentiveness so there was little hope they would address the way Sara represented the segments. Nevertheless Eve did ask Sara to repeat her statement, but no request for

elaboration occurred.

Eve: Say it again.

Sara: These two triangles that you have drawn right here—the length on AER-- we could assign the length up here—we know this is 260 going across here minus some length—so we could do like 260 minus x —give that name to this side [that is, side AE]—and 300 minus x over here [that is, side ED]—and then find the length of this bottom piece for each of the triangles in terms of—I mean we might have to have a variable in there but—and then we could set these two together—[pause]—I don't know if that would help us though solve for x .

As Sara explained, she turned her paper so Eve and Abby could see and Eve responded with several, “uh huhs.” Sara's representation of the segments was illogical but no one corrected her or tried to make sense of her suggestions. There was no communication—no request for justification, no request for elaboration, no request for further explanations. Getting no response from the others not only left Sara with misconceptions of her own but also had to be very frustrating for her. Still wanting to get feedback from the others, Sara asked, “Does anyone think it is worth trying or should we try something else?”

Jane: Anything will work—hopefully something's gonna work.

Eve: We should try it...might come up with a brilliant idea.

Jane's and Eve's responses, forms of social collaboration, really gave Sara permission to continue with her illogical assumptions. I decided to intervene to help them make sense of the concept of variable. This decision to intervene probably had significant influence on the problem solving. In retrospect I perhaps should have dealt with their misconceptions more directly or should have let them figure out their misconceptions as they encountered problems with the work. They needed the opportunity to face the consequences of these illogical assumptions. Instead, I think they were frustrated by my questions because they did not yet see their relevance. From the very beginning my intervention was plagued with poor choices.

My initial question was not mathematically sophisticated so I got an unsophisticated response from Eve.

LBC: I'm still a little confused about this x . If you do this 260 minus x and 300 minus x , why did you use x —why not y ?

Eve: [giggling] First variable that comes to your mind.

The following exchange shows my attempts to help them make sense of the concept of a variable. I believed progress was being made—at least for Sara and Eve. Neither Jane nor Abby engaged in the conversation so any progress in their understanding was not evident.

LBC: Ok—so what is a variable?

Sara: An unknown...measurement of some sort that would—I mean right here we don't necessarily know that this 260 is divided directly in the center so I can't say that this is you know—

Eve: [interrupting] half

Sara: [continuing] this is half of 260, I don't know that for sure—but I know it is 260 minus some length that I'm not exactly sure about.

LBC: So what is that length? In that picture, where is that length?

Eve: E to C, is that it? [said this tentatively]

Sara: I guess this would be x .

Sara wrote x adjacent to the segment EC on her paper. I then asked about the length of AE. After a brief pause Sara, with a slight rise of her shoulder as if not sure, replied, "260 minus x ." Eve said, "Yeah," in agreement.

I wanted them to realize the inconsistency with having ED labeled $300 - x$ while having AE labeled $260 - x$.

LBC: Ok—so now what about that 300 minus x ?

Sara: I guess that could be y —it's not necessarily the same.

Sara misinterpreted my question. I was hoping for an “ah hah” moment whereby she realized $300 - x$ conflicted with $260 - x$. Instead she said “that could be y .” The nebulous “that” in her statement created confusion. Sara merely changed the $300 - x$ written next to segment ED to $300 - y$ as shown in Figure 9 whereas Eve assumed Sara let segment ED be y .

Eve: So E to D would be equal to y ?

Sara: Uh huh.

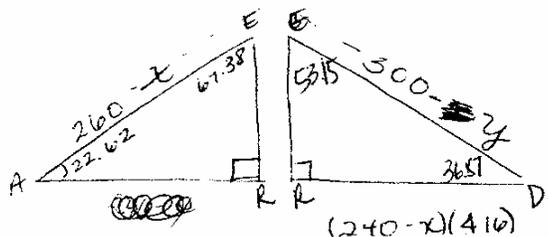


Figure 9. Sara's drawing showing how she changed $300 - x$ to $300 - y$.

Here is another example of the lack of a common workspace contributing to inaccurate assumptions. It is also an example of ineffective communication resulting from misunderstandings. I also contributed to the ineffective communication. When I asked about $300 - x$ and Sara claimed that could be y , she only changed the x to y on her drawing; this changed the representation of ED to $300 - y$. (See figure 8.) Eve, however, heard Sara say ED would be y , and Sara verified this. Eve initially wrote $ED = y$, but later changed it to $300 - y$. Standing between Sara and Abby, I did not see that Eve wrote $ED = y$ on her paper. A common workspace, effective communication, and a sense-making perspective could perhaps have eliminated this confusion. Instead, Sara and Eve both made changes but with little justification of explanation offered. When I asked them about the changes they made, only Sara responded.

LBC: What do ya'll think?

Sara: I think that makes more sense now because our x —they'd be equal to each other and we don't necessarily know that they are equal to each other.

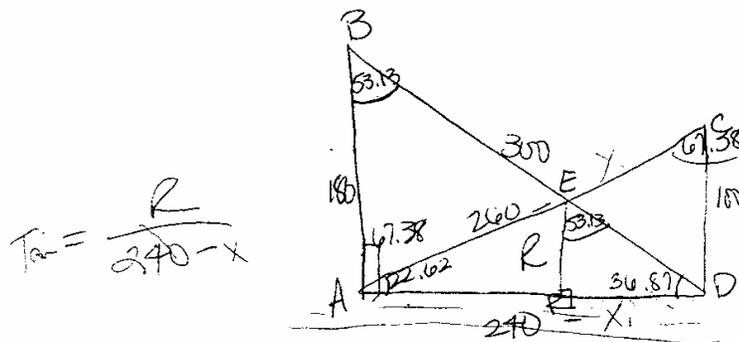
LBC: What do you mean? What do you mean they'd be equal to each other?

Sara: Well, if we're—if x is in a problem then x has one and only one—or should have one and only one true value and we don't know by looking at the picture and the information given that

the length $300 - x$ and $260 - x$, we don't know for a fact that both x 's are equal to each other.

Sara's explanation led me to believe she had a better understanding of the concept of variable representation. In the final interview, however, I discovered Sara's understanding of variable representation was still flawed so the lifting I thought occurred for her in this session was actually incomplete. Although Eve contributed to the above conversation and made changes to her work, I was not certain of her understanding of the variable concept. Abby, the constant pupil, just accepted what she saw and heard so her understanding was questionable. Jane paid no attention at all to the discussion about the variable, and her later suggestions show the discussion had no impact on her understanding of this concept.

I was optimistic that progress could be made because Sara, and possibly Eve, appeared to have a better understanding of variable representation. At this time I believed they were representing the unknowns in the problem correctly (although during analysis I realized there were inconsistencies). Exactly how they would have used these representations is unknown. The group did not have the big picture in mind; furthermore, Jane frequently separated herself from the group and worked alone. It was now that Jane decided to share her work with the group. Any progress the group could have made from the assigned representations was put on hold as the group dealt with Jane's suggestions. As indicated earlier, Jane was constantly in search of "the answer" so when she found a value for x , she was eager to share her work (in Figure 10). Jane's tendency to implement procedures without mathematical justification was very evident in the episode that follows. She committed vandalism with the mathematics in her pursuit of the value of x .



$$\tan 22.62 = \frac{100}{240 - x}$$

$$\tan 22.62 (240 - x) = 100$$

$$240 \tan 22.62 - x \tan 22.62 = 100$$

$$174.57 - .72736x = 100$$

$$74.5657 = .72736x$$

$$102.52 = x$$

$$240 - 102.52 = 137.48$$

Figure 10. An example of Jane vandalizing the mathematics.

Jane: I have something that I tried—might work, I'm not sure. I took the triangle ACD and you know all the angles and sides—and you know that the side CD is 100 and if we do that 260 minus x as AC and AD as 240 minus x—Right?

Jane contradicted herself by first saying they knew the sides of triangle ACD and then representing AC and AD as variable expressions when the lengths of AC and AD are known. As Jane explained, Sara acknowledged what Jane said by periodically saying “uh huh” which may be a form of social collaboration. I asked Jane to repeat what she had said.

Jane: AC is 260 minus x; AD is 240 minus x. And you know that CD is already 100, right? You know the angles and side. If you know the angles and side, using the angle 22.62 and do the

tangent of it, you can do the opposite over the adjacent which is tangent of 22.62 equals 100 over 240 minus x —and then solve for x . [pause] Would that work?

As Jane explained, she kept looking at Sara probably because she saw Sara as the leader. When Sara did not respond, Jane handed her paper to Sara and saving face said, “I know it’s hard to think about it.” Sara did not question Jane’s logic but instead wondered if Jane was able to get an answer.

Sara: Did you work it out yet? Is this it?

Jane: Yeah, this is it.

Jane still wanted to save face so she admitted she was not completely sure about the approach. She continued to commit vandalism with the mathematics.

Jane: It’s just a shot. Cause then you would know this measurement and then you could take this and subtract it from 240 and hopefully that would be the side of AD —and to back it up we could do the Pythagorean Theorem and see if it checks—in some weird way.

Jane committed vandalism by claiming they would subtract to find the length of AD although the length of AD was known to be 240. The following exchange shows Sara doubted Jane’s reasoning but did not engage her in cognitive disagreement. Instead, Sara tried to avoid a confrontation which showed her reluctance to be a leader. She also doubted her own understanding of variable so that she could avoid a confrontation with Jane. Jane also tried to save face. My subtle attempt to help Jane realize the length of \overline{AD} was known was wasted as Sara answered my question and Jane was unaffected. The following dialogue illustrates these dynamics.

Sara: [pointing to Jane’s paper]...this x , is what?

Jane: What we have to subtract from 240 to get the length of AD —or to get RD by itself.

LBC: How long is AD ?

Sara: 240

Jane: RD would be 137.48—cause this is the whole length, right?

Sara: uh huh

Jane: And we want to find out what x is—and x would be RD. [But this contradicts her previous statement that RD is $240-x$ or 137.48.] So if we take this x which is what we are gonna—what we isolated from the 240, subtract this from 240, you get 137.48 which would be side RD. I don't know if it works yet.

[Jane's expression of doubt was a way to save face.]

Sara: I think—I mean I think it makes—

Jane: [interrupting] I can't explain it.

Sara: [continuing] --some sense—

Jane: I can't explain it—

Sara: But—like what I'm having trouble seeing is—we were just talking about how we—I don't know if this is true or not because it's one triangle that we are working on, but if x here and x here are necessarily going to be the same?

Jane: Oh, I don't know about that. I didn't think about that.

Jane's explanation was full of errors. Sara chose, however, to focus on Jane's use of x and $240 - x$ as segments or parts of segments in triangle ACD. Sara pointed out the x 's on Jane's paper—Jane has $260 - x$ next to side AC and had $240 - x$ next to side AD. Rather than helping Jane understand her faulty thinking, Sara expressed doubt about her own use of the variable, claiming she was unsure whether Jane's tactic could be done since she was using one triangle only. Sara's doubt was either genuine indicating her concept of variable was still undeveloped or was an example of social collaboration that helped avoid a confrontation with Jane. Either way, Sara's reluctance to engage in cognitive disagreement did not help Jane confront her misunderstanding. Nevertheless, Jane heard Sara's question about whether the x 's were different values, and Jane suggested making one x_1 and the other x_2 ; she then

changed $240 - x$ to $240 - x_1$ on the drawing adjacent to AD. (See Figure 9 above.) Although the suggestion was appropriate, I was unsure whether Jane fully understood why the differentiation was necessary.

Jane chose not to pursue this approach any further. Perhaps Sara's doubt caused her to abandon the approach in order to save face. Sara then questioned, "What if we do the same thing with the hypotenuse—that would be the same thing, though, right? [*pause*] Or what if we just broke it down to a smaller triangle?" I did not understand what Sara was suggesting. Again, no one questioned her nor did she try to engage the others in a discussion. However, after a twenty second pause, Sara asked me, "Are we on the right track?" At this point in the investigation, they did not have a defined plan so when she asked if they were on the right track I suggested, "Read the statement that you wrote last week" (that is, the reminder Sara wrote to herself at the end of session one).

Sara: [*reading*] I know we are going to have to set two equations together and solve for a variable.

Ultimately the approach Sara suggested above was the one they used. Getting those two equations, however, took some real twists and turns. Eve asked what Sara first thought was such an obvious question but one that ultimately led Sara to the system of equations. Abby also made good observations about relationships among the sides and the angles of the triangles. Jane made cryptic remarks throughout the remainder of this session that showed her understanding of the mathematics was incomplete and flawed. Although Jane's cryptic remarks were not useful and sometimes detrimental, the group allowed Jane to have her say. At these times, the group engaged in tasks that got them no closer to the solution nor was Jane's understanding enhanced because she often tried to save face.

By the end of this second session a correct system of equations was determined. When the session ended, however, they were still unable to solve the system. Sara and Eve were

doing similar work but were not collaborating so they wound up doing double the work. When the session ended, real progress was being made. However, because Sara and Eve were the two who understood best what was being done, when Sara missed the third session, the work was stymied. The third session took a turn from what they had been doing in the second session when Jane suggested a different approach. For now, however, a discussion of the twists and turns to get the two equations is important.

The following dialogue shows Sara continued to have misconceptions about the concept of variable.

Sara: Could we use the triangle on the left—and do 240 minus x over here?

LBC: What's x ?

Jane: It's going to be the length of AR.

Sara: It should be the length of EC also though—

Jane: How do we know they are congruent?

Sara: We don't.

When Sara asked about the triangle on the left, it was not clear whether she was assigning $240 - x$ to segment AR or RD. Regardless, this representation conflicts with the decision she and Eve made about x representing EC. When Jane claimed x would be AR, Sara claimed that EC would also be x (as she and Eve decided earlier). The inconsistencies were not problematic for Sara, signifying the concept of variable was still misunderstood. Jane, however, asked an insightful question, "How do we know they are congruent?" Perhaps the discussion of variable had been more of a learning opportunity for her than I realized. Sara merely answered Jane's question with, "We don't." No elaboration, no alternate suggestions, and no verification were offered. The communication was ineffective and was followed by a 47-second period of no conversation.

Eve then revisited a question she had in the first session. She questioned whether angle A in triangle ACD was the same angle A in triangle AER. This issue was problematic for

her in session one so the explanation offered during the first session apparently did not lift her. Perhaps her uneasiness during that session prevented her from fully understanding the explanation given. Nevertheless, her attempt to seek additional information was indicative of her tenacity in understanding the problem and also indicative of her level of comfort during this session. The dialogue that occurred was a learning opportunity for Sara and Eve and provided lifting for the two of them as was determined during their final interviews. The episode was also one of the rare occasions where Abby contributed.

Eve: I have a question—uhm---for angle A, that is 22.62, right? And if we were going to solve it, 22.62—that angle is for [triangle] ACD, right? So we would have to...that's not the angle for [triangle] AER, is it? (See Figure 11.)

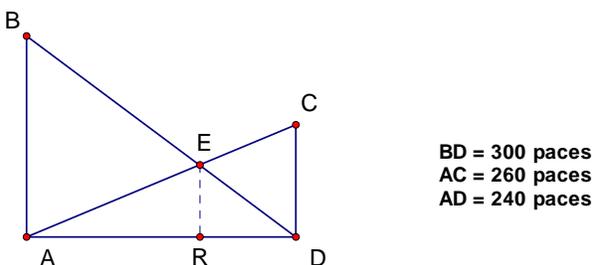


Figure 11. Schematic of Buried Treasure Problem with point R labeled.

[Sara and Abby both say that yes it is.]

Eve: It would be?

LBC: You mean triangle AER?

Eve: yeah, in triangle AER—so angle A would be 22.62 in triangle AER and in ACD?

Sara: uh huh.

LBC: Did you think that would be the case, Eve?

Eve: I thought it uhm—that this angle would only be for ACD since we solved for only angle A cause this is—that's one triangle and the triangle angle A is—it's in two —so it would be different in triangle AER—I mean, I'm not sure, I'm just—I don't know—I was just wondering. [pause] Will it be the same in both triangles?

Sara: Yes.

Abby: Cause the angle—the angle is [using her fingers to show an angle]—I don't know what I'm going to say—it's still the same angle, it's just like that the triangle—if you just cut that part off [that is, the trapezoid RECD], the angle is still the same—[Eve said ok]--wouldn't it be? I mean wouldn't—it's the same angle measure it's just that how far the sides are going out— [Abby used her arms to illustrate]—like this one is—I don't know—

Abby gave a good visual demonstration of angle by first using her fingers and then using her arms. Eve listened carefully and even said “Ok” as Abby explained. I was not sure Eve understood the concept of measurement of an angle so I intervened.

LBC: What's meant by the measure of an angle?

Sara: How far it opens up [used her arms to explain].

Abby: It's open the same amount; it's just when you go up that far, the sides are just longer but the angle is open the same amount. [Eve listened to Abby and eventually nodded her head in agreement]

LBC: So the lengths of the sides have nothing to do with the measure of the angle? [Eve picked up her calculator]

Sara: [tentatively] They do—I mean—

[When Sara said nothing more, I interjected.]

LBC: [using two pencils to show the sides of angle A in Abby's drawing] Alright, in triangle ACD the sides of angle A are longer than they are in triangle AER----but would the lengths of those sides affect the measure of the angle--do they influence the measure of the angle?

Sara: No, it's all proportional—to the angle.

LBC: What do you mean?

Sara: I mean if you have angle 22.62 and a triangle AER—the measure of AE and AR are the same in proportion to—if you have triangle ACD—the lengths AC and AD are—should be the same proportion to these two sides, AE and AR.

I wanted to know why Eve thought the angle A of 22.62 degrees belonged only to triangle ACD. Eve's explanation showed a misconception, but it was not until the final interview that I understood the source of her misconception.

Eve: Because when we solved it, we were using angle ACD and that angle would just go with that triangle [that is, ACD] and in here it would either have to be—that this would not be the angle for AER—since this a different size in a different triangle—I wasn't sure so that's what I was thinking that it just wouldn't be 22.62.

Eve showed a sense-making perspective in trying to understand the angle measure in the overlapping triangles. Her tenacity in understanding this concept provided lifting for her. It also provided unexpected lifting for Sara. Eve's question was an example of the indirect collaboration that Cobb (1995) identified in his study. Eve's question was an unexpected learning opportunity for Sara in that at the time Eve asked the question, Sara believed the answer was obvious and yet later (in the final interview) she admitted it had a significant impact on her thinking about ratios.

I believe it was my question "What do you mean?" that encouraged further explanation. It was while Sara explained that she saw relationships that would prove useful. Without my question, the communication that proved to be effective may not have occurred. During the session, however, Sara did not acknowledge how Eve's question and subsequent explanations provided an epiphany for her. She merely suggested they consider ratios.

Sara: Is there any way that we can use the ratio idea?...Like shouldn't these lengths right here—these lengths right here should both be proportional, right? Seeing as how, I mean, this

is a triangle—this is the triangle right here that we are trying to look at [pointing to her new drawing on her paper]—I mean, is that true, that these lengths should be proportioned in length to...

Abby: I don't know how to say what—yeah, I'm thinking like what you're saying [Abby pointed to Sara's paper]—like—the angle is open the same amount, it's just—

Sara: like proportion wise—

Abby: yeah, yeah.

Between the first and second sessions, Abby discussed the Buried Treasure Problem with her colleagues at work who observed the smaller triangles were just smaller copies of the bigger triangles (as told to me in the final interview). It was not until Sara pointed out the proportional sides, however, that Abby indicated she also saw this relationship. As typical of Abby, she did not contribute unless she was certain her contribution was correct; getting some verification from Sara encouraged her to tell Sara that she was “thinking like what you're saying.”

Sara and Abby seemed to have similar ideas but the opportunity to clarify and extend their thinking was lost. Abby backed down when Sara gave her no more encouragement. Sara was interested in pursuing her ideas but was not sharing what she was doing with the group. In fact, she withdrew cognitively from the group as she explored the idea of ratios. I again intervened and asked them to share their thoughts. My request did not initiate effective communication. Sara merely responded with a quick report of what she was doing but sounded unsure about her work.

Sara: What I'm doing is--I'm giving ER the name ER and I'm solving the tangent of 22.62 and setting that equal to ER over 240 minus x—[pause]—I don't know why, but I'm doing it.

Jane explained, “Cause you are going to eventually set it equal to ACD—or subtract it from ACD, I think.” This remark was another of the bizarre comments for which Jane was notorious, and I saw it as a form of social collaboration. They were working independently, and

I tried to encourage collaboration by asking, "Are you working together?" Sara again was the only one who acknowledged my interjection and explained that she was "just trying to figure this out real quick and then see if I can give any insight or see if it is just a waste of my time." The work she generated during this time is in Figure 12.

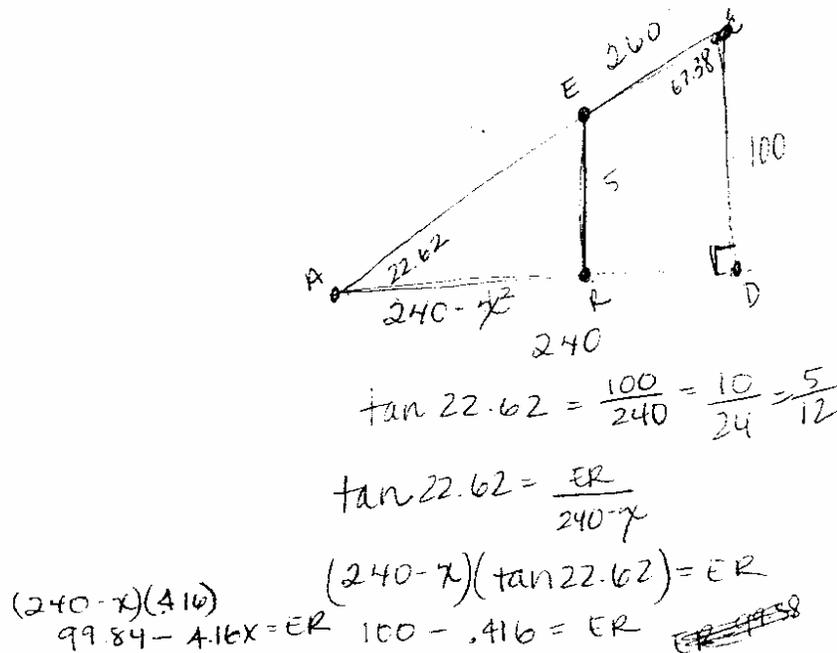


Figure 12. Sara's work to find the length of ER.

Numerous errors were made in this work. They included: not defining x , distributing $\tan 22.62$ over $(240 - x)$ incorrectly, and misplacing the decimal point in $.416$. When Sara obtained $99.84 - 4.16x = ER$ and saw it as no help, she showed interest in Jane's work. This interest in Jane's work was surprising to me because Jane had taken the group on some unproductive paths. In the following exchange Sara asked Jane about her work but Jane's tendency to save face prevented effective communication.

Sara: Did that help at all—what you just did? [asked of Jane]

Jane: Not according to my numbers, but—

Sara: Well, what is it that you did?

Jane: I did what you had said—take the tangent of 22.62 and set it equal to ER over 240 minus x.

Sara: What did you come out with? What did you solve for—x?

Jane: I had x and ER in there and then I set them equal to each one—but I didn't get the numbers you got at all.

LBC: You set it equal to each one? What do you mean?

Jane: [ignoring the question] Are we supposed to be in degree mode? [looking up at the group]

Sara: uhm—I am.

Jane: Oh, well that is probably why!

Jane would grasp at any procedure. Hearing Sara say “ $\tan 22.62$ equals ER over 240 minus x,” Jane wrote the equation and began implementing. No discussion or sense-making accompanied the decision to use this equation. The work Jane generated is shown in Figure 13.

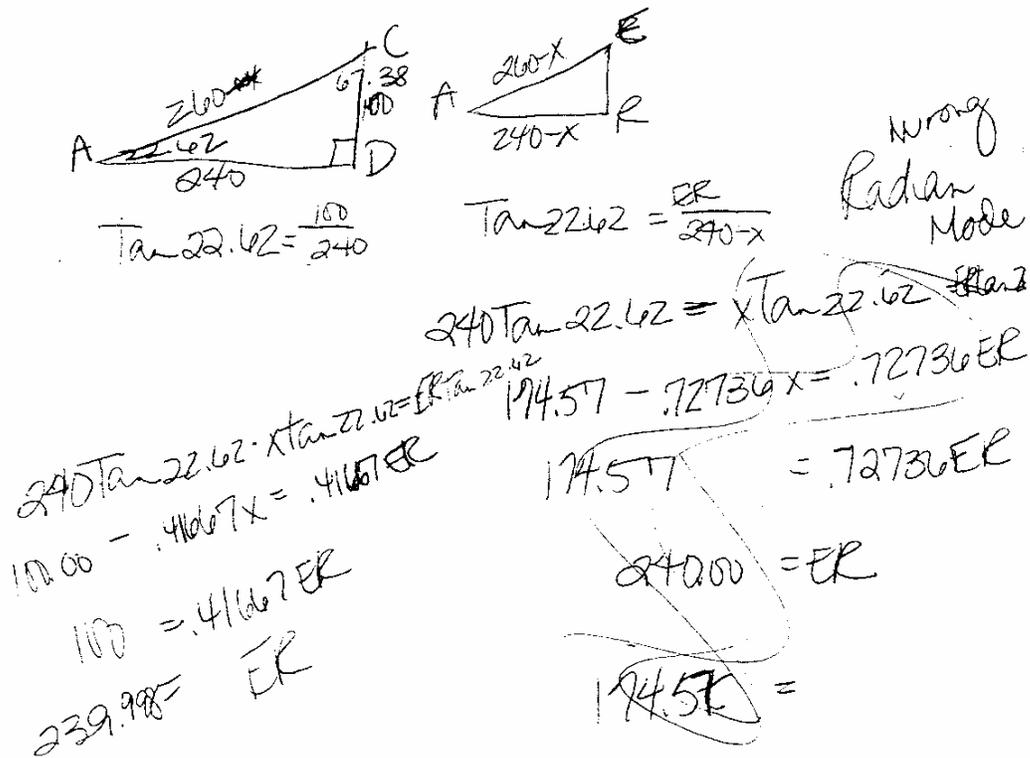


Figure 13. Jane's work to find ER.

As can be seen she marked this work out when her answer was not as expected. Not only was she in radian rather than degree mode, she was also unable to perform correctly the procedure students frequently refer to as “cross-multiplication.” She then performed the same procedure again using degree mode and eliminated the term $.41667x$. Her claim that she “had x and ER in there and then...set them equal to each one” perhaps was her explanation for dropping the terms. Her work shows how she committed vandalism in her pursuit of an answer.

The remainder of this session had the potential for some real progress. A description of the twists and turns for the remainder of this second session follows.

Assigning 5 and 12 to the sides AR and ER respectively of triangle AER , Sara concluded the hypotenuse AE would be 13. Her drawing is shown in Figure 14 below.

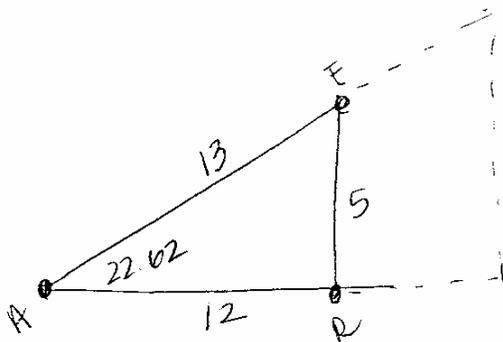


Figure 14. Sara's drawing of triangle AER to show the ratios of the sides.

Although Abby contributed nothing in the way of paper and pencil work, her comments showed she could see the relationships between the similar triangles.

Abby: Like that looks like that is the smaller version of that—and then that looks like it is the smaller version of that—and that's the thing with the angle, too, about it being the same or whatever [I think she is referring to Eve's earlier question about angle A]—this looks like it is just a smaller version of that one right there—it's like you've cut 'em, just cut part of it off or whatever—these are smaller versions of the big ones.

I asked the group about Abby's comment and Sara and Jane both responded.

LBC: What do you think Abby means when she says, 'they are smaller versions'?

Sara: uh—that they are going to have the same angle measures and their lengths will be proportional.

Jane: [as soon as Sara completed the above statement] proportional different

Sara responded with an appropriate mathematical explanation so perhaps Abby's comment about "smaller versions" helped Sara focus on not only the sides but the angles of the triangles as well. Jane's comment, spoken on the heels of Sara's comment, added little and was probably spoken to give the impression that she knew what was going on—another instance of saving face.

Sara drew two right triangles on her paper (see Figure 15) and explained:

Sara: This is what I've got now—I did the proportions of the other triangle, too...these are the proportions of the triangles back to back and these are the lengths right here—this is the length we are trying to find out—[Sara propped on her elbow on the table and rested her chin on her fist as if she was trying to sort out what she was thinking]—uhm—does that necessarily mean though that this length right here and this length would be the same?

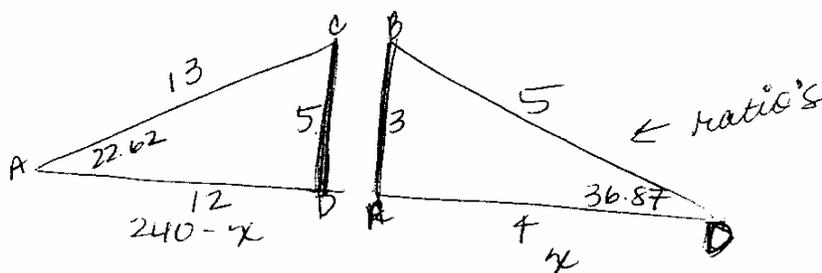


Figure 15. Sara's drawing of triangles ADC and ABD.

Initially Sara did not label the vertices and did not have points E and R identified. When she said “this is the length we are trying to find out,” she pointed to the segments with the numbers of 5 and 3 written next to them. Pointing to both of these segments but saying “this is the length we are trying to find out” led me to believe Sara saw these segments as the same segment ER. But if she saw them as the same segment, asking “does that necessarily mean though that this length right here and this length would be the same?” made no sense. She was perhaps assuming the two triangles above were proportional but could not make sense of the different numbers on the sides.

I suggested Sara label the points in the drawing. Her next question showed she was still confused about which sides were proportional.

Sara: Does that make sense there? I mean—are these proportioned together? I guess—well, are those two angle measures the same?No.

From her preceding work, Sara got the measurements of angles ACD and ABD and wrote them on the triangles ACD and ABD shown in Figure 15 above. Because the angle measures were not the same, Sara concluded, “Well, since the triangles themselves are not proportioned together then I’m guessing the sides aren’t either—cause the lengths of the sides were only proportional when the angle measures were the same.” Her comment showed her understanding that congruent angles are necessary for similarity.

I watched Abby listen to Sara and sensed she was trying to muster the courage to make an observation. She finally made the following comment.

Abby: Can I ask a dumb question? This may not make any sense--but like if you think about— if you think about that or whatever and you just think about it like that being from E to R---or whatever—[sighing]—then that and that would have to be the same so—

Abby pointed to Sara’s paper as she explained. Because her fear of being wrong made her reluctant to share her ideas, she must have been confident of her observation. I believe Abby was trying to show Sara the location of \overline{ER} on the two triangles so that Sara could see the triangles whose sides were proportional. Unfortunately Abby had a difficult time expressing herself so Sara paid little attention to Abby’s comment. Sara responded, “Right!” to Abby’s comment. However, Sara’s next comments indicate Sara did not know where \overline{ER} was located on the drawing in Figure 15 above.

LBC: So where is ER in here?

Sara: It’s—it’s somewhere, I mean, it can be this right here—[pointing to segments CD and BA and drawing heavy lines over them]--cause right now the measures of the sides are all just proportions; they’re not actual lengths—

Abby was perhaps frustrated because she was unable to make the location of \overline{ER} known to Sara. She was staring at Sara's paper, and I asked her what she was thinking. Without removing her eyes from Sara's paper, she sighed, paused, and then responded, "I don't know—what I was thinking I'm not thinking anymore cause—I don't know." Giving one last deep sigh, she still looked as if she wanted to say something but nothing was forthcoming.

Eve had been working independently and I tried to bring her back into the group by asking her to share her work. She explained that she set $\frac{ER}{240-x}$ equal to $\frac{100}{240}$ because she heard me point out they were each equal to $\tan 22.62^\circ$. [I made this observation earlier in the session.] The following explanation was somewhat confusing as she referred to 240 as 24 and used a variable of r without defining it (but later explained it was the length of \overline{ER}). She explained, "[I] solved for x ...then I plugged x back in to 24 minus x and solved for r and got r to equal five-twelfths and now I'm plugging in r ..." Eve solved for x in the proportion

$$\frac{r}{240-x} = \frac{100}{240} \text{ and correctly obtained } x = -\frac{12(r-100)}{5}.$$

She then substituted the expression for x into the expression $240-x$ but rather than just getting an expression in terms of r , she solved for r and concluded $r = \frac{5}{12}$. Her work is shown below in Figure 16. Thus, Eve did not note the difference between an algebraic expression and an algebraic equation.

$$240 = \left(\frac{-12(r-100)}{5} \right)$$

$$240 + \frac{12(r-100)}{5}$$

$$\frac{240}{5} + \frac{12r - 1200}{5}$$

$$\frac{5}{12} \cdot \frac{12r}{5} = \frac{5}{12} \quad r = \frac{5}{12}$$

Figure 16. Eve does not differentiate between an expression and an equation.

Jane gave Eve only a cursory glance but the following exchange shows Sara's interest was piqued as she heard similarities between Eve's work and her own.

LBC: Now what is r standing for now?

Eve: ER, segment ER. [said confidently] And now I was plugging in segment ER in for x —I don't know, I was just trying it.

Sara: So you got five-twelfths for what?

Eve: ER—cause I solved for x and x came out to be negative twelve parentheses r minus 100 over 5 and then I plugged x into 24 minus x —right there—[she showed her work to Sara who said "uh huh"]—I plugged that equation into 24 minus x .

LBC: 240 minus x ?

Eve: Yeah, 240 minus x .

Sara: I got five-twelfths also as the ratio of ER to RA.

It was difficult to follow Eve's explanation because she talked about a variable r that she did not define, and she said "24" when she meant "240." It was also difficult to follow how she was substituting. The other three students did not ask Eve for clarification, and regrettably I

failed to ask them during the final interviews why they did not seek clarification. I can only wonder: Did they not hear the confusing statements? Did they hear but made assumptions about their meaning? Did they hear but didn't believe it was important to seek clarification?

Moreover, poor communication and the lack of a common workspace contributed to Sara's misunderstanding of Eve's explanation. Although Eve said that she got five-twelfths for ER, Sara said, "I got five-twelfths also as the ratio of ER to RA." These two statements were not the same but the differences were not noted. There was potential to discover the difference as Sara sought more information from Eve.

Sara: And the equation that you set up—what were you setting up? You were setting what equal to what?

Eve: The tangent—remember in the beginning we had tangent of 22.62 of both of the triangles and it was r over 24 minus x and 100 over 240—

Sara: Will you show me where you did it? [an opportunity for a common workspace]

Eve: Let's see. [finding her work and turning it so Sara can see it] Right there.

Eve turned her paper so Sara could see where she had written $\tan 22.62 = \frac{r}{240 - x}$ and

$\tan 22.62 = \frac{100}{240}$. Abby also looked at Eve's paper, but Jane was engaged in her own work.

Sara: So you worked both of these out?

Eve: I set them equal to each other.

Sara: Oh, ok, so basically—I see what you did! [pause] That's basically—and you said you got five-twelfths then after that, right?

Eve: uh huh

Sara: Yeah, that's what I got, too, for the—I just did it a different way—

LBC: You got five-twelfths for?

Sara: The ratio of the segment ER to the segment RA.

Although Eve and Sara both obtained a value of five-twelfths, Eve claimed it was the value of ER while Sara claimed it was the value of the ratio of ER to RA. I was unsure whether poor communication was the culprit--specifically poor listening skills—or whether they failed to see these as mathematically different.

As I looked back at the tapes, I realized there were several learning opportunities for Sara and Eve that were not addressed. One was recognizing that ER and a ratio involving ER were two totally different concepts. Had they acknowledged and understood the difference in these concepts, they could have questioned how ER and the ratio of ER to RA were both equal to five-twelfths. Further exploration could have also led them to note the difference between an algebraic expression and an algebraic equation. Unfortunately these learning opportunities did not occur.

At this point Eve reminded me she had to leave so the session needed to end. Progress was being made but Sara's absence from the third session and Eve's absence from the fourth session slowed their progress in the subsequent sessions.

Session 3

Sara was absent for Session 3. As I corrected a problem with the video camera, Jane, Eve, and Abby chatted comfortably. They talked about former teachers, about the math courses they will take next, and about their pseudonyms for this project. I especially noticed how relaxed Abby appeared; as an example, consider the casual banter between Eve and Abby that occurred about halfway during this third session.

Eve: Abby, what are you thinking?

Abby: I don't know—I'm lost on one of them roads on that map somewhere.

Abby felt comfortable admitting she was lost—this admission was unusual.

Early in the session Jane located a textbook problem that she thought “is the one that guy worked out in class.” The problem she referred to is in Figure 17.

Find x correct to one decimal place.

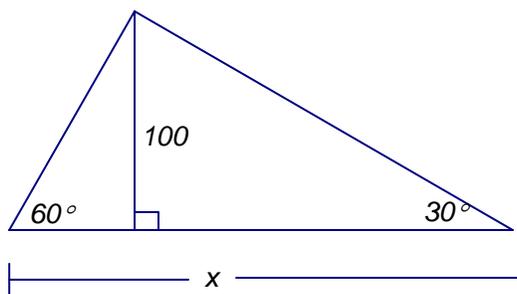


Figure 17. Problem 53 from the textbook.

Jane's plan was to work this problem and "by looking at the back of the book to try to figure out the answer—and compare how we worked it." Problem 53 was similar to the Buried Treasure Problem in that the drawing consisted of two juxtaposed right triangles with known angle measures. In problem 53 the side common to the two triangles was known so finding the other sides of the triangles was straightforward. Eve and Abby watched as Jane found the segment lengths of one triangle but they did not do any work themselves. Jane apparently recognized problem 53 as different from the Buried Treasure Problem because she asked, "Why was it that easy?" Abby also saw the difference in the two problems.

Abby: This [problem 53] is like in reverse—like you know the whole, you know the big parts but you don't know—like there—they tell you that—but like that's what we want to know—so we need to do it in reverse.

Undeterred by Abby's comment, Jane continued to solve problem 53 while Eve ignored them and instead looked through her work from the previous sessions. Perhaps feeling abandoned by Eve, Abby nervously tapped her chin as Jane persisted in working problem 53. When Jane confirmed her answer matched that in the back of the book, she asked, "Why can't we just do the back route and get that?" Abby sighed and extended her arm on the table. It

was obvious that Abby wanted to say something, but she was so nervous about doing so. I broke the silence with a question which motivated Abby to express her doubt.

LBC: So you got problem 53 to work?

Abby: But the-- but the—the whole thing about this one is—it gives you the whole length—

Jane: the whole length

Abby: Yeah, you don't have the little pieces—you've got to figure out what the little pieces are first.

It was insightful of Abby to recognize the difference in the two problems, but she still had to be encouraged to speak up. Jane spent nine minutes exploring problem 53. She had a procedure and was happy implementing that procedure even though differences were acknowledged. In the previous two sessions, Jane explored ideas on her own while Abby watched Sara work. With Sara's absence, Abby watched Jane. Although Sara was not an effective leader, Jane tended to back down to save face if Sara expressed any doubt—backing down curtailed the amount of time Jane would pursue “wild goose chases.” Even though Abby expressed doubt in Jane's exploration of Problem 53, Jane did not back down as she typically did in Sara's presence.

It was Eve who finally directed the group's attention back to the Buried Treasure Problem by recalling the work from the previous session.

Eve: Didn't like last time we like broke the triangles up?

They were making progress when Session 2 ended so referring to their previous work was promising. Following the conversation in Session 3 was especially difficult. Jane frequently made inaccurate statements. The following four statements were spoken by Jane in Session 3 and support my observation.

1. "AD equals 240 minus x"
2. "what ...if we went ahead and tried to find the length of AD"
3. "And then x would give you the AR and then you could take that and subtract it from the 240 to find AD"
4. "We already know the whole length of BD is 240."

I could not be certain whether her mistakes were slips of the tongue or were evidences of misunderstanding. The others in the group either did not notice or chose to ignore her statements. Because Jane's tendency was to save face, correcting her would not necessarily be a learning opportunity for her.

Also making the discussion difficult to follow was the group's tendency to talk past each other. There was a lot of conversation but with little understanding. Jane ignored the discussion of variable representation in Session 2 so her misconceptions were an issue in this Session 3. It was obvious these misconceptions existed for her throughout these problem sessions. In fact, her misconceptions were addressed again in her final interview. (See the description of her final interview.) Consequently, the communication in this Session 3 was difficult to follow as inaccurate statements, talking past each other, and unresolved misconceptions made much of the communication ineffective.

Although (as noted above) Eve mentioned breaking the triangles up in Session 2, no one acknowledged her statement so after one minute of silence I asked Eve, "What are you doing there?"

Eve: Uhm last time I know we—we were trying—our ultimate goal was to try to find E [sic] and we labeled the segment ER as unknown. And we had—I think A was 22.62 and I think A to E was 260 minus x.

They worked independently for 1 minute and 35 seconds before I suggested they share what they were doing. The following exchange reveals Jane's continued misconception about

the concept of variable. Eve was apparently more confident of her understanding of this concept as she was able to share her doubt about Jane's representation of the lengths.

Jane: I just redrew ABD—that's what we know—we know that BD is all 300—we know that BA is 180 and we know that AD is 240—alright then I took the triangle ERD and labeled what we know—we know that RD is 240 minus x because we know the whole length of AD is 240 but you want to subtract this part so we can find out what RD is so we can in turn find ER—we don't know what ER is—and ED we know is 300 minus x because it is part of BD which we know is 300, so it is 300 minus x —we know the angle measurements of the little triangle and we know the angles here; so now I'm just looking to see if there is a way to set them equal, in some respect. (See Figure 18 for the schematic drawing.)

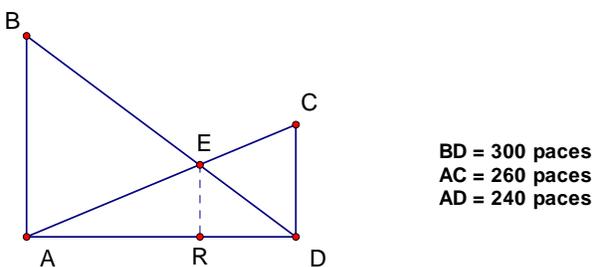


Figure 18. Schematic drawing of the Buried Treasure Problem.

Eve: You want to use like different variables so we won't get confused?

Jane: Yeah—I'm just used to using x .

Eve did not offer Jane an acceptable mathematical explanation for using different variables so I asked her why she asked the question about using different variables. Jane answered first, but her explanation showed limited understanding of the concept of variable representation.

Jane: It gets confusing.

Eve: Yeah, and if x is whatever one would choose like 240 minus x , that is just for that x —if we make them all x then whatever we find x to be in that one will have to be in the rest of them—and that's not true. It's just—

Eve's explanation showed her understanding that even though x is a variable, its value is constant within a problem. Eve disagreed with the way Jane represented the lengths of the sides, but the conflict was resolved by Jane conceding "so we can change those variables-- that's fine." I do not believe Jane understood it was a mathematical necessity to use different variables. Her concession had nothing to do with recognizing the flaw in her representation. Her concession was typical of her procedural approach to mathematics; she fixed a problem without making sense of the mathematics related to the problem. Furthermore, Eve felt no obligation to explain to Jane why different variables were necessary.

Jane drew triangle ABD and triangle ERD as shown in Figure 19. She used variables x , y and z as the lengths of \overline{RD} , \overline{ED} , and \overline{ER} , respectively.

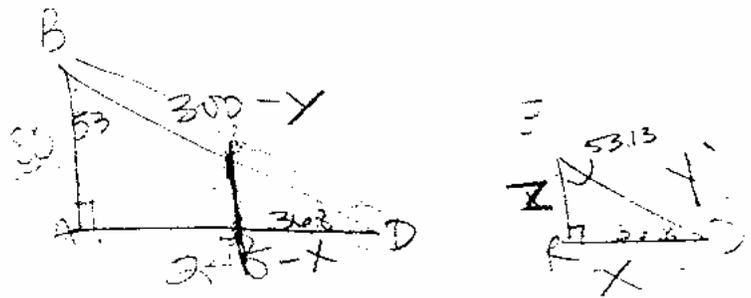


Figure 19. Jane's drawing of triangles ABD and ERD.

Jane had $300 - y$ written adjacent to segment BD in the figure above so I pointed to her paper and asked, "So if ED is 300 minus y , and BD is 300, then what's y ?" Jane responded immediately and confidently.

Jane: We don't know yet—that's what we have to figure out.

My question was ambiguous. Certainly we did not know the value of y so her answer was not incorrect. Jane had not engaged in the conversation during Session 2 related to variable representation so I rephrased my question to gain a better understanding of this concept for them. The following rather lengthy exchange especially reveals the struggle Jane had with variable representation. This struggle for her was not resolved during the problem sessions and will be addressed later during the discussion of her final interview.

LBC: Alright, when I ask what is y , what do I mean?

Jane and Eve both offered explanations. Jane appeared to see y as a value rather than a representation.

Jane: What is the length of that side. [in answer to my question]

Eve: What does y represent. [in answer to my question]

LBC: uh huh—that's really what I want to know—what does y stand for, what does it represent.

Jane: DE, the length of ED.

Eve: Yeah. Segment ED.

LBC: Segment ED. Alright, now, so you are saying 300 minus y is what?

Jane: BD. The whole length is BD, is 300—but there is a segment right here, this part right here, that's where ER would come in--so it's a part of the 300. [Jane pointed to where ER intersected BD to break BD into two segments.]

LBC: So what does y represent?

Jane: A part of BD—an unknown length of BD cause we know it's cut somewhere but we don't know the length.

[Abby and Eve are listening to Jane but are being very still.]

Eve: [tentatively] y would be ED, right?

[pause]

LBC: Eve says y would be ED....So what does 300 minus y represent?

Abby: The length of B to E. [said confidently]

Eve: yeah—uh huh.

LBC: Does that seem reasonable?

[I heard “yes” and saw nodding heads.]

They worked independently for 45 seconds and then I intervened.

LBC: What are you thinking? Think out loud for me. [Abby gave a nervous giggle.]

Jane: I’m just rewriting what we just said—that BE is 300 minus y; ED equals y; AD equals 240 minus x; and RD equals x...Right? Is that what we just said?

It was apparent that Jane still did not understand how to express the lengths of the segments with variable expressions. Although the length of \overline{AD} was known to be 240, she represented its length as $240 - x$. Eve was not listening and mindlessly validated Jane’s incorrect statement.

Eve: uh huh.

LBC: AD is 240 minus x, you said, and RD—

[I repeated what Jane said, and Eve and Abby both realized the mistake.]

Eve: uh uh.

Abby: No, it’s the same thing—A to D is 240 but A to R is like—

Jane: [interrupting and her comment really contributed nothing] AR

LBC: [to Jane] Why did you think that AD was 240 minus x?

Jane: I don’t know.

Jane was unwilling to make sense of her work and chose to save face by saying she did not know why she thought AD should be $240 - x$. Although she changed the representations to $AD = 240$, $AR = 240 - x$, and $RD = x$, I was not convinced she understood.

At this point in the session they have represented the lengths of segments AR, RD, BD, BE, and ER but did not discuss how they would use this information. Eventually Eve suggested another procedure.

Eve: Well, can’t we just go on and find like say cosine of 36.8 would be uhm—

Abby was concerned that none of the lengths of the sides of triangle ERD was known “Every side is a letter, though.” She then explained how the Buried Treasure Problem is different from problem 53.

Abby: Yeah, and like in the little one—and that’s the thing that makes that harder than that cause you don’t—there [that is, problem 53] they gave you at least one thing and you could figure the rest of it out but like in that little one every side is a letter.

Abby noted each side of the triangle was a letter whereas a better observation would have been that each side was an unknown. To deal with Abby’s observation, Eve suggested, “We could change it—instead of y it would be 300 minus y and let the other one equal y —I mean, I don’t know.” I believe Eve was suggesting they represent BE as y and ED as $300 - y$ although making the change would not reduce the number of unknowns. Eve’s suggestion indicated the variable was still problematic for her.

In the episode which follows, Jane suggested they find the length of AD even though AD’s length is 240. Jane also convinced Eve and Abby to commit vandalism with the mathematics as they tried to solve for x .

Jane: Now what about if we went ahead and tried to find the length of AD. We already know that AB is 180—then we could find that length for AR—or AD.

Eve: AR?

Jane: So we are taking this angle right here—[Jane pointed to her paper; Abby and Eve looked and listened.]--and doing the tangent which is opposite over adjacent—or this whole length—we are doing the whole triangle—we’re taking this whole side—and then solve—like what you were trying to start to say—

Eve: What are we finding now?

Jane: x —the x . And then x would give you the AR and then you could take that and subtract it from the 240 to find AD...possible?...maybe?

Eve: So we take the tangent of 36.8 is 180 and 240 minus x.

Jane: Just work it and see if it works?

Jane and Eve were talking past each other again. When Eve finally suggested “the tangent of 36.8 is 180 and 240 minus x,” the inaccuracy was not detected. Jane was glad to have an equation for which she could solve for x. With no recognition that Eve’s statement was

incorrect, each of them wrote $\tan 36.87 = \frac{180}{240 - x}$. They worked for 1 minute and 15 seconds

before Eve realized there was a glitch in what they were doing. After entering some data into her calculator, she propped her chin on her calculator as if trying to make sense of the result. The work on her paper gave no indication of what she entered. However, she recognized their faulty reasoning.

Eve: uhm—Ok, we took 180...then how come uhm like 240 minus x—wouldn’t it just be 240? Cause we took 180, AB--so—

Jane: But I know--and, this is what bothers me cause we know that AD is 240.... That’s why it’s not working.

Eve realized the tangent of 36.87 should be $\frac{180}{240}$ and not $\frac{180}{240 - x}$, but I could not tell how she made this determination. Jane was the only one of the three to record her symbolic manipulation, and her work is shown in Figure 20. Earlier Jane suggested they find the measure of AD so in all probability Jane said AD should be 240 only because Eve first said it. Jane, however, knew something was wrong when her symbolic manipulations did not work out as she explained below.

$$\tan 36.87 = \frac{180}{240-x}$$

$$180 = .750x = 180$$

Figure 20. Jane's work for the tangent of 36.87.

Jane: Because when you multiply the tangent of 36.87 times 240, you get 180 and you already have 180 on the other side and then when you subtract it out you get zero—not very smart.

The above procedure did not work so it was time to try something else. When one procedure did not work, they would move on to something else. Eve again mentioned the work they did in Session 2, suggesting they consider using ratios with triangles ABD and ERD. Not only did Abby agree, but she also finally picked up Sara's paper. Although Sara's work was placed on the table at the beginning of the session, no one referred to it. Perhaps they did not use Sara's work earlier because they saw themselves as individuals working this problem and not as a collaborative unit. Jane and Eve discussed which lengths were proportional but reaching a consensus was difficult. They often talked past each other, and no common workspace made understanding difficult. Jane particularly did not understand the proportionality of the sides and was unable to discern which segments should be proportional. The following exchange shows how they struggled with getting the ratios.

Eve: I think that is x —so it would be—can't we like do the ratio/proportion so we can find some of these angles? [She means sides.]—240 minus...

Jane: [looking at Eve] You mean for AD?

Eve: uh huh—and we can use AB also cause AB to RE—

Jane: Ah, yeah, it all comes back slowly—so you are saying taking the whole AD—

Eve: Took—wait a minute—took AB to ER

Jane: to ER—Ok, and then take—take what? Take AD to AR?

Eve: To AR or RD?

They were talking past each other so the communication was ineffective. They did not try to make sense of what the other was saying and they did not create a common workspace to help them understand. Jane thought the ratio of AB to ER would equal AD to AR. Her suggestion showed she did not understand the concept of proportionality of the sides of the triangles. As Eve and Jane talked, Abby used Sara's paper to draw the 5,12,13 triangle and the 3,4,5 triangle shown in Figure 21.

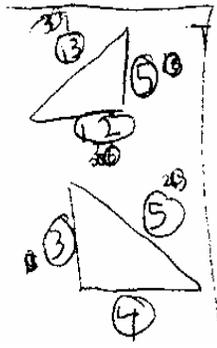


Figure 21. Abby's drawing of the 5,12,13 and 3,4,5 triangles.

There was a period of no talking so I again tried to encourage collaboration by asking what they were thinking about doing. The exchange below shows an intuitive understanding of similarity.

Eve: Doing the ratio—it's the ratio or proportion—cause see these it's AB, it's like the bigger version of ER—and if that's x then there—I can't find the words uhm—similar—[Eve, smiling, gestures with her hands indicating a loss for words]

Jane: [suggesting] proportionally

Eve: Yeah.

LBC: You mentioned 'similar'—what do you mean by similar?

Jane: They are congruent cause they are parallel but--their sides are different.

Abby returned Sara's paper to its original location and then slid her paper toward Eve. She pointed to the right triangles she had drawn (see Figure 21 above) and said, "That's the thing that we did last time about the ratio of them or whatever...she [Sara] had it wrote [sic] down but I didn't." For twenty-five seconds Eve merely glanced sideways to look at what Abby showed her—she did not pick up the paper to study it nor did she make a comment. Getting no response from Eve or Jane, Abby eventually retrieved her paper. Eve did not copy the triangles Abby showed her so she perhaps saw them as no help. It may not have been clear to Eve (or to Abby) how Sara concluded 5,12,13 and 3,4,5 because this information was never part of their discussion during Session 2.

The next episode was so promising but broke down when they had trouble with symbolic manipulation. The episode began with Eve and Jane struggling to write the proportions as the following exchange shows.

Jane: Are you putting them proportionally?

Eve: I was trying—I think if—since we have 180 so it will be 180 over 240 or 240 minus x?

It was clear to me part of their struggle was ineffective communication. Even though they could not understand the references each one made, they made no effort to seek clarification. During the conversation, Eve asked, "So ER is z?" Although no one responded to Eve's question, it was this question that may have helped Abby decide on the proportions to write because as Jane and Eve were trying to decide on the proportions, Abby quietly wrote the

proportions $\frac{180}{240} = \frac{z}{x}$ and $\frac{100}{240} = \frac{z}{240-x}$. The work from Abby's paper is shown in Figure 22.

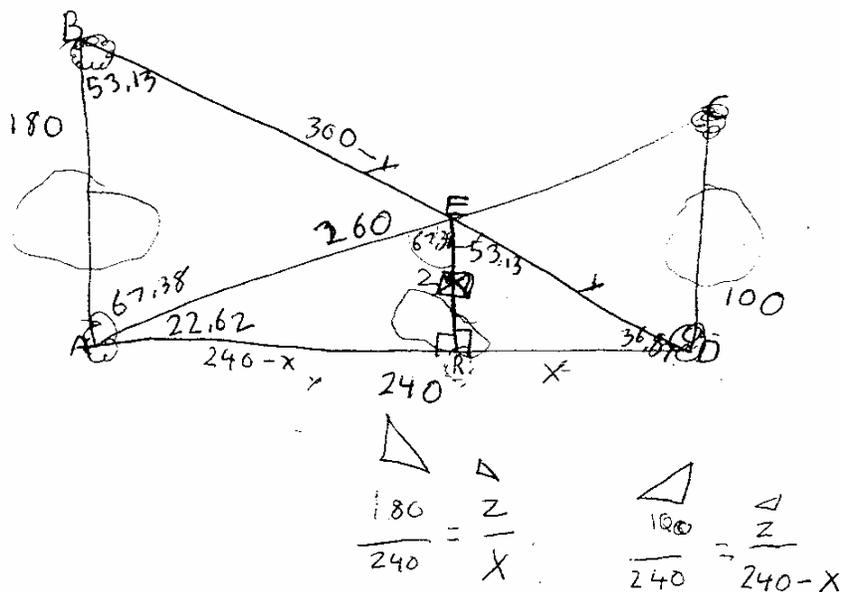


Figure 22. Abby's drawing and proportions.

Abby listened to what Eve said and eventually said, "I did—I wrote that [pointing to the proportions in the figure above] down—the same thing you are thinking about." Abby showed her work to Eve who looked at it with more interest this time. As was typical, Abby did not share her work until she was sure her work was similar to that of Eve's. Looking at Abby's work in Figure 22, I was pleased to see how she separately drew the similar triangles and then looked at the measures on the more detailed drawing to help her get the proportions. Although Abby's symbolic manipulation skills were sometimes lacking, her ability to visualize was more developed. Her confidence in her work was evident as she explained how she wrote the proportions. Lacking, though, was an explanation of why these proportions made sense in relation to the drawing.

Abby: I did the 180 over 240 and then I did the z over x—like that—and then I did the other side, I did 100 over 240 and then I did z over 240 minus x—same thing as what you were thinking.

Eve listened attentively to Abby’s explanation and then became more animated as she explained what they could do next. No justification was offered as to why these proportions were appropriate, and Eve’s next suggestion was a procedure they could implement.

Eve: And then we can cross multiply or whatever...we can solve for like one of the variables...like solve for z or solve for x and then just go back...let me work it first.

Eve had a plan in mind but wanted to work out the details alone before sharing with the group. Although at the time I did not realize it, I must admit I interfered with Eve’s work. I was standing between Eve and Jane and decided to quiz Abby about her representations of z and x. When I began talking to Abby, Eve quit working and started listening to my conversation with Abby. Although Eve said nothing, she was unable to make much progress before Jane started asking questions. Jane showed Eve she had $\frac{180}{240} = \frac{z}{x}$ followed by $180x = 240z$. Poor

communication is evident in the following dialog. Eve and Jane had trouble discerning to which of the two proportions the other was referring; this misunderstanding created confusion in their communication. Additionally, Eve and Jane were not precise in their language, saying “z x” when they meant “z divided by x.” Surprisingly, “z x” was not a problem for Jane or Eve; in fact, they were innocuous to it and only Abby acknowledged the inaccuracy. A major hindrance in this episode was the inability to “see” what the others were talking about; with no common workspace they were really just “guessing” they understood. They did a lot of talking past each other.

Jane: [to Abby] We are going to isolate the x.

Abby: Ok

Jane: We have $240z$ divided by 180 so we can get x by itself and then take in to the other equation....we are doing 100 over 240 .

Eve: This is 180 over 240 ?

Jane: Oh, 180 over 240 —I wrote 100 —it's 180 though-----so it's going to be 180 —like that?

Eve: Yeah, 180 to 240 and then z to x . Is that it?

Jane: 240 minus x ? Wait a minute—is it 240 minus x ?

Eve: Yeah—yes, it is 240 minus x .

Jane: So it's 180 over 240 equals 240 minus x over z ?

Eve: It's 180 over 240 equals z x —Right?

[Abby began to say something but backed down.]

Jane: z x ?

LBC: z x ?

Abby: [realizing Eve's mistake] Like z over x .

Eve: Cause ER is z and then-- RD is x .

Jane: Oh yeah, 180 over 240 equals z x —you are right—OK.

Eve: And then 100 over 240 minus x —is it 240 ?

Jane: 100 over 240

Abby: Yeah, 100 over 240 and then—

Eve: Yeah! Equals z --

Abby: [completing Eve's statement] And then the z over 240 minus x .

Eve: 100 over—

Jane: The z over—

Eve: z over 240 minus x ?

Abby: Yeah—and I wrote that. I don't know if that is going to go anywhere—but I wrote it just to remember what triangles or whatever—cause I drew the little triangles up above but I don't necessarily know whether it's going to go to anything.

Part of the problem in the above episode was that Eve had the proportions and was trying to solve whereas Jane was still trying to obtain the second proportion. Abby, lacking confidence in herself, offered little help. Finally, however, they solved $\frac{180}{240} = \frac{z}{x}$ and got $x = \frac{240z}{180}$. Even though Jane said in the dialogue above they would “get x by itself and then take it to the other equation,” she did not use the other equation but instead wrote at the bottom of her page 6 the equation: $\frac{180}{240} = \frac{z}{\left(\frac{240z}{180}\right)}$. When Jane read this equation aloud, Eve apparently listened because she pointed to Jane’s paper and commented, “I thought x would be like four thirds—it’s like if you reduced it.” Jane, agreeing, said, “Oh yeah, if you reduce it. I didn’t reduce it.” Jane changed $\frac{240z}{180}$ to $\frac{4z}{3}$. Unfortunately, both Jane and Eve substituted $\frac{4}{3}z$ for x in the proportion $\frac{180}{240} = \frac{z}{x}$ rather than into the proportion $\frac{100}{240} = \frac{z}{240-x}$. Thus, Eve and Jane wrote $\frac{180}{240} = \frac{z}{\frac{4}{3}z}$, a statement that is an identity and of no use in obtaining the value for z. Unwittingly, Jane and Eve continued their symbolic manipulations. Jane had trouble with the fraction.

Jane: You got three-fourths, right? Cause you had to do the reciprocal.

[Eve was writing but Jane’s question got her attention; as if making sense of what Jane had said, Eve hesitated before she commented.]

Eve: So you just did the reciprocal first?

Jane: I’m not sure—give me a minute—I haven’t done any of this in so long.

As I looked at the tapes and watched the subsequent interactions, I inferred Eve’s comment was more of a statement than a question. Eve understood they had at least two options: one

option was to “cross-multiply”; the other option was to multiply both sides by the reciprocal of $\frac{4}{3}z$. Eve was “cross multiplying” but when Jane asked her question about the reciprocal, Eve assumed Jane was using the second option. Instead, it seems Jane was having trouble multiplying fractions, confusing the procedure to multiply fractions with the procedure for dividing by a fraction. The opportunity to resolve the misunderstanding was lost when Jane tried to save face and Eve showed no further interest in Jane’s question. Although Jane, through her errors, obtained a value of 1.78 for z , she never reported this value. Eve finished her work and obtained $240z = 240z$. Her work is shown in Figure 23.

$$\frac{180}{240} = \frac{z}{\frac{4}{3}z}$$

$$180\left(\frac{4}{3}z\right) = 240z$$

$$60(4z) = 240z$$

$$240z = \frac{240z}{240}$$

$$z = 1$$

Figure 23: Eve solves the equation.

When Eve obtained $240z = 240z$, she was uncertain of its meaning and said, “I think maybe I’m at a dead end—I don’t know.” I asked her what $240z = 240z$ implied and she

tentatively said, “z is 1?” When I asked her why she concluded z is 1, she explained “240 over 240, I guess...I don’t know.” Neither Jane nor Abby commented on Eve’s conclusion so the learning opportunity to realize $240z = 240z$ is an identity and why she obtained this identity was missed.

It was obvious Jane was becoming disinterested with the problem. She left the room to get a tissue and when she returned she was less involved. While Jane was gone, Eve solved the equation again and once again obtained $240z = 240z$. Eve reported she was still getting $240z = 240z$ but wondered if it would be one or zero. She still did not recognize this as an identity, and I did not explain this to her. When Jane returned to the room, Eve explained she had gotten $240z = 240z$ and wondered if it would be one or zero. Jane quickly replied zero but Eve questioned, “zero?” Eve’s doubt prompted Jane to reconsider.

Jane: Well, no wait a minute—it would be zero because if you take it over and subtract it, it would be zero.

Eve, accepting Jane’s explanation, said, “...so it would be zero.” I asked Eve to tell me how she got zero. She explained “that [z] cancels out so that’s zero, zero.” Abby pointed out, however, “z couldn’t be zero anyway cause we know it’s a distance—we just don’t know the distance.” Eve concluded, “So that doesn’t work.”

Eve was willing to accept Jane’s explanation that z should be zero. It was not until Abby pointed out that z was a distance that Eve acknowledged “that doesn’t work.” Although Abby’s symbolic manipulation skills were weak, her ability to relate variable expressions to their physical representations was quite strong.

This last procedure also did not work. It was unfortunate they did not try to determine why the procedure did not work. They did not try to make sense of what they were doing so they could figure out why they were having trouble. When the last procedure did not work, they spent seven minutes without talking. Eve and Jane wrote, but Abby only sporadically glanced at

Eve's work. These glances were brief, and she never leaned over into Eve's workspace. This

time Eve took the proportion $\frac{100}{240} = \frac{z}{240-x}$, solved for z , and substituted this expression for z

into the same proportion—again obtaining an identity although the symbolic manipulations were much more complicated. Eve's work is shown in Figure 24.

$$\frac{100}{240} = \frac{z}{240-x}$$

$$240z = 100(240-x)$$

$$\frac{240z}{240} = \frac{24000 - 100x}{240}$$

$$z = 100 - \frac{100x}{240}$$

$$\frac{100}{240} = \frac{24000 - 100x}{240} \cdot \frac{1}{240-x}$$

$$\frac{100}{240} = \frac{24000 - 100x}{240(240-x)}$$

$$100(57600 - 240x) = 240(24000 - 100x)$$

$$5760000 - 24000x = 5760000 - 24000x$$

$$5760000 - 5760000 - 24000x = -24000x$$

Figure 24. Eve's calculations with the proportion $\frac{100}{240} = \frac{z}{240-x}$.

Eve's work shows that she had forgotten how to use substitution to solve two equations with two unknowns. The procedure of solving a system of two equations with two unknowns was certainly one with which Eve should have been familiar. At first I believed it was the format of the two equations that gave Eve trouble—after all, the two equations were not set up as they are traditionally set up in textbooks. In Session 4, though, Sara asked a question that led me to believe it was the physical representation that forced Eve to limit herself to the single proportion only.

Eve did all of the work shown in Figure 24 during the seven minutes of no talking. Realizing that $-24000x = -24000x$ was of no more help than her other conclusions, she asked a question not unlike those I had been asking.

Eve: What are you doing, Jane?

Jane: I have no idea.

Eve: What are you thinking?

Jane: I'm at a dead end. Cause I took the whole length of BD and put it proportionally to BE—and then did 240 over 240 minus x which would be the same on the opposite side and then—tried to solve for x—and then I don't know—I'm just trying anything—but I know it didn't work.

Eve: Abby?

Whereas Eve had trouble with a procedure, Jane's trouble stemmed from a conceptual misunderstanding—specifically, Jane did not understand the proportionality of the sides of the triangles. Her work is shown in Figure 25.

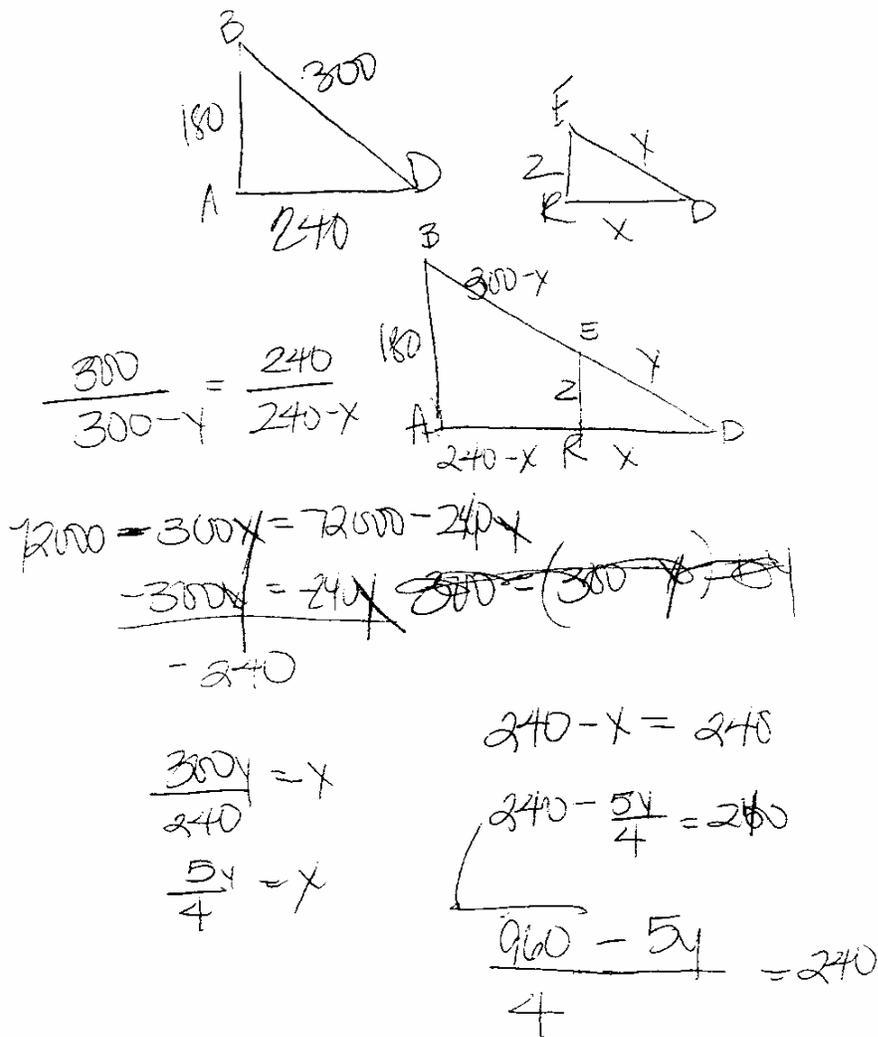


Figure 25. Jane's attempt at expressing proportional sides.

Jane made at least two conceptual errors in the work above. First, her statement of proportionality was incorrect. Subsequent work also confirms Jane did not understand the concept of proportionality. Secondly, she illogically stated in the figure above that $240 - x = 240$. These errors lend support to my observation that Jane did not connect the mathematical symbols to the physical situation.

Jane further explained that she knew the procedure did not work because she “can’t get down to a zero or one.” Although I cannot be certain, perhaps she was relating her comment to the “zero” and “one” discussed when Eve had the identity. Nevertheless, Eve said to Jane, “So you went and you did the proportion thing—and you got zero.” Curiously, Jane replied, “Uh huh,” contradicting her earlier statement. I then asked her to show me where she got zero. Again, she tried to save face.

Jane: I don’t even think I got zero—I just tried something and it didn’t work at all—I don’t even know why I did it—I just, for lack of a better—

We concluded the session with the problem still unfinished. I wrote in my field notes that I should have stopped the session sooner than I did. Jane was particularly unmotivated at the end; in her defense, however, she may not have felt well. Although Abby’s written work was minimal, she made several good observations: she recognized how problem 53 was limited in its usefulness to the Buried Treasure Problem; she also determined the correct proportions that would have led to a solution had there been no procedural errors; and she recognized why the value of z could not be zero. Procedural errors hindered Eve’s progress. All of these problems were compounded by their poor communication skills, having no common workspace, and lacking a sense-making perspective.

Session 4

Eve did not attend this session and did not let me know she would be absent. Sara was back. There was a noticeable tension in this session. They were certainly frustrated because the problem was still unsolved. Jane was very sleepy and later admitted during the final interview she had trouble staying focused on the problem. Abby appeared very uncomfortable. She and Eve seemed to have developed camaraderie during Session 3 that she missed with Eve’s absence. Too, Abby may have been intimidated by Sara. Abby was very reluctant to share the work they had done in Session 3. Sara, having missed the third session, first considered the work she did in the second session but struggled with the concept of

proportionality. Only with my insistence did Abby finally show Sara what was done in Session 3. I really believe it was only because I pushed them along during this session that they finally solved the problem.

At the end of the third session, I saw that Jane had trouble with the concept of proportionality. Not only did Jane struggle with the concept of proportionality, but Sara, too, had some misconceptions that became evident as the session progressed. At the beginning of Session 4, Sara looked at the work she did in Session 2 and said, “I was trying to figure out if the ratios were of these smaller triangles inside of the bigger triangles—but I think we did the big triangles.”

I was not sure they understood why proportionality was even an appropriate concept in this problem. In sessions 2 and 3 Jane explained segments were proportional “cause they are parallel.” Abby explained that one triangle was just a smaller version of another triangle. These explanations were not mathematically appropriate so I felt an obligation to be sure they understood why the sides of the triangles were proportional. By drawing and cutting out triangles, I was eventually convinced they understood for two triangles to have sides that are proportional, it must be known the angles of the triangles are congruent.

Having established this conjecture, Sara looked at the triangles ACD and ABC she had drawn in Session 2 (see Figure 26). She claimed they are not proportional because the “angle of elevations” were different. This realization made her question whether they should proceed with using ratios.

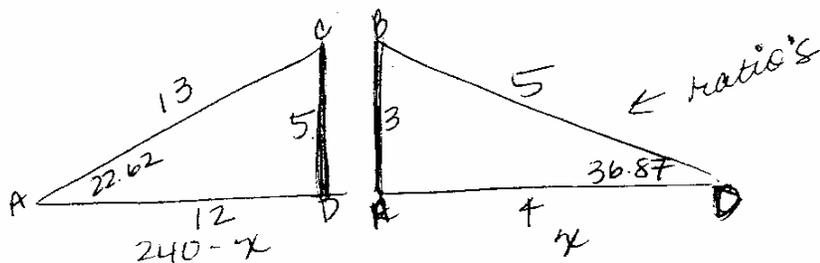


Figure 26. Sara's drawing of triangles ADC and ABD from Session 2.

Sara: So is it not a good idea to work—like is the ratio thing not a good idea seeing as how they aren't—they don't have the same angles? [Sara looked up at me when she asked this.]

LBC: Well, these two triangles are not—their sides are not in proportion, but do you have triangles whose sides are in proportion?

Sara: Uh huh.

Although Sara acknowledged there were triangles with sides that are proportional, neither she nor the others identified these triangles. Knowing they did considerable work with proportions in Session 3, I suggested Abby and Jane look at their work from the previous week; I even pointed to Eve's work. Strangely, neither Jane nor Abby picked up Eve's work; in fact, both Jane and Abby just sat—writing nothing and saying nothing! Although Sara had said the sides of triangles ACD and BAD were not proportional, her next question showed she did not yet understand what this meant.

Sara: So can you say that---240—240 is this entire length right here—can you compare this triangle to this triangle in terms of this proportion to that proportion? [Sara pointed to the triangles in Figure 26 above and looked at me as she asked this.]

LBC: Are they proportional?

Sara: They are not proportional but—[pause]

LBC: So if they are not proportional why would it make sense to set up ratios between them?

Do you have triangles that are proportional?

Sara: Uh huh—the small ones within the larger ones.

My language in the above dialogue was not accurate. Rather than “triangles that are proportional,” I should have said “triangles whose sides are proportional.” There was a period of silence during which time Sara drew the triangles shown in Figure 27 below. Her initial drawing had $240 - x$ rather than 240 on the triangle on the left; her initial drawing also did not include the dashed segment and the length of $240 - x$ on the triangle on the right.

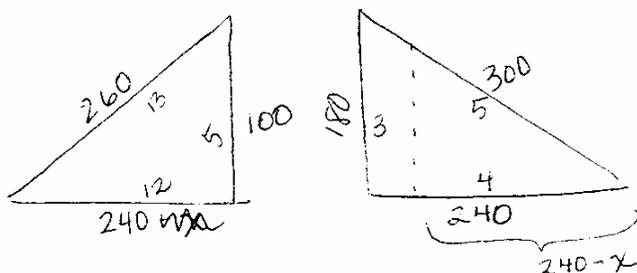


Figure 27. Sara draws triangles ACD and ABD.

Adjacent to the triangles Sara wrote the proportion $\frac{5}{100} = \frac{12}{240-x}$; she looked perplexed when she obtained $1200 - 5x = 1200$. A learning opportunity to make sense of her work was missed for Sara when I decided to see if I could point them toward the work done in Session 3. They had made some progress during Session 3, and I selfishly did not want their efforts to be wasted. Frankly, I was growing tired of the problem just as they were. Interestingly, Jane and Abby were very reluctant to share this work with Sara. Their reluctance may have been an indication of their not fully understanding the work that was done. Finally, however, I pushed them to share.

LBC: Abby, share what we did last week—to kind of be a springboard for that. What was being done last week?—and Jane you worked on the same concepts last week, too. What was going on last week? Why did ya'll make the decisions you made?

Abby: I—well—I mean we tried stuff but we kind of tried stuff but—that we had tried before but—I don't know—to see if---

Neither Jane nor Sara appeared to be listening. Sara was still puzzled by her statement $1200 - 5x = 1200$. I encouraged Abby to talk about “the stuff” that was tried, hoping she would explain how the proportions were set up. Instead, she referred to the failures they had with the procedure.

Abby: Uhm—when she went and tried to do that 240 minus x and then the x and tried to do that and then uhm we tried a different way like if we did it uhm—that way—

I had to encourage Abby to continue with her explanation given below. As she explained she drew the triangles shown in Figure 28 and confidently explained how she was representing the lengths. (At this time Abby did not write 3, 4, and 5 on triangle ABD and did not write 5, 12, and 13 on triangle ACD; these numerals were added to the drawing later.) Sara and Jane both attended to what Abby was saying and doing but did not get in Abby's personal space to have a better vantage point.

Abby: Cause that's 180—that part right there and it is proportional to that little part—it's like-- that's 180 and that's 240—but ...and then that was z and then that was x. And then for the other one—that was 100 and that was 240 and then-----that was z and that was 240 minus x.

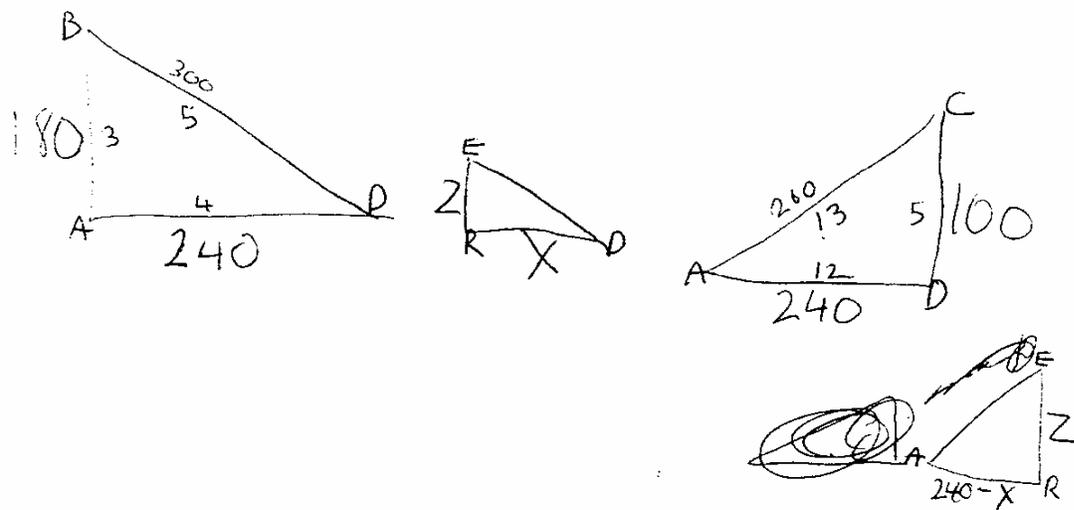


Figure 28. Abby's drawing of the triangles.

I pointed to triangles ABD and RED and said “so Abby's claim is that this triangle and this one right here...are proportional.” [I later realized my carelessness in speaking—it is the sides that are proportional, not the triangles.] I also pointed to triangles ADC and ARE and said Abby claimed these were also proportional. Sara confidently exclaimed, “Uh huh! I agree with that.” I pointed out the angles of the triangles in the two sets were equal and explained these triangles are termed “similar.” [When the Buried Treasure Problem was completed, I cut out quadrilaterals to help them see that congruent angle measures in any two polygons is not enough to guarantee that the sides of a polygon are proportional; I explained, however, congruent angle measures in triangles is sufficient to conclude the sides are proportional.]

Abby had two sets of similar triangles drawn on her paper. Their vertices were labeled and each side was labeled with its length or a representation of its length. I asked if they believed the sides of these triangles would be in proportion, and Sara correctly pointed out on Abby's paper “the smaller triangles within the larger ones but not this one [ABD] to this one [ADC].”

I expected them to set up correct proportions. After all, in Session 3 Abby had determined two correct proportions, and certainly the groundwork was carefully laid for Sara to set up the proportions. Unfortunately, progress was again waylaid as misconceptions about proportionality surfaced. I cannot be certain why Abby was reluctant to share the proportions she confidently volunteered in Session 3. Perhaps she was intimidated by Sara; perhaps, too, she was unsure about the proportions in light of the difficulty they had in Session 3 with using those proportions.

It was now that Sara drew the dashed segment in the triangle on the right in Figure 27 and also designated the length $240 - x$ in this triangle. Using this drawing, she wrote the proportion $\frac{240 - x}{4} = \frac{300}{5}$ as she said, “This length right here would be 240 minus x and then the ratio of this 240 minus x over the ratio of its length would be 4—set that equal to--goes like this—maybe? And then you could solve for x like that and then find the smaller length which [unclear].”

Abby, noticing that Sara had 3,4,5 and 5,12,13 on her drawing, wrote these numerals on her triangles shown in Figure 28 above. Sara asked Abby, “Do you see where I got that from?” [that is, the proportion $\frac{240 - x}{4} = \frac{300}{5}$]. Abby claimed to understand but when I asked her to explain why she added the numerals to her drawing, she stammered, “I wrote that just because—but like I’ve got the—that part right there, where the 4—I see where the 4 and 5 came from—“ Abby was unable to explain because she had merely copied from Sara’s paper. Her fear of being wrong kept her from volunteering the proportions she set up during the previous session.

Even though Abby claimed she understood, Sara realized the proportion she wrote was incorrect. In the following quote she explained why it must be wrong and suggested how to rectify the problem.

Sara: I don't think that makes sense because it's now giving me x equals zero...maybe I should just put x ...both these sides are 240 though. I wonder if it would work if just using....would it work just using x ?

To rectify the problem she arbitrarily changed $240 - x$ to x in the proportion $\frac{240 - x}{4} = \frac{300}{5}$ to

obtain $\frac{x}{4} = \frac{300}{5}$. She also changed $240 - x$ to 240 in the triangle on the left to obtain the

drawing in Figure 27. These changes were completely arbitrary and made without

mathematical justification. Concerned about these changes, I asked, "Where are the triangles

that are similar to each other? What triangles are similar?" Although Sara answered, "You

have to do the smaller ones within the bigger ones," she seemed unsure of how to use this idea.

They appeared to be making no progress so I pointed to Abby's drawing in Figure 28 above and

suggested they use the idea Abby's drawings represented. Sara took Abby's paper and looked

at her drawing. She then returned Abby's paper and then drew at the top of a new page the

triangles shown in Figure 29. When Sara first drew this figure, she did not name the vertices

and the segment RD was labeled as $240 - x$ and later changed to x .

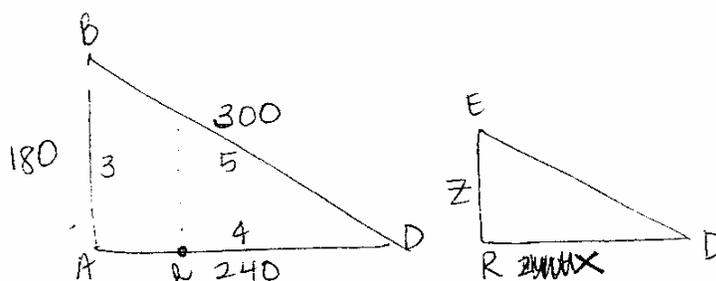


Figure 29. Sara's drawing of triangles ABC and RED.

Again there was silence so I pointed to Abby's paper and suggested they look at the work that was done in the previous session. Specifically I pointed to the proportions and the work Abby did in Session 3 (shown in Figure 30).

$$\begin{array}{l} \triangle \\ \frac{180}{240} = \frac{z}{x} \\ 180x = 240z \\ x = \frac{240z}{180} \\ x = \frac{4}{3}z \end{array}$$

$$\begin{array}{l} \triangle \\ \frac{100}{240} = \frac{z}{240-x} \\ \text{[Crossed out]} \\ \text{[Crossed out]} \end{array}$$

Figure 30: Abby's work from Session 3.

Sara reached for Abby's work, compared the work to her own, and read from Abby's paper, "x is four-thirds of z." Sara returned the paper to Abby after 25 seconds but then leaned across to read from Abby's paper again. Strangely, Abby did not hand her paper to Sara but instead made an excuse.

Abby: That part somebody else was working it out and I was trying to write it and keep up but after that part I stopped writing and I was listening and I don't remember.

Abby apparently was afraid the proportions $\frac{180}{240} = \frac{z}{x}$ and $\frac{100}{240} = \frac{z}{240-x}$ were not correct. These were proportions she had suggested in Session 3 and yet today she attributed them to someone else and trivialized her own contribution. I believe Abby was trying to save face in front of Sara.

After looking at Abby's work, Sara first wrote $\frac{240}{x} = \frac{180}{z}$ but immediately marked through this proportion and wrote $\frac{180}{240} = \frac{z}{x}$. When I asked her why she discarded the first proportion, she replied, "Because--I think I just set it up wrong." Because the proportions are equivalent, Sara apparently had some confusion about the concept of proportionality. I missed a learning opportunity by not helping Sara understand the two proportions she wrote were equivalent.

Furthermore, although Sara wrote $\frac{180}{240} = \frac{z}{x}$, Abby's work apparently influenced her writing of this proportion. On Sara's original drawing (as explained above), x 's representation was not identified; instead $240 - x$ was the only variable expression shown. Later she changed $240 - x$ to x to obtain the drawing shown in Figure 29. Frankly, I did not realize the discrepancy between her drawing and Abby's. (When Sara began her symbolic manipulations later in the session, she used the contradictory statement, $\frac{z}{240 - x} = \frac{180}{240}$. This proportion was probably determined by her original drawing that had $240 - x$ instead of x .)

Sara then drew the triangles in Figure 31 below but labeled point C as point E. She wrote the proportion $\frac{100}{240} = \frac{z}{240 - x}$.

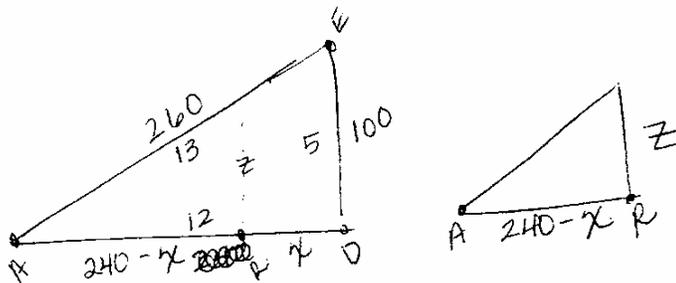


Figure 31: Sara's drawing of triangles ACD and AER.

During the session I believed Sara now had the same proportions the group had in Session 3. I

saw her write $\frac{180}{240} = \frac{z}{x}$ and $\frac{100}{240} = \frac{z}{240-x}$. Neither Abby nor Jane made the task of getting

these proportions easy. Jane probably did not understand the work from Session 3, but it was troubling that Abby was so unwilling to volunteer any information. Again, I believed she was fearful her work was wrong. Armed with the two proportions, Sara now explained what the plan would be.

Sara: We are going to set up two equations for each of the two triangles, solve for z on both of them which should give me an answer with a variable of x in there and then set those two together and then solve for x .

The proportions were already established so she had the two equations. Because Sara struggled with expressing physical relationships symbolically, it appeared she did not understand a proportion was a symbolic way to state a physical relationship. The next exchange not only showed this struggle but perhaps also explained Eve's trouble in Session 3 with solving a system of two equations and two unknowns.

Sara: But can we do that since the triangles aren't proportioned?

LBC: Do you mean because you've got two different sets of proportions?

Sara: Yeah.

LBC: But at that point are you saying anything about those triangles being proportional?

Sara: I guess we are just solving-----no, I don't think so.

Because one proportion made a statement about one set of similar triangles and the other proportion made a statement about a different set of similar triangles, Sara was unsure about using information gleaned from one proportion in the other proportion. Sara's question perhaps explained Eve's difficulty with solving the system of equations in Session 3. Recall that even though Eve had two proportions, she used only one of them. The physical representation perhaps interfered with Eve's ability to solve the system.

Sara's question raised a learning opportunity that was reduced to resolving "how" rather than "why." Although I answered her question with a question, it did not initiate exploration; instead she assumed she could proceed with the symbolic manipulations. I asked Abby if she could share the work that was done the previous week. She tried to save face by claiming she did "not work it out like that...stopped writing and started listening...didn't know so much how to put it together like that."

Sara was understandably frustrated. She worked for twenty seconds while Abby and Jane watched. Jane was so sleepy at one point it looked as if she would fall from the chair. Sara's frustration and my effort to encourage participation were evident in the following exchange. Also evident was Abby's and Jane's lack of obligation to the problem solving.

Sara: Does anybody agree or disagree that what I'm doing is going to help at all or any other ideas?

[10 seconds of no response]

LBC: What do you need for them to do? What would you like for them to do, Sara?

Sara: Just respond.

[Abby giggled but still no other response—20 seconds of silence.]

LBC: What do you think?

[pause]

LBC: Do ya'll trust her arithmetic?

Sara: It's been a while.

[Abby giggled.]

Abby: That makes sense to do that, though.

Jane: I trust her a heck of a lot more than I trust myself right now.

There was silence for 75 seconds while Sara worked. It was amazing that neither Abby nor Jane felt any obligation to help. Finally Sara fell back in her chair with a frustrated look on her face and announced, "x equals 240." Jane and Abby have offered Sara no help, but they immediately realized x could not be 240. They were not motivated, however, to reexamine the thought process used to reach " $x = 240$." It was easier to place the blame elsewhere.

Jane: [interrupting] That would reiterate like everything that we've done that doesn't work.

Abby: I know last week everything—cause she [Eve] did two or three things and everything equaled zero and I know you [Jane] tried two or three different things and everything equaled zero—like everything both of ya'll did equaled zero—

Sara had been requesting help from Jane and Abby. I had also encouraged them to get involved. Jane was not involved because she not only was very sleepy but I believed her understanding of the work done in the previous session was very limited. Abby, however, was intimidated by Sara; her fear of being wrong prevented her from sharing. However, when Sara got the answer of 240, Abby and Jane did not doubt Sara's work; they, in fact, showed their support by faulting the problem.

Admittedly, I was surprised Sara got $x = 240$. I believed her proportions were correct, and I had confidence in Sara's manipulation skills but something was wrong.

LBC: Alright, so do you have some faith in what you did?

Jane: yeah—

LBC: At what point—at what point? Where do you feel confident?

Abby: It looks right—to me that looks good—that looks like—

LBC: Alright, what looks good?

Abby: The idea of doing that—and how it looks—and it's just when you get the answer zero—cause you know it can't be zero—I mean you know it can't be zero.

Again Abby and Jane did not question whether the work that was done was reasonable or mathematically correct; Abby's claim that "it looks right" was even questionable as there had been no common workspace so her ability to see everything was doubtful. Whereas Jane and Abby expressed their approval of the work Sara did, Sara studied her work to discover what had happened. It was then that I saw she used the incorrect proportion $\frac{z}{240-x} = \frac{180}{240}$ rather than

$\frac{z}{x} = \frac{180}{240}$. Her work is shown in Figure 32.

$$\frac{100}{240} = \frac{z}{240-x}$$

$$100(240-x) = 240z$$

$$\frac{24000 - 100x}{240} = 240z$$

$$\frac{24000 - 100x}{240} = z$$

$$\frac{z}{240-x} = \frac{180}{240}$$

$$240z = 180(240-x)$$

$$240z = 43200 - 180x$$

$$z = \frac{43200 - 180x}{240}$$

$$\frac{43200 - 180x}{240} = \frac{24000 - 100x}{240}$$

$$43200 - 180x = 24000 - 100x$$

$$19,200 = 80x$$

$$x = 240$$

Figure 32. Sara uses incorrect proportions to represent the situation and obtains $x = 240$.

I could not understand why Sara's proportions were not correct; after all, I saw her write the correct ones. As I reflected on the session, I concluded Sara apparently wrote the correct proportions $\frac{180}{240} = \frac{z}{x}$ and $\frac{100}{240} = \frac{z}{240-x}$ after seeing these on Abby's paper. Unfortunately, though, she did not make sense of the connection between these proportions and the sets of similar triangles. No mathematical justification was given as to how the proportions were determined so Sara missed an opportunity to deal with her misconception. Because I saw Sara write the correct proportions, I incorrectly assumed she understood their physical representation. When she started the symbolic manipulations shown in Figure 32 above, she used her drawings to get these proportions rather than using the proportions she had already written. Unfortunately, her drawings had RD and AR both labeled as $240 - x$. I pointed out the contradictory proportions to Sara.

LBC: Alright let me ask you this: Why up here do you have this [pointing to $\frac{180}{240} = \frac{z}{x}$] and

why down here do you have this [pointing to $\frac{z}{240-x} = \frac{180}{240}$]?

[pause]

Sara: Well-----[looking at her paper]-----I think I might have just named it wrong.

Although Sara claimed she "just named it wrong," she did not sound convinced. In fact, as I worked with Sara through the remainder of this session and in the final interview, I realized that even though we spent time in Session 2 working on variable representation, Sara still had misconceptions. She had segments RD and AR both labeled as $240 - x$. In the next exchange Sara's confusion about this concept can be seen.

LBC: Alright, if you said that RD is $240 - x$, then what would x have to be?

Sara: uhm

Jane: less than 240

[There was silence, and Sara frowned while she studied her work.]

LBC: Why did you go back and put 240 minus x right here when you had x right there?

[Sara shook her head as if wrestling with why.]

Sara: Just because I know it's the-----the length is going to have to be 240 minus some unknown length which is x -----now that I'm looking at it, maybe it should just be x —cause x is the unknown length.

It was not until the final interview that I fully understood Sara's trouble with variable representation. During Session 4 I believed the concept of proportionality was most problematic for her, but her final interview showed the concept of variable representation was her real nemesis.

I spent five minutes helping Sara label the vertices of the triangles and changing the length of RD to x on the triangles in Figures 31 and 33. As I talked to Sara, Abby was listening and writing. My conversation with Sara must have validated Abby's thinking because she redrew the triangles she had in Figure 28 above and wrote the proportions expressing the correct relationships between the two sets of similar triangles. The expression on her face indicated her confidence in what she was doing. Abby explained in the final interview how she realized that often the ideas she had were correct, but she was fearful of sharing them.

After getting Sara's drawings corrected, I asked her to look at her proportions. I hoped she would realize why the proportions she used in Figure 32 were incorrect. As Sara studied her work, Abby leaned toward Sara's paper as if she had something to say. Finally after 35 seconds Abby pointed out the mistake. She was still somewhat timid about confronting Sara because her explanation was in the form of a question; nevertheless, her willingness to point out the error (without my encouragement) revealed her confidence in her mathematical thinking.

Abby: uhm—in one of them, though, like would that—would that one just be 100 over 240 is that, but the 180 over 240 wouldn't it just be z over x ? Would the 240 still be there?

Sara: No. [said confidently]

LBC: Why would it not be?

Sara: Because you're saying that 180 to 240 is z to x and so this is not right.

LBC: Does that make sense?

Sara: Little bit.

I should have probed deeper because "little bit" did not convey a real understanding. It was during the final interview that I realized Sara did not understand why the ratio of 180 to 240 should be z to x. I missed this learning opportunity in the problem-solving session although I had the opportunity during the final interview to address her misconception.

Jane interrupted the flow of conversation by referring to the work she did earlier in the session. Her tendency to throw out irrelevant ideas at inappropriate times was her way of feigning involvement.

Jane: Well, if it makes you feel any better, have you tried working it like 47?

Jane tried to pattern the Buried Treasure Problem after problem 47 in the textbook. Her work is shown in Figure 33 below.

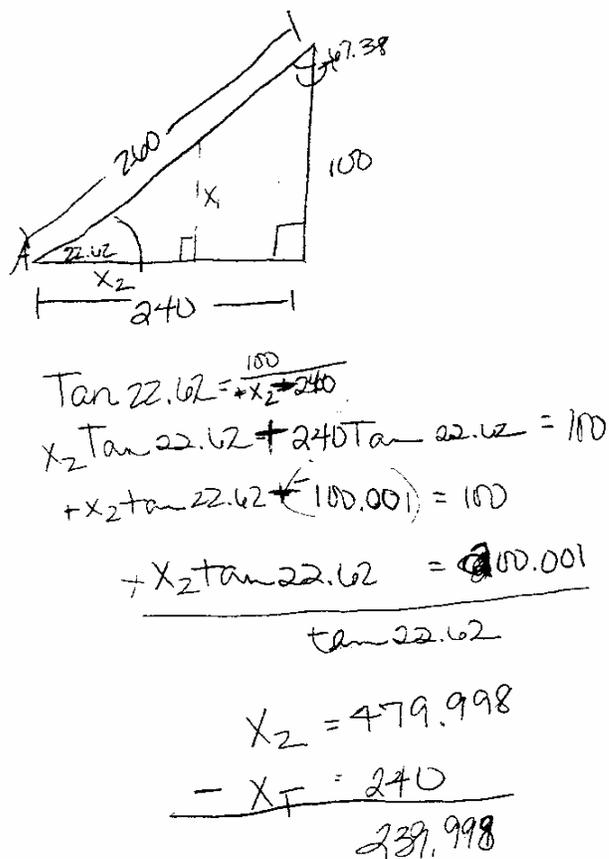


Figure 33. Jane commits vandalism again.

This work is evidence of Jane's tendency to commit vandalism with the mathematics in her quest for an answer. Although Sara and Eve looked briefly at Jane's work and made several comments, they were eager to return to their own work. Although Jane told me she understood their comments, I was certain she was merely saving face. I explained why her initial statement was incorrect, but she made no response and gave no indication she understood. Jane's tendency to save face prevented her from confronting her misconceptions.

For 2 ½ minutes Sara worked to solve for x . There was no discussion of the approach Sara was using. She did not explain she was using the same approach used earlier but with the corrected proportions. Abby discreetly copied from Sara's paper. Sara and Abby both solved for

z in the proportions $\frac{z}{x} = \frac{180}{240}$ and $\frac{100}{240} = \frac{z}{240-x}$. Obtaining $z = \frac{180x}{240}$ and $z = \frac{24000-100x}{240}$,

they equated these two expressions for z and determined $x = 85.7$. Sara's work is in Figure 34 and Abby's work is in Figure 35.

$$\frac{24000-100x}{240} = \frac{180x}{240}$$

$$24000-100x = 180x \quad \frac{z}{x} = \frac{180}{240}$$

$$24000 = 280x \quad 180x = 240z$$

$$E \quad x = 85.7 \quad \frac{z}{240} = \frac{180x}{240}$$

$$\frac{100}{240} = \frac{z}{240-x}$$

$$240z = 100(240-x)$$

$$240z = 24000 - 100x$$

$$z = \frac{24000-100x}{240}$$

Figure 34. Sara's solution.

$$\frac{z}{x} = \frac{180}{240}$$

$$\frac{180x}{240} = 240z \qquad \frac{100}{240} = \frac{z}{240-x}$$

$$z = \frac{180x}{240} \qquad 240z = 100(240-x)$$

$$\qquad \qquad \qquad 240z = 24000 - 100x$$

$$\frac{180x}{240} = \frac{24000 - 100x}{240} \qquad z = \frac{24000 - 100x}{240}$$

$$24000 - 100x = 180x$$

$$24000 = 280x$$

$$x = 85.7$$

Figure 35. Abby's solution.

Particularly telling that Abby copied is noticing that each moved from the equality of $\frac{180x}{240}$ and

$\frac{24000 - 100x}{240}$ to the conclusion that the numerators of these expressions are equivalent.

When Sara announced she had x equals 85.7, Jane cryptically said, "That's better than zero." There was a brief silence so I asked if they were finished.

Abby: No, you've got to figure—

Sara: [interrupting] Ok, so then--I just don't know if maybe—I just don't want to go ahead and not have my math be right.

Sara wanted verification. Abby reached for a calculator and said, “Let’s see.” Apparently Abby only divided 24000 by 280 to verify the answer would be 85.7. Getting the answer of 85.7, she compared it to Sara’s answer.

Sara: Ok, and if this—looking at the smaller triangle—I figure if you just set up a ra [sounds like she is going to say ratio] like a—

Jane: [interrupting] And do the tangent of the—well, couldn’t you? Since you know what that side is now.

Sara: You could or we could just set a ratio.

Jane: That works, too.

It was not clear whether Jane understood how to use the tangent or to use ratios. Sara drew the triangle shown in Figure 36 but without any explanation. Notice the introduction of the variable x for the missing length even though this length had been represented by z .

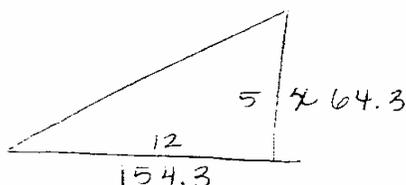


Figure 36. Sara’s drawing of the triangle to find the length of ER.

Sara began explaining how to set up the proportion but expressed doubt. Jane offered help but no justification was offered or requested.

Sara: It would be 12 to—it would be 12 to 154.3 equals 5 over x —or is it 12 to 5 equals—

Jane: 12 to 5.

Abby: yeah

Jane: 12 to 5 equals 154.3 over x .

Sara did not understand that $\frac{12}{154.3} = \frac{5}{x}$ and $\frac{12}{5} = \frac{154.3}{x}$ were equivalent. Instead, she

believed only one of the two was correct and was willing to accept Jane’s answer without any

justification. Earlier in the session Sara also struggled with the same situation. She did not, however, ask at either time for justification.

Accepting Jane's recommendation, Sara wrote the proportion $\frac{12}{5} = \frac{154.3}{x}$. Abby wrote the same proportion on her paper. When I pointed out "so now you are letting x be really what z was in the first problem," Sara said, "Right." Abby, though, changed the x to z in her proportion which may indicate Abby's understanding of the concept of variable representation. Eventually Sara announces, "So apparently this side [ER] is 64.3."

Sara looked at me and asked, "Is that right?" We were all weary of the problem so I chose not to insist they verify the answer. Instead, I told them the answer was correct, and then we talked a bit about the solution process and what hindered and what helped them.

Jane and Abby appeared not to appreciate the complexity of the problem. Jane claimed they had the idea the whole time but "just didn't know how to set it up." Abby said "it almost seemed like that was just too simple—that was the idea but that was too simple...cause all the other stuff we tried something harder and more—like that just seemed too simple to do." I asked Jane why she kept returning to the former problems in MATH 1113 and she explained **Jane:** I don't know—cause that's the way I was taught the whole way through. And I just figured that there had to have been a way like that—and I'm pretty sure there is but I just figured since the ratio—it just seemed way too easy to go back to the ratios—you know go back to the elementary geometry, geometric way of figuring out triangles and solving it.

Abby also observed they had the approach "written down just like the second day...and it almost seemed too simple to solve it that way." Thus, Jane's and Abby's assumptions about the problem hindered their progress.

Sara pointed out how the question about angle measures in overlapping triangles helped her think about setting up proportions. Although Eve asked the question, Sara attributed the question to Jane who immediately claimed credit.

Sara: The reason why I thought of it was when Jane had initially asked a question--on the second session she asked if the angle measures--like the smaller triangle within the bigger triangle if the angle measures were going to be the same [Jane begins talking at the same time] And it almost seemed like a silly question at first but then actually looking at the triangles to answer the question you see that—

LBC: When Eve asked that?

[Jane interrupted Sara.]

Jane: I don't know—it was one of us. I think we both had the same question—we had the same angle and it was cut in half and you still had the big triangle cut into two—they both had the same angle measurement so wouldn't they have a similar side and talking about them being parallel and having the same length.

[Jane's explanation did not make much sense and was not really what Sara was referencing.]

LBC: Ok, so that's what made you think about it when that question was posed?

Sara: Yeah. And I wouldn't have thought about it unless the question was asked and that ended up getting us to go into setting up the ratios... cause I think visually and when she asked the question, I looked at the triangles and I was like well looking at the triangles you can see that these lengths just keep extending at a steady rate—proportioned to the thing and I was like well the sides are going to remain proportioned and I was like—so that just kind of helped.

I asked, "What made it hard to work together?" At first there was no response, perhaps indicating their reluctance to share in front of the others. Finally Jane answered.

Jane: Being skeptical about what could work and not. It took a lot of assumptions to go ahead and go with the ratios. Because it sounded right with the ratios work but I haven't taken geometry in way too long. And after everything we did in trig, I don't really remember going over anything with ratios and the right triangles—I really don't remember. I remember you do ratios in class maybe once or twice but I didn't—it really didn't click until everyone started talking about

it. It's just being skeptical and then going back and trying something—trying something is better than nothing.

Jane was not confident with the ratios so she resisted suggestions about this approach. She was more confident with the trig and kept searching for a trig-based approach. Sara explained what made the problem solving easier with a group.

Sara: I think what made it easier for me was—I kind of get in a rut and I'll keep thinking about the same things I've already been thinking about over and over and over—hoping that something will pop out of it and then with having everyone else here they ask questions that make you kind of look at it from a different point of view—and it just kind of makes you think to ask yourself—yeah—and that's how that whole thing came in to play and had that question not been asked we might still be sitting here trying to do tangents and what not.

Although Sara would solicit their input and claimed that she benefited from the questions others would ask, she could be egocentric at times and tune the others out. Even when Eve asked the question that Sara found so helpful, Sara made no mention of the epiphany this question had for her.

Reflections and Final Interviews

Abby

Abby wrote in her reflection after the first session that she understood the problem after reading it “two or three times to really see what it was that I wanted to solve and find.” She also stated that she wondered if finding the angles and sides would be necessary but did not ask the question because she “felt that since the other girls agreed in what we were doing that we were doing the right thing.” Furthermore, she wished she had talked more during the session but “let Jane and Sara do all the talking [because she] felt like they knew more about what to do than [she] did.”

I identified Abby as the constant pupil. As the constant pupil, she made a concerted effort to hear every comment that was made; in fact, you could count on Abby's looking directly

at whoever was talking. However, she was unable to make sense of the mathematics because she was reluctant to ask questions or to make suggestions. Abby shared with me in the final interview that it was difficult for her to make a suggestion or to ask a question because she doesn't like to be wrong. She then clarified this statement by saying, "not because I don't like to be wrong but because I don't want people to know I'm wrong." She explained that "if somebody else was like 'oh, I've got the idea' instead of me trying to work it out I was trying to listen to what they were saying cause if I started working it I would have been still working when they were explaining so if they had an idea of what was going on I kind of...I listened to...what you know and then if it didn't work out I might have tried to do something else whatever." Abby's work during the session confirmed her tendency to listen rather than work. Her work was very sketchy and could not be used to verify that she understood the procedures that were done.

Although she was reluctant to make suggestions or to ask questions, in Session 2 Abby pointed out that one triangle was just a smaller version of another. She and I watched the clip where she explained, "that line right there just cut off...some part of it...that line cuts off.....that line just cuts off that part and then like this one over here that line just cuts off that part. But if you look at them...I don't know...they are smaller...this one is the smaller one of that and then that one is the smaller one of that." Abby had a hard time communicating her ideas, and as she and I watched the clip she admitted that it took her about two minutes to get up the courage to share her observation. I believe her courage to share this idea resulted from help she got from her work colleagues with whom she discussed the problem after the first session. I assumed they helped her see the relationship between the triangles because mentioning such an idea was unlike Abby. She talked more about how difficult it was for her to share her ideas:

even if I have an idea I don't trust myself enough...to say something to them or to...it's like even if I thought or was thinking what and then somebody said like five minutes after I thought or two seconds after...I didn't trust...and.....I didn't say anything...and a

lot...but I mean I didn't trust myself to say it...I didn't...I never...the only thing I think I ever said that I was thinking was the thing about the triangles...and I said I know this may sound stupid but and I said that...but...

Abby, I believe, wanted to be a contributing member of the group but was unwilling to risk being wrong or saying something that would be perceived as stupid by someone else. In other words, she had a saving face attitude and saved face by keeping quiet. In the final interview she shared that she tried to find something to say.

Abby: ... sometimes I was like...oh I hadn't said, I hadn't said very much or I feel like I am not saying enough, I'm not helping out enough...I would always try to think when we get started what can...what do I see that I can...think is important or something.

It was this need to save face that motivated her to seek help from her colleagues. Nevertheless, Abby's getting their help was encouraging to her, and she was eventually able to use this information to help the group set up the correct proportions. It became clear as the sessions progressed that Abby clearly understood the relationships between the sides of the similar triangles and could accurately set up the proportions. Abby's poor communication skills and her lack of self-confidence impeded her ability to share this information with the group in a timely and effective manner. Furthermore, she needed encouragement from the others to continue her explanation. Because the group lacked a sense-making perspective, they failed to discuss Abby's observation about one triangle being a smaller version of the other. As a result, the discussion was dropped, and the group missed the opportunity to benefit from a very powerful observation.

The little recorded work that Abby had was often the result of her discreetly copying from either Eve or Sara. Perhaps this, too, was a sign of the constant pupil—one who takes rather than gives. Ironically, I believed Abby's understanding of the concepts of proportionality and variable far surpassed those of the other group members although she had little confidence in her ability to perform procedures. This was probably due to her very weak algebra background.

As a result, she copied from the others. For example, when I asked Abby if she recalled how the group found the angle measures that were recorded on her paper from the sessions, she did not hesitate to admit that “one of the other girls wrote it down...I can’t remember what we did to do that.” Her admission confirmed my observations of her copying from her group members’ work.

During the sessions I believed Abby’s understanding of the concept of variable was better than that of the other three students. Social collaboration and perhaps egocentrism prevented her from initiating cognitive disagreement concerning their misconceptions about the concept of variable. However, I wanted to be sure Abby did understand why I kept asking about the variable. In the exchange below, Abby showed that she understood about variable representation. She was very confident in her explanation and showed the frustration she must have felt with her group’s persistence in misusing this concept.

LBC: One of the things we kept dealing with a lot was the whole concept of what is x . And I kept asking ya’ll, “What is x ? What is x ?” And Jane would say 240 minus x and 300 minus x or something. Did you understand why I kept asking what is x , what is x ?

Abby: Yeah...cause...well see and to me like when they would do stuff and they would....sigh...like they kept...they were saying...I think they were calling part of it...I think they were calling part of it 240...they were leaving out a part of it and I can’t remember what part of it it was...instead of part of it, like if that is x then that’s got to be whatever 240 minus x is...if that is 16 then that would be 240 minus 16...that’s what’s left...But they were leaving out...I think the whole time...or instead of calling it x ...like they were doing...it was something and I was like, “No.”

...

LBC: Does it make sense if you say this is 240 minus x , could this be 260 minus x ?

Abby: No...like it had to be a different...a different variable...it had to be something...and that was one thing...it couldn’t be the same...they were calling it x but it couldn’t...just like that...just

like that had to be z and then that had to be x and even over here just when I was writing I just made up like you know if that is z and that's x and everything had to be different letters.

LBC: And why is that?

Abby: Sigh...Cause if you say...if you call that x and you call that x it's like you are saying they are the same thing and they are not. Cause if...when you solve for x...you have to say they are both the same and they are not...you'd have to call them something different cause they are different...and...

LBC: And that x stands for?

Abby: what...just one...just one of the values...it can't...they are not the same, you can't call them both x because x can only...it's just one, just one value and then z is one value and then....

It was unfortunate that Abby did not use her understanding of the concept of variable to help the other girls understand this concept better. The misconceptions related to variable representation impeded the progress of the group, and Abby could have helped the group deal with this misconception. Abby's fear of being wrong in front of others prevented her from sharing her understanding. This fear, in fact, perhaps impeded Abby's progress in the group as she squelched her ideas.

Abby: ...to me, just...if you are doing something by yourself and you try it and it doesn't work, let's try something else...but if you tell, if you are working with somebody and you tell 'em your idea and it didn't (sic) work, you....I almost, I feel bad because I've told...cause I've told them something and we've tried it and we've kind of wasted time and it didn't work and then I feel like it makes me look like I don't know what I'm doing... If you are working by yourself, if you do something and it didn't work, you know it didn't work, you just pick something else and then if that didn't work it's no big deal cause it's just you...but ...with somebody else I feel bad if I tell them something and it didn't work and I feel like it makes me look like a... "she doesn't know what she is doing" ...you know?

Whereas group work can be a vehicle to get many ideas on the table, Abby's fear of being wrong caused her to squelch her ideas until she was sure that someone else in the group had the same idea. In the problem sessions, Abby would piggyback on someone's suggestion and claim that she was thinking the same thing (for example in Session 3). She, thus, saw the group work as a way to validate her own thinking even though her thinking was private.

Abby: You had somebody there that you could ask questions or...and almost the same thing if you were thinking something and they said it, they said it you felt reassured about what you were doing...you didn't just try it and say "Well, I don't know if..."....if they said something and somebody was like "Oh, yeah, I think that sounds good" and you had been thinking that... then there is two other people that think that too you know...that must have been a good idea, I must have been thinking on the right track and you know....

This view of how the group work helped spoke to the constant pupil in Abby. She did not say that the group afforded the opportunity for ideas to be generated but rather that a person's own thinking could be validated provided another person in the group had the same idea. Abby was there to learn and learning to her was being told how to think or receive confirmation that her own thinking (albeit private) was correct.

In the fourth session there was an episode where Jane and Abby were offering no response to Sara's requests for help, and it was obvious Sara was becoming quite frustrated. Abby and I watched this clip, and I asked her to share with me what she believed was going through Sara's mind. Abby's response showed her belief that social collaboration was important.

Abby: I think she wanted...she wanted somebody...and not so much she wanted somebody to say something that was (unintelligible)...she wanted somebody to say something that would make her feel better about that she was doing was right...like if we had said something "Ah, yeah..." you know I think that would be good and "we do this " and she would have felt better about what she was doing.

Because Abby wanted the group to validate her own thinking, she saw the lack of support given to Sara as a hindrance for Sara. Even superficial support just to make someone feel better was perceived by Abby to be helpful. Abby's habit of giggling when someone would make a comment was perhaps her way of offering support.

Abby's fear of being wrong was such a hindrance for her. She claimed that she did not trust herself in doing mathematics which caused her to keep her thoughts to herself.

Abby: To me...I didn't...I don't...I didn't think...I don't think the math part of it...like I drew the picture or whatever and I knew what we...but I don't necessarily trust myself with doing the math...I don't...even if I have an idea I don't trust myself enough...to say something to them or to...it's like even if I thought or was thinking what and then somebody said like five minutes after I thought or two seconds after...I didn't trust...and.....I didn't say anything...and a lot....some of the stuff especially like Sara would say something that I was thinking but I mean I didn't trust myself to say it...I didn't...I never...the only thing I think I ever said that I was thinking was the thing about the triangles...and I said I know this may sound stupid but and I said that.

Abby believed that one of the factors that helped the problem solving was the fact that they "all got along together...it was balanced out....for the people that were quiet there were people that would make us talk...[we] balanced each other out." Furthermore she said that she liked all the girls in the group. For Abby, getting along was important. The group work was effective because the problem was solved and everyone "got along."

Abby did not appreciate the complexity of the problem. During the final interview she said that she "didn't realize it would be as simple as it was." She said one of the hindrances to their solving the problem was assuming the problem "was going to be so hard." Instead Abby said "it was just very simple and not hard at all."

Jane

Jane's approach to mathematics was very procedural. For Jane there was no such thing as a novel problem. Her view of mathematics was limited to working problems similar to

problems she had already solved. This view explains her tenacity in looking for a model problem they could use to solve the Buried Treasure Problem. She focused on the strategies recently studied in MATH 1113 and failed to consider less recently studied strategies.

In MATH 1113 Jane resented my asking her to explain her thinking. In the final interview she mentioned how she disliked having to give explanations in MATH 1113.

Jane: I couldn't explain it...No. I'm not one for explaining stuff. I do it and this is why...that was a big thing in class [that is, MATH 1113]...I didn't know how to explain why I did...like you asked me how to explain how to figure out the degrees inside...you just do what you know you are programmed to do...explanations only do so much...in math...you know, this is how you are taught, this is why you do it...that's always been the thing...you know I was never told to question or explain why in words...I could never explain things in words...still can't to this day...it doesn't matter.

Jane had somewhat of an inflated and inaccurate opinion of her mathematics ability. She believed that she knew how to “do” mathematics but just did not know how to explain “why.” The problem sessions and the final interview confirmed that her “doing” of mathematics was even lacking. Jane often committed vandalism with the mathematics in her pursuit of an answer. Thus, not only was Jane's view of mathematics procedural, she lacked the conceptual understanding of mathematics. During the final interview I asked her about finding the measures of the angles since this was a procedure the group struggled with during Session 1. I pointed to the statement $\sin A = \frac{100}{260}$ on her paper. As shown below she first suggested using cross-multiplication as she had during the session but then finally suggested using the inverse function. Furthermore, she had no clue that her original suggestion of cross-multiplication was of no help.

Jane: You would take your sine A equals 100 over 260, crossmultiply.....260 sine of A equals 100...here you would divide by 260...this is the long way of doing it of course but uhm...then

you do a hundred divided by 260 and you get the sine of A. But you would have to do the inverse sine, I think, in my calculator, I think...[pause]...yeah, you have to do second sine...yeah, you do the inverse sine.

When I asked Jane if she saw that the crossmultiplication was unnecessary, she justified the procedure by claiming it was “just a way of checking yourself.” This example was typical of Jane’s approach. She knew procedures but had no real understanding of why they worked nor did she know what procedure was needed in a given situation. There were many gaps in her procedural understanding of mathematics, and she was incapable and unwilling to recognize these gaps. To compensate for the gaps she often used face-saving strategies.

To Jane the solution of the problem using ratios was just too easy. She was convinced the problem should be solved using trigonometry because “in the trig classes you always see...two triangles drawn together.” Although Jane claimed using the ratios was too easy, her understanding of what it meant for the sides of triangles to be proportional was still inaccurate. Although she claimed that she listened and claimed to understand, her explanations during the final interview indicate that she failed to make sense of the relationships between the similar triangles. First of all she claimed it would be triangle ABD and ACD whose sides are proportional. She explained how to set up the proportions but inserted face-saving techniques.

Jane: This is when I started to stop...cause I was just paying attention to everybody else. Uhm...like you would know that uhm BA over AD is proportional to CD over AD. The thing is that you know they share the common side [that is, \overline{AD}]. That these are proportional here...right?...I’m not functioning right now.....

[Jane wrote on the paper $\frac{BA}{AD} = \frac{CD}{AD}$.]

LBC: So this would imply...if these two ratios are equal then BA and CD must be the same length. ...What do you think?

Jane: They are not the same length because one is 180 and one is 100.

LBC: ...so would this....what do you think about what you wrote here?...

Jane: Well, you'd have numbers there...so if you did that you'd have 180 over 240 equals 100 over 240. And you bring them down...I think you had three-fourths...equals five-twelfths...[She

wrote $\frac{180}{240} = \frac{100}{240}$ and concluded $\frac{3}{4} = \frac{5}{12}$.]...and then you would know that BA is three....cause

we separated the triangles out...we got the triangles outand then you did the

Pythagorean Theorem and you can figure out the other side...and then when they were trying to find the ratios--they kind of got me confused first when they were talking about it—because they were looking for an easier way to solving the triangles...to try to figure out these sides...well the thing was we already knew the big triangles...the problem was trying to figure out the little triangles...do you have Sara's work? ...Cause I didn't write any of that down...

[I handed Sara's work to her.]

Little of what Jane said above made any sense. She was not alarmed to conclude

$\frac{3}{4} = \frac{5}{12}$. These were just symbols to her that carried no meaning. As Jane looked at Sara's

work, she maintained a barrage of her interpretations of the procedures that were used. I noted in my field notes that it was obvious to me that she was “not understanding what is going on and yet never confesses that she is confused—is it pride or no realization of confusion?” Jane told me what was done (that is, she was reading the symbols from the paper) but she had no understanding of what the mathematics meant. Furthermore, her procedural explanations were often lacking. For example, observing that in Session 4 Sara solved for z in the two proportions, Jane explained, “she [Sara] set it equal for the z ...took z and put it in for x and set it equal so you didn't have z any more... z didn't exist cause we knew what it was.”

Jane believed that although she did not write down the work dealing with ratios, it was enough that she “could follow her [Sara] and...was watching her do it.” I think she believed she understood the process as it was occurring during the sessions. However, when I asked her if

asked could she write up the solution and tell me what was done, she quickly said, “Not with the work I have...I couldn’t explain it.” True enough, her personal work did not support the solution of the problem. Furthermore, the understanding she had of the solution was reduced merely to knowing ratios were used. Thus, Jane’s lack of a sense-making perspective prevented lifting for her and possibly caused falling.

Jane’s misconceptions about the concept of variable were evident during the problem sessions. Because she often tuned me out when I was trying to help them make sense of this concept, Jane missed the opportunity to make sense of this concept. In the final interview I questioned her about the concept to see what her understanding was. To my chagrin, I realized as I was doing the analysis of her final interview that her misconceptions were even more pronounced than I initially thought.

I began my exploration of her understanding of the variable concept by reminding her that I kept asking, “What is x ?” As I analyzed the following exchange there were times, however, I was unsure Jane and I were talking about the same thing.

Jane: x is a variable. It is a variable for the length of that side that we don’t know...of the unknown side subtracted from 240.

LBC: Alright, so...what...what length?

Jane: That was the question...we didn’t...we were trying to figure out...some of us were trying to figure out what RD was...and some of us were trying to find AR...

Jane still did not acknowledge the importance of relating x to a physical quantity and maintaining that relationship.

Jane: We knew that BD was 300—that was the whole length—BE was 300 minus y ...which is another variable, we want to know what this section was right here...to know this length...so if we knew this length...you could subtract it from 300 and figure out what ED was.

LBC: Alright, so what is...if BE is 300 minus y

Jane: ...then ED is just y . [I was excited that Jane had this understanding.]

LBC: Why?

Jane: Because...that would be what was left from whatever uhm...300 minus y was. But it would actually be like y , too. [...unintelligible]

LBC: Alright, why do you say y , too?

During the interview, I understood Jane to mean “ y , too” but during analysis as I looked back at Jane’s work from the problem sessions, I concluded she was saying “ y_2 ” and not “ y , too.” Thus, she claimed ED should be y_2 . Her reasoning for the y_2 was rather bizarre as the comments below show.

Jane: Because then it would be just the same variable...and that would be inferring that whatever we got for the y with BE...was going to be equal to whatever ED was...that was inferring that this 300 was divided equally...so this would be 150 and this would 150...but you don’t know that...

Unpacking Jane’s thinking was a real challenge. Only later did I realize she said BE was 300 minus y because this meant to her that when you solve for y , you get BE. At this point, however, in the interview, this realization was not clear to me. So I plodded along still trying to make sense of Jane’s comments.

[Jane appeared to realize her mistake in saying that BE and ED are both represented by y .]

LBC: So are you saying that if you say ED is y and BE is $300-y$...

Jane: Then you say ED equals y , then you’re trying to say that that y is going to be equal to ED...

LBC: Alright...should it be?

Jane: Not necessarily...cause you don’t know if it is drawn to scale...you don’t know if this was intersect--if AC intersected BD evenly—

Jane’s justification for assigning y_2 to ED was because BE was $300-y$ which meant for her that BE would be the y she would find. Thus, if she said ED was y , that would imply BE and ED would each have to be 150 and “you don’t know....if AC intersected BD evenly.”

LBC: Alright so are you saying then that...if you say that ED is y , then this says that BE is 300 minus ED...

Jane: ...now I'm confusing myself, but yeah.

Surprisingly, Jane admitted her confusion. This was unusual. Although she responded to my question with the correct answer of "yeah," I was not convinced she understood.

LBC: So would these two variables be the same? Would...if ED is y , should BE be 300 minus the y ...the same y ?

Jane: No, it shouldn't....

LBC: Why?

In the following explanation, Jane had an epiphany. Being asked to explain allowed her to realize her misconception about the variable y . Explaining her thinking caused lifting for Jane.

Jane: Cause you don't know if they are the same...you don't know if they are equal...that's the point of having 300 minus y ...you wanted to find out...like let's say that y was 40...for whatever reason...if you were to subtract that from 300 to find out what BE was--that would be 260—so that would mean this is 260...and just because...wait a minute.....yeah, y would be the same.

LBC: Why are you thinking that?

Jane: Because if you subtract the 300 from 40, that'd give you BE...so 300 minus 40 is 260 which equals BE...that's the line for BE...so I'm trying...in order to make that 300 if you subtracted the 260 from 300, y would be 40. So ED would be 40--the y is the same thing.

LBC: Does that make sense?

Jane: Yes, I'm just...I'm brain dead right now.

Jane was forced to look her misconception in the face. However, she was unwilling to admit her misconception and instead used a face-saving technique by claiming she was "brain dead" rather than admitting her misconception. It was satisfying for me to see that Jane made sense of the mathematics as she explained. It would have been more satisfying had she

recognized the benefit of her explanation especially since she claimed explaining how and why you do mathematics was of no value to her.

During the sessions the group let “ x ” be the generic variable for any unknown length. I wanted to get Jane’s perspective on this misconception. First she acknowledged that if RD is x and AR is 240 minus x , then the x values must be the same; this acknowledgement led me to believe our prior conversation had been helpful. She saw that once a value of x was determined, this value was constant, but she was not at all clear on the unique physical representation of x .

LBC: Ok. Now...could you have said if this [RD] is x and this [AR] is 240 minus x ...could you have said that BE is 300 minus x ?

Jane: Yes.

LBC: Why?

Jane:either way you are going to set it equal.....could you repeat that one more time?

Jane would typically provide an answer to a question even when it was quite clear she did not understand the question. Thus, her asking me to repeat my question was surprising. Perhaps her realization of the y 's being the same caused her to question her own understanding. I repeated my question.

LBC: Ok...if you say that RD is x ...and AR is 240 minus x ...could you have said that BE is 300 minus x ?

Jane:you didn’t want to make it the same variable though so could figure out...putting the same variable makes it difficult...and we...

LBC: Alright, what do you mean makes it difficult?

Jane: To try and uhm differentiate which problem you are trying to work...by putting the same variable on it...that was the thing cause normally you use x when you work problems...when it is just a single variable problem...and uhm I know with me that’s a big deal when you are trying to come up with like different variables you’ve got to keep in mind that on the 240 you are using

x as the variable and when we were using 300 you were using y...to try to remember that they are separate problems...they are not the same lengths of variables...

[The idea of variable representation was quite elusive to Jane.]

LBC: OK...do you think you just use different variables just so that you...do you think it is incorrect...

I did a poor job with this question. As a result Jane interrupted me and misinterpreted my meaning. However, from her response I concluded she believed it was acceptable to use the same variable for different segment lengths.

Jane: ...it's not incorrect [that is, to use different variables]...it's just a way of making it easier for you to see what you are trying to work...so you won't confuse yourself...

LBC: But if you say that...if you had said that this is x [RD] and this is 240 minus x [AR] and you had said that BE is 300 minus x ...then how would you find out the length of BE? What are you really saying that BE.....

[Again, Jane interrupted me before I could make my question clear.]

Jane: BE is whatever that x is subtracted from 300.

The lifting that occurred earlier for Jane has not been lost because she sees that if BE is 300 minus x, then BE is not x but is the difference between 300 and x. However, she has not grasped the concept that if x is the length of RD, then physically representing BE by 300 minus x is inappropriate.

LBC: OK...is that physically true in that problem?

Jane: I don't know...the sides...[Jane thumbed through her work.] I didn't write down anything that Sara had worked the last day, but uhm...

LBC: Alright but BE is...is this whole length...300 minus something.

Jane: uh huh

LBC: What do I have to subtract from 300 in order to get the length of BE?

Jane: This other side. You need to subtract this length from the side. [I did not record in my notes what she was referring to.]

LBC: Alright. If you say to subtract x , and x is RD ...if you say that BE is 300 minus x , then what are you really implying?

Jane: That they are the same length...if you use the same variable. If I understand the question right...if you use the same variable you are thinking that the same length.

LBC: That what are the same length?

Jane: ED and RD .

LBC: OK...OK...does that make sense?

Jane: No, cause you don't know that...they are on two separate different lengths...

LBC: That's right...so you can't make that assumption...

In retrospect I believe I misunderstood Jane's reasoning. Through further analysis of this transcript I now believe her claim that it was not possible to assume ED and RD were the same length had to do with one being a part of 240 and one being a part of 300 . Thus, her reasoning was not mathematically appropriate.

Jane: Right.

LBC: And if you say that RD is x and you say that BE is 300 minus x , then physically you are saying that BE is this length minus this length [that is, BD minus RD].

Jane: Right. That's why we used a different variable.

LBC: Right. Do you follow?

Jane: Yeah...

LBC: So it really is a major...

Jane: It is just difficult to remember that because in most problems you only have one variable and you always use x ...and that's a lot of problems, a lot of people get them confused...but in the problem that we had, we remembered that you had 3 different variables in this case...you had uhm...uhm... x , y and z ...and that you had to keep them all straight...in between...

LBC: Ok...cause they represented...

Jane: They represented different things, different sides.

The concept of variable representation was still not clear to Jane. Although she appeared to have made some gains, quite frankly I believe I put words in her mouth. Jane's tendency to save face did not allow her to disagree or to express doubt. Instead, agreement with me was a face-saving strategy.

Jane frequently made irrelevant and illogical comments during the problem sessions. I questioned Jane about one of these instances. I showed her the clip where she said "Cause tangent and 180 at pi is 0...because on the unit circle it's one zero [i.e., the ordered pair (1,0)]...and if you do the tangent, it's undefined." I asked if her response made sense to her and she quickly replied:

Jane: Not at all...I don't know why I would say that...I have no idea why that was said at all...cause that really had no point to it...that was more about the degree and radians, I think...

LBC: And I didn't even know where the pi came from...

Jane: I don't know either...

LBC: I wondered if you were thinking pi, 180 degrees, 180....

Jane: I don't know...that's a stretch, I don't remember...

LBC: Ok, alright...so you can't shed any light on that?

Jane: No.

Jane was very quick to dismiss this clip. I saw her refusal to confront her illogical statements in the clip as an effort to save face.

Jane does not see group work as very helpful. She said in the final interview that she wonders if you "really learn anything working in a group." She said that it is "better if you do it independently first and then come back to the group." However, when I asked if she believed my giving the Buried Treasure Problem to her ahead of time would have been better, she said "might have been better but probably wouldn't have changed anything." Furthermore, she

admitted that had she had this problem to work on her own she “would have thrown it in the trash and said, ‘Quit.’” She also acknowledged that she probably would not have been able to figure out the problem on her own.

Jane believed that a hindrance to their problem solving was the fact they had forgotten some mathematics. In the reflection after the first session she wrote that she did not “remember the exact way to work problem easily” and wished she had “looked at book before actually working problem.” After Session 2 she wrote that she has “forgotten the tricks in using triangle trig...to figure out the length of the sides.” In the final interview she again said she had not looked at trig since the MATH 1113 class and had “not looked at any geometry...had completely forgotten about the ratios and what not because it was 10th grade and that seems too long ago.” I asked, though, if the group was able to fill in some gaps for Jane, and she admitted they did because they had a “better grasp on the ratios and proportions of a triangle.” Jane claimed that everything she remembered in geometry was not what was important to figure out the Buried Treasure Problem.

Jane claimed in the final interview that “everyone had the ratio or proportional problem concept...the whole way through...but the ratios didn’t come into play until Sara said something about it.” Because Jane kept looking for a trigonometric approach to solving the problem, I was surprised to hear her claim that the ratio approach was a common idea so I questioned her.

LBC: But even when you were doing the trig...

Jane: It’s a ratio within a ratio...I mean...you have two sides with tangent and that’s opposite over adjacent...that’s setting up a proportion but actually breaking down the numbers to easier numbers to work with and seeing the ratios easier...Sara did that ...when she took like the 100 over 180 or 100 over 240 equals 180 over 240...putting them in smaller numbers made a lot easier...

LBC: So do you see even trying to do the trig still being a ratio and proportion? Do you see your strategy that you were striving for being basically the same strategy that...

Jane: Except for it was instead of using easy...concepts...yeah...except for mine was more trig based than them...

Jane kept looking for a trigonometric approach to solve the problem. Even though she recognized the trigonometric functions express ratios, she did not see a trigonometric approach as being different from using similar triangles. Ratios are a part of both approaches so for Jane the two approaches were not different.

Jane's belief that every problem has a "textbook example" was reiterated at the end of the interview when I asked what suggestions she might have for another group of four people who might work on a problem together. Her comments also show how she allows her most recent experiences in mathematics to dictate the way she views subsequent problems.

Jane: Need to look at the ratios, look at the sides, look at the proportions of the triangles.

LBC: So if I came in next week...and I were to ask you to work another problem...would you immediately think about doing ratios?

Jane: Probably...probably...it would make a definite impact on how you work a problem...I wouldn't forget it.

Finally, Jane admitted that it was hard for her to stay focused on the problem because not only was it the summer but it was also very early in the morning. She said the last day was the hardest because she "was starting to fall asleep."

Sara

I began the final interview with Sara by asking her to explain how the group went about finding the angle measures in the triangles. I sensed that she was really not in the mood to do this interview and especially not in the mood to do math; in fact, she said "I thought I wasn't going to have to do math today." Her responses during the interview were not always very thoughtful and reflective, and there were times when she and I appeared to be talking past each other. A couple of times her response was very short and cryptic—almost rude. For example, at one time she mentioned an episode in the first session but could recall no particulars about it.

The following dialogue conveys my attempts to trigger her memory as well as her somewhat recalcitrant attitude.

LBC: Do you remember anything about it?

Sara: Uhm...no, I don't.

LBC: Or who asked it?

Sara: I've no idea.

Perhaps part of the problem was my own bias about Sara's mathematics ability. Although her work in MATH 1113 could have been better, I perceived her to have a more conceptual understanding of mathematics. My bias, however, clouded my ability to interpret fully Sara's thinking at the time of the interview. I must admit there were times in the interview when I assumed Sara understood and yet when I analyzed the data, I found myself questioning her understanding. Episodes where I misinterpreted her understanding will be discussed.

Sara was perceived by the other group members as the leader of the group. It was easy to see why the other group members would consider her to be the leader. Jane had worked with Sara during the MATH 1113 class and was aware of Sara's mathematical ability. Sara was also very articulate. Furthermore, she wanted to be sure everyone in the group participated so she would ask them for suggestions. Sara, however, did not want to be the leader. In the final interview Sara acknowledged that it was difficult to avoid the leadership role while trying to be sure no one felt left out.

Sara: I don't like to be the type of person that seems like...well she just wants to be the leader...but at the same time...I just want to make sure that if you are in a group situation that everybody is interacting so that nobody feels left out and at the same time that everybody does actually understand what is going on...you just don't want anybody to be left out—you don't want anybody to not understand what is going on—especially if them not understanding can slow the process down—you know, ask a question, we'll answer it, you know.

Sara was the student who was most involved with the problem solving throughout the three sessions she attended. She was the most vocal of all the group members and as Jane put it, "Well, we know Sara figured it out." And yet Sara did not have a complete understanding of all the pieces. For example, because the group had difficulty determining the angles of the triangle during the first session, I asked Sara in the final interview if she could tell me how they determined the measure of angle CAD. She at first identified AC and AD as the hypotenuse and opposite sides, respectively, and said, "You could take sine." She then realized her mistake and changed the function to the cosine. She said "cosine of the angle is going to be ...240 over 260." Picking up a calculator, she said, "Make sure it is in degree mode first of all....dividing 240 by 260...uhm...cosine of...I forgot how I did it.....I forgot...I don't know." Only when I prompted her with "using the inverse cosine" did she recall how to determine the degree measure. Thus, lifting occurred for her during the session, but the procedure was not one she retained.

Another concern was whether or not Sara fully understood why it was important to define the variable.

LBC: ...and one of the things I kept harping on was what is x ?...what is x ?...do you know why I kept saying that?

Sara: I think I realized it towards the end of the problem...when we were like ...we knew that we had to name the smaller triangles with certain sides and we were getting confused about which part of the triangle to name $240 - x$ and which part to name x and which part to name z ...it was like we knew what we were supposed to do but not which...exactly which piece was x and...

LBC: Does it matter?

Sara: uhm....

LBC: ...what you let x be? Or does it matter that you **know** [emphasized] what x is?...I guess is my...

Sara: I guess as long as you just keep it...as long as you keep it in a firm spot and you don't go name other things x that wouldn't be the same value...I guess it doesn't matter what piece you name x .

Sara appeared to understand that once the value of a variable was determined in a problem that value was constant. At this point in the interview I believed that my intervention in the problem sessions had lifted Sara in her understanding of the concept of variable representation. Later in the interview, though, I realized Sara still had a misconception about using a variable to represent a physical entity. (The analysis of her conversation that occurred later in this interview convinced me that Sara still struggled with the concept of using a variable to represent a physical entity. During data collection I believed Sara had a misconception about proportional reasoning, but as I analyzed the transcripts I concluded her misconception was with variable representation rather than proportional reasoning. Her trouble with variable representation surfaced later in the interview as we talked about setting up the proportions. My personal misconception was that during the interview I believed her trouble was with proportional reasoning and only through detailed analysis of all the data did I realize her primary misconception was with variable representation.)

An epiphany for Sara during the session was when Eve asked the question about whether or not the angle would remain the same in a triangle if this angle is an angle of two overlapping triangles. Eve's question allowed Sara to see the problem in a different light so that she could formulate a potential plan of solution. In the reflection following the second session Sara wrote that at first the answer to Eve's question seemed obvious, "but after looking at the picture and answering the question I was able to see something I had not seen before. Although the lengths of the sides on the two triangles are not the same, the angles within the two triangles are [and] it is (or should be) safe to assume the lengths of the triangles are proportional." Thus, Eve's question helped Sara formulate a plan. Sara realized how Eve's

question helped her because she wrote in her reflection that answering Eve's question contributed to her seeing the relationships among the triangles.

In the final interview Sara again shared how Eve's question was helpful. She again made reference to the sense-making that she did as the question was answered and how that sense-making helped her formulate a plan.

Sara: And at first when she asked the question I was you know looking at the picture and I thought you know the answer should be obvious but then looking at the answer and actually explaining it you know to myself like internally or whoever explained it out loud...you know you are looking at it and then you realize that...the answer is that it is going to be the same because the triangles are still proportioned no matter how small or how big it is as long as it shares the same acute and right angles...and from that I thought it would be a good idea to maybe fill the ratios in...uhm...cause I thought if you had the ratios and you could somehow figure one length out, you could just set up an equation and fill in the rest of them...so that was my thinking on that...I wasn't sure whether it would help or not...but like I said just fill in the information that you can and if it doesn't work you haven't lost anything.

Sara was prompted to consider ratios by Eve's question in Session 2 as explained above. The group (minus Sara) revisited the ratios in Session 3. During the final interview Sara admitted the difficulty she had in getting information from Abby and Jane about the group's work in Session 3.

And then I came back and I really didn't feel like I got a good grasp of what they had done...you know...like I didn't feel like they...like it was explained very well to me...and a lot of times it is harder for me to understand things unless I actually work them out myself instead of somebody just telling me...cause actually I have to see that that is being done to be like "OK, I know why you are doing that" and uhm...but...then...then I think this is the point where we all kind of knew that ...we were going to have x's and other variables that we didn't know and...I knew in the back of my head I knew the whole time that we were going to

have to have two different variables, solve for one variable, and then set those two equations together to get the right answer but we started getting confused about which piece should be x and which piece should be z and how to set up the ratios and the equations and all that good stuff and then...I guess....on this page...you kind of helped us realize that we needed to like...that we were on the right track...just not completely....

Sara acknowledged they had trouble deciding “which piece should be x and which piece should be z and how to set up the ratios.” She also said she had “problems with naming the pieces correctly.” Based on my observations of her work in Session 4 and her comments above, I thought proportional reasoning was problematic for her. The following excerpt, however, shows Sara’s primary misconception is with the concept of variable representation instead of proportional reasoning.

I pointed to Sara’s work from Session 4 where she had written $\frac{z}{240 - x} = \frac{180}{240}$. She asked me if this was where she came up with x equals 240, and I confirmed that it was. (During Session 4 when Sara got $x = 240$, the group immediately claimed this answer was inappropriate.) Immediately Sara said, “I don’t know why I was using the two different triangles...that’s not even....” (Sara’s drawings are shown in Figures 29 and 31.) Sara’s proportion looked as if she were using triangles ABD and AER, triangles that were not similar to each other. She realized the inappropriateness of this statement by questioning why she was using the two different triangles. Her questioning indicated an understanding of which triangles were similar. She then remembered that her initial drawing had RD as $240 - x$, and she had changed it to x during the fourth session. “Yeah, that’s what I was doing...this one was z and I had originally marked that as 240 minus x .” Thus, the proportion she wrote on her paper was stating a relationship between the similar triangles ABD and RED. Sara’s making sense of her work was helping me verify that her error was not related to a misconception of proportional

reasoning. Instead, her trouble was related to the concept of variable as our continued conversation revealed.

Sara explained that she initially labeled RD as $240 - x$ in triangle ERD. (Later in Session 4 at my leading she changed RD to x .)

Sara: When I was doing this right here...I had that [RD] as 240 minus x ...that was the triangle [ERD] I was using...uhm...I guess the way I was thinking about it was...I think I just got confused with I knew this whole length [AD] was 240 ...and I...you know that because this [RD] was...a portion of that length then it would be 240 minus a portion and that's why I named it that...

[It was during analysis that I realized she was using the word *portion* to define two different lengths—this observation explained why she also named AR as $240 - x$ —because it, too, was a portion of the length of 240!]

In the following excerpt I tried to lead Sara to realize that labeling a segment with $240 - x$ was dependent upon the representation of x . I pointed to the $240 - x$ on segment AR (see Figure 31). I was trying to get her to make sense of the physical representation of x . [As I analyzed her final interview I realized Sara still saw x as a value and not as a representation for a physical entity. I unfortunately chose to use her term *portion*, but during analysis realized Sara used the word *portion* ambiguously so my choice of the word was perhaps confusing.] The following dialogue shows that Sara's understanding of variable as a representation of a physical entity is not clear. I was hoping she would realize her initial labeling of RD and AR as $240 - x$ was meaningless. Ultimately she was concerned with whether or not the answer was correct.

LBC: But what would the portion have to be if you do 240 minus x here? [I pointed to \overline{RD} in triangle ERD which she had labeled as $240 - x$ in Session 4.]

Sara: x ?

LBC: Alright, so where would x have to be?

Sara: I thought it was right there. [She pointed to \overline{RD} because she had changed it to x in Session 4. She missed my point in getting her to see that if RD was $240 - x$, then AR would have been x . I let it drop and continued.]

LBC: OK. Alright so if RD is x , then what would be 240 minus x ?

Sara: This whole length.

LBC: AD ?

Sara: Or....actually that AR would be 240 minus x .

LBC: Why?

Sara: Because this length right here, RD , is x and then smaller portion would be 240 minus x . [Sara's drawing had the length of AR shorter than RD in triangle ABD —thus, she referred to AR as the smaller portion.]

LBC: Does that make sense?

Sara: It does.

LBC: OK, so down here [in triangle ACD which Sara incorrectly labeled as AED] you've got 240 minus x and x ...and if RD is x here--you've got RD being x here—

Sara: Right...

LBC: So then you set up a ratio, 100 divided by 240 is z divided by 240 minus x .

Sara: Uh huh...

LBC: Does that make sense?

Sara: No. [And yet this proportion states a correct relationship.]

LBC: Why?

Sara: Because...this whole length is 240 ...so if I were to name RD x 240 minus this portion...did I get the right answer from that? I don't think I did.

LBC: Does this seem to be making sense? Is this logical?

Sara: It seems logical to me....

LBC: Ok, that looks good...alright, now over here you've got z over 240 minus x is 180 over 240...now here's the 180 and here's the 240...and you are saying that's the ratio of z to 240 minus x...

[I was too quick to assume Sara did understand.]

Sara: Right...and that's where I messed up...

LBC: Cause if you say 180 to 240 then what should this ratio be?

Sara: uhm...it should be z to x.

[Again, the problem with the above exchange is that I thought Sara was confused about proportional reasoning when she was really confused about the concept of the variable. Thus, my questions were not as probing as they needed to be.]

In the above exchange, Sara did state that if RD is x, then AR would be 240 minus x. Furthermore, she claimed the ratio of 180 to 240 would be equivalent to z to x. Although I heard her make these correct statements, I was not fully convinced she understood the concepts [of variable representation] fully. I decided to push the issue no further as I sensed her weariness with the topic.

This final interview was informative in that it gave me the opportunity to see that the variable concept rather than proportional reasoning was the real stumbling block for Sara. I had incorrectly concluded during the fourth session that a misconception of proportional reasoning was one of the cognitive factors that hindered her ability to carry out the approach Sara suggested.

I asked Sara her perspective on the factors that helped and hindered the problem solving for the group. She pointed out that "some of the questions that were asked were helpful in figuring out...you know taking us to the next step." She specifically cited Eve's question about the angle in the overlapping triangles as such a question that moved the group forward. A related realization for her was that "silly questions can sometimes help you get a step further and it just kind of makes you step back and put yourself in somebody else's shoes and try to

remember that...look at it from a different point of view.” Although Sara initially saw the answer to Eve’s question as obvious, her consideration of the question directed her thoughts toward the use of similar triangles.

Sara cited a number of factors that she believed hindered their progress in solving the problem. In her reflection after the first session she wrote that “it would have been nice if we would have first discussed what approach we would each like to take and then discuss which approach sounds the best. I guess I could have attempted to make that happen, but I did not. Perhaps that will be what I do next time.”

Sara noticed that the groups not having a common workspace was problematic. In the second reflection she observed that having “one consolidated workspace so that everyone is looking at and working on the same thing instead of working on separate things” would perhaps be best. During the final interview she again mentioned that having one workspace would be helpful but explained that working on different ideas would be useful as long as you communicate to the group what you are doing.

Sara: Yeah, I think from the get-go I would....maybe suggest working on one space...maybe suggest instead of just people going off and trying something... to talk about, “ Well, I think this is a good idea,” or “She thinks this is a good idea,” “She doesn’t understand this,”...”He doesn’t understand this.” Answer any questions that may be had and maybe “OK, why don’t you two...try this ...the two of us will try this and ...then we will see where we are and what information we had...or maybe even put 2 different people working on the same thing together so that you get maybe 2 different answers or two different ways of looking at it...uhm...

Sara said that the group never knew whether what Jane was looking for in the book “was going to be helpful or not.” She also pointed out that at one time she and Eve were unknowingly working on similar ideas but because “it took time for her to explain what she had done to me and she was just throwing out numbers and letters and... then after she explained it I realized that we had both just done the same thing...it took up time you know us doing it separately and

then having to explain it to each other and realizing that we had come up with the same answer.” [Sara was referring to the episode in Session 2. Actually the same answer was not obtained because Eve arrived at $ER = \frac{5}{12}$ whereas Sara had the ratio of ER to RA as $\frac{5}{12}$.]

Sara also observed that Abby was “trying to figure out what all three of us were doing.” Sara summed it up by saying “I think there was a little lack of communication there.”

Related to the lack of communication was Sara’s frustration with the group’s unwillingness to give any feedback. She referred to the episode in the fourth session when she made a suggestion and asked if anybody agreed or disagreed with her idea.

Sara: Nobody said anything and I was like...you know...in the back of my mind I’m thinking well should I go ahead and do it...I don’t want to seem like I’m trying to lead the group...I don’t...you know what if they don’t agree with what I’m doing...but nobody said anything so I was like...Ok...you know, some response or just feedback from anybody would have been helpful.

Sara was confident in her ability to solve the problem. She claimed, “I know I can get to the end...I think I could have eventually gotten the answer [by myself].” She was conflicted in her desire to involve the others while unwilling to be the leader.

Eve

Eve missed the fourth problem session so we talked only about the work that she contributed. Because she had very little to say in the first session, I wondered what was going through her mind during that session. Her comments indicated she was engaged in the problem but was having trouble making sense of the problem in that environment.

Eve: I mean I was thinking things...I was just trying to get the feel for what we were doing, why we were doing, and a solution to get there...

LBC: Were you uncomfortable that day?

Eve: Kind of... because I'd never been used to working in a group...like for problem solving so I was like I'd just let everybody else talk...and...I'd jump in here and now...

....

Eve: Because I was....because the atmosphere was different so I was just trying to feel around...I mean cause we were in, we were working all together...and you know we couldn't just run off and do our own thing...you know we work as a team...so I was just trying to feel how everybody else was feeling so I just laid low....

Eve also shared that when the group was finding the angle measures in the first session, she “didn’t see the point of some of the stuff that we were doing...and I kept thinking about that...” This admission confirms my assessment that Eve’s lack of participation on that first day was not due to arrogance but was instead a struggle to make sense of the meaning of the problem. Although she did not understand how the angle measures would help (and even raised the question to the group) she agreed to go along because it was something they could do. Consequently, she was swept up in implementation before she fully understood the problem.

Eve had been a strong MATH 1113 student. Her work from session one (shown in Figure 5) to find the length of segment AB was not at all typical of the work I had seen from her in class. This work was generated early in the first session before she got a calculator and when her level of discomfort was probably greatest. During the final interview I pointed to the second line of the work in Figure 5 and asked “How did you get this step right here?” She quickly responded as if reading, “I subtracted 240 from both sides,” but then thoughtfully studied her work and corrected her statement. “I guess it should have been squared...still. It should have been squared...AB squared and then this is 300 squared minus 240 squared.” She then explained that to find AB you would take “the square root of the difference of that.” I was satisfied that Eve did understand the procedure for using the Pythagorean Theorem to find the length of a third side of a right triangle. Regrettably, I did not give her the opportunity to explain

why she produced the work she did in Figure 5 so I can only assume the lack of a calculator, her level of discomfort, and her need to keep up with the others all contributed to her incorrect statements.

I also asked Eve if she understood why I continued to ask, "What is x ?" I believed during the problem sessions Eve began to understand the significance of the representation of x . Her comments in this final interview verified her understanding.

LBC: I kept asking, "What is x , what is x ?" Did you understand why?

Eve: Kind of...cause I know at first...we were like..."well, this is 240 minus x "...we just put that...and then we just...I know at one time we assigned x to another value and then we had x for that one....they are not equaling the same thing... so if we were to...and when we did find x ...it would not be the same in both cases...so x couldn't have the same value cause they weren't equal to the same thing....

Eve was also able to identify the triangles whose sides would be proportional and explained their sides were proportional because the angle measurements were the same. As she explained she was reminded of her confusion about an angle common to two overlapping triangles. She first brought up the question in Session 1 and again in Session 2.

Eve: ...cause I remember me asking a question about that.....and if the angle measurements were the same in the triangles.....

LBC: Cause you asked if we found this angle, CAD, by using this triangle, CAD,would the measure of EAR be the same as CAD.....why did you ask that question?

Eve: Because for some reason...I thought....that in a....ACD is one triangle...and AER is a triangle so I thought the measurements would be different since one is bigger than the other one...

[I decided to probe her thinking about the measure of an angle.]

LBC: What does it mean to say, “The measure of an angle”? If I’ve got an angle that looks like this [I gave her a triangle cut out of paper]...What am I talking about when I talk about the measure of that angle?

Eve: The.....the.....I know how to say it....but it’s just not coming out....it’s...it’s how wide.....

LBC: Does the length of the sides have anything to do with....

Eve: [interrupting] Yes.....

LBC:with the measure of the angle? [I was surprised by her response and knew I needed to find out more about her thinking.]

Eve: Yes...I know the rule...it’s uhm...if it is a small angle then it’s going to be opposite of a long side or either it’s that way or I’ve got it backwards...if the measurement of the angle is big then it is opposite of the big side....I got it confused.....I remember if you have a triangle.....and.....the....largest angle I think is opposite of the...largest side.....yeah....and that’s how it’s going to be....and the rest of them are going to be...like if this is a small angle then it’s going to be...if it is the smallest angle out of all of all the angles then this is going to be the smallest side...so the angle in a triangle affects the side....how long the sides....

I believed Eve understood what was meant by the measure of an angle but her confusion was the result of trying to apply an inappropriate and misunderstood theorem. I asked her:

LBC: Was the idea of the measure being wideness new to you...with an angle?

Eve: Not really...it’s just...uhm....for some reason I remember...about angles changing....and I must have got all confused and I just thought it would be different since it’s in a different triangle but the wideness of it doesn’t change so....

Because angle CAD was in two triangles that were obviously different sizes, Eve was unable to see that if the measure of angle CAD was found using the sides of the larger triangle CAD, the measure of this angle in the smaller triangle EAR would necessarily be the same. Her

confusion resulted from trying to apply some memorized rule for which she had no understanding. The “rule” she was apparently referring to was the theorem that states the longest side of a triangle is opposite the biggest angle of the triangle; the smallest side is opposite the smallest angle. Eve’s explanation showed that not only did she not understand the meaning of this theorem, but she was also unable to quote it accurately. This episode illustrated how a rule memorized with no conceptual understanding interferes with the ability to apply that rule.

It was clear to me that Eve’s question about the angle provided the opportunity for Eve to confront her misconception. Although she first asked the question during Session 1, her desire to understand motivated her to seek an additional explanation during Session 2. Her focus was not merely on getting the problem solved but on making sense of the problem as well.

During the third problem session, the group had correct proportions that could have led to a solution of the problem. The proportions represented two equations in variables z and x , but they were unable to solve this system of equations. Of the three, Eve was particularly involved in the symbolic manipulations. Her error was failing to use both proportions; thus, she wound up with an identity. I questioned her about this work during the final interview and she recognized where her procedure broke down.

LBC: Now look at what the problem was...you had this proportion...and you had this proportion...alright here is your first one and here is your second one...you used the second proportion—right here—and you got x to be $\frac{4}{3}z$ now when you got x to be $\frac{4}{3}z$, what did you do then?

Eve: I plugged it back into the....into the second one.....I plugged it back into the problem I had used originally.

LBC: So what does that mean to you? What should you have done?

Eve: I should have plugged it into the first problem....yeah....

LBC: Why were you thinking to substitute right here? Why did you do that?

Eve: cause I thought some way if I substitute...I substitute x back in and I'll find z...but it didn't work out like that...but since I had the two of them I think I was just doing...I think I did that one and found z or something and plugged it back in...it should have been flip flopped...whatever I found for z or for x right here, plug it back in to the other equation.

Eve explained "how" she should have performed the procedure. It was not that she did not know the procedure for solving a system of two equations with two unknowns; rather, I don't believe she recognized that she in fact had such a system. Too often the systems of equations students encounter in textbooks are neatly packaged with curly braces grouping the two. The procedure then becomes automatic. In a problem-solving task, though, the relevant pieces often look quite different from those textbook problems.

Eve said she was unaccustomed to working in a group for problem solving. As a result, she said she "let everybody else talk...and ...I'd jump in here and now." In her first reflection she wrote, "I wish I would have suggested more ideas to the group instead of trying to figure out the problem by myself." In the final interview she pointed out that helpful suggestions were made that she probably would not have thought about on her own. On the first day Eve struggled with trying to make sense of the problem and said it would have been helpful had they talked about the problem "instead of just jumping in...talk a little bit to understand." In the first reflection Eve wrote "How is finding angle B going to help us?" She later wrote "we should first figure out what is going to be accomplished when figuring out the unknown angles. I didn't suggest them because we were already finding the unknown angles so I kept quiet." Although in her reflections Eve wrote that she kept quiet, she did ask in the first session how would finding the angle measures help them and in the final interview she explained why she questioned the benefit of finding these measures.

Eve: Cause I was thinking before then and I guess I was like well, how is this going to help us...cause you know I was trying to figure out how to solve the problem so I didn't want to just be doing something that I wasn't for sure about or didn't know why we were doing it.

Eve believed her contribution to the group was to keep them in line and "directed toward finding the answer to the problem." She saw Sara as the leader, describing her as the one who got the group started and who would make a suggestion of "something that might work." She called Jane the researcher because she kept looking for information in the book, and interestingly recognized Abby's contribution about the proportions and ratios.

I asked what suggestion she would make to another group about solving a problem. Her reply shed some insight on her perspective of the factors that can influence problem solving in a small group.

Eve: I'd say review over...review over some word problems so you can get your mind stimulated...cause I know a lot of us we just hadn't been doing math...cause I know on that first day we were out of it....I was like review over it and conversate [sic] a lot more...

First of all Eve acknowledged how important prior experience and knowledge is to effective problem solving. The group had forgotten some mathematics but her suggestion to "review over some word problems" may be a suggestion to get the group in a problem-solving mode as opposed merely to practicing mathematical procedures. She indicated effective communication was important but stopped short of recognizing that conversation and discussion could help the participants remember forgotten mathematics and could help them understand the problem.

A positive outcome for Eve as a result of working this problem was recognition of the role visualization skills can play. She explained:

Eve: Because a lot of times I would just go write things down instead of visualizing and drawing it...we did a lot of drawing....I think drawing, drawing things and diagramming helps in solving a word problem....

LBC: And that's not something you had done before?

Eve: No, I would just do more of the math things, not just draw it out and see everything that was going on...

Synthesis

Group 1 had little prior experience working problems in a small group and therefore had no clear understanding of the expectations for the experience. They instead had their own personal goals and needs which made it difficult for them to work collaboratively. For example, Jane disliked group work and preferred to work alone. Although Sara and Eve claimed they wanted the group to work the problem together, they were accustomed to solving problems alone and needed quiet time to make sense of the problem. They did not, however, communicate this need to the group. To satisfy this need, Sara and Eve would cognitively withdraw from the group but without explaining what they were doing. Jane's preference to work alone meant that she, too, would withdraw from the group. Cognitive withdrawal caused a break in the communication. Jane's goal was to solve the problem; she was not interested in working together, and she committed vandalism with the mathematics to obtain an answer. Abby was there to learn from the others but also wanted to contribute. She even had some good ideas but was fearful of being wrong so she often kept these ideas to herself unless she received some verification that her thinking was correct.

There were times when Sara, Eve, and Jane would withdraw cognitively from the group to think about the problem on their own without explaining what they were doing. Jane especially worked alone quite a bit and was perceived by Abby as not liking the group. When she returned to the group, it was usually with an idea that was either not related to what the others were working on or was full of vandalism. I frequently had to encourage Sara and Eve to share the work they did independently. Again, a common workspace could have minimized the opportunities for keeping ideas and work private.

The students had little prior experience solving non-routine problems. As a result, Jane continued to search for a model problem to help them solve the Buried Treasure Problem. Jane, for one, believed the problem would be solved using trig since they had right triangles. When the problem was finally solved using similar triangles, Jane and Abby both expressed surprise that the solution was so easy.

There were at least three misconceptions about big mathematical ideas that created problems for them. The first was that of the concept of variable representation as they tried to represent physical relationships symbolically. The misconception became evident at the end of the first session and was dealt with in the second session but continued to be problematic for Jane and Sara even until the final interview. Jane completely ignored the discussion of this concept in the second session so was not lifted at all. I believed the discussion provided lifting for Eve, Abby, and Sara but realized Sara continued to have problems with the concept in Session 4 and during the final interview. The second misconception dealt with proportionality. The group used similar triangles to solve the problem but the concept of proportionality of the sides was not well understood. Abby understood the concept of the similar triangles but lacked the confidence to convey this information. The concepts of variable and proportionality were concepts Jane and Sara may not have believed they misunderstood.

A third misconception for Eve dealt with the measure of an angle in two overlapping triangles. This issue was resolved for her as a result of her tenacity in making sense of this issue. She first asked the question in Session 1 and revisited it again in Session 2.

There were mathematical procedures the group had clearly forgotten. Ideally, the other group members would be able to help those who had forgotten but egocentrism and lack of a common workspace made this help difficult. Jane had trouble using the Pythagorean Theorem in Session 1 but did not ask for help. Furthermore, there was no common workspace so she could not see Sara's work to be reminded of the correct procedure. A procedure that all four students had forgotten was that of finding the measure of an acute angle of a right triangle using

inverse trigonometric functions. Jane's tenacity in searching the textbook finally reminded them of the correct procedure although finding the procedure took considerable time because of misunderstandings. A third procedure that gave them trouble was that of solving a system of two equations with two unknowns. I believe the trouble with this procedure was related to the two equations expressing relationships between pairs of similar triangles. It was the physical representation that restricted their ability to perform the symbolic manipulations.

Missing in this group was a commitment to a sense-making perspective. Rarely did a group member ask for further explanation, clarification, or justification. There was, in fact, little discussion of ideas. If doubt was expressed or a question asked, egocentrism in the form of a face-saving strategy or camouflaging strategy may be used to avoid the question. There were times when one student would clearly misunderstand another which would lead to wrong assumptions. There were times when Jane committed vandalism with the mathematics as she tried to find the answer. If Jane was questioned, it was more likely she would save face rather than trying to make sense of the mathematics.

Related to the sense-making perspective was my observation that Abby often had ideas that she was unwilling to share. Her reluctance to share these ideas was influenced by her fear of being wrong (egocentrism). When Abby realized her ideas were confirmed by another member of the group, she would tentatively share her ideas—but only when she was sure another person had the same idea. In this respect Abby could be seen as making sense of another's ideas and relating them to her own. In the final interview she told me she noticed there were times when "some of the stuff that [she] was thinking was right...and somebody else said it." Abby needed the affirmation that her ideas made sense before she would share them. For example, in Session 2 it was not until Sara pointed out "it is all proportional" that Abby shared her observation that one triangle was just a smaller version of another. Also Abby appeared to have a good understanding of proportional reasoning and in Session 3 wrote correct proportions for the relationships among the triangles. Only when she noticed that Eve

had similar proportions did she point out that she had written the same thing. In Session 4 it was only when I insisted that she did finally share their work from Session 3 with Sara.

Cognitive disagreement rarely occurred in Group 1. Cognitive disagreement could have led to lifting but such disagreement was avoided. There could be various personal reasons for avoiding disagreement. Abby perhaps saw such cognitive disagreement as “not getting along.” Cognitive disagreement for Jane would have meant confronting her misunderstandings. If Sara engaged in cognitive disagreement, she may have been perceived as taking over.

Related to a sense-making perspective were the communication factors that influenced the problem solving in Group 1. An example of effective communication occurred in Session 2 when Eve asked about the measure of an angle that belonged to a pair of overlapping triangles. When Eve asked the question, Sara’s initial answer was brief so I asked Sara what she meant by “it’s all proportional—to the angle.” In the final interview she claimed having to elaborate on what she meant helped her see the relationships among the triangles. Thus, it was Eve’s question as well as Sara’s more detailed explanation following my insistence that led to lifting for Sara. It was also significant to note this relationship was ultimately the one they used to solve the problem.

Most of the communication in Group 1 could be termed “ineffective.” There were instances of misunderstandings that were ignored, doubts that were not expressed, and occasions of talking past each other. Their poor listening skills also contributed to the ineffective communication. Also contributing to the ineffective communication was the lack of a common workspace. In her reflection after the second session and in the final interview Sara also observed a common workspace would have been useful. The communication could have been improved had they at least shown their work as they explained. The listener could also have asked to see the explainer’s work; given that this rarely occurred confirmed their lack of a sense-making perspective. Listening to the explanation of another without looking at her work resulted in their talking past each other as there was no way they could understand what the

other was saying and could at best only guess they understood to what she was referring. For example, “this is x ” and “this part is 240 minus x ” were misinterpreted as the lack of a common workspace prevented their being able to see to which part the speaker was referring. Much time was wasted as they engaged in talk that was incoherent. Although Eve would occasionally ask for clarification, even she was guilty of nonsense banter.

Watson and Chick (2001) acknowledged how important it is for the group to stay focused on the big picture of the task as the group works on subsidiary tasks. According to Schoenfeld (1983), generating or considering plausible approaches is important in problem solving. Missing in this group was the discussion of a qualitative approach to the problem. An effective leader could have focused the group’s attention in this direction. The work in this group, however, consisted of a series of procedures with little discussion of how or why these procedures were relevant. When one procedure did not work, they would try another rather than trying to make sense of what they were doing. For example, in Session 1 when finding the measure of angle A with the tangent function was not fruitful, Sara and Jane opted for the sine function even though the same dilemma existed for them. And in Session 3 when Eve got the identity $0 = 0$ as she tried to solve the system of equations, the group did not look for a mathematical explanation of this result. It would have been helpful for the group to take a step back to examine their initial proportions and be convinced these were correct. Instead, the time in this session ended before they could make this determination. As a result, they may have left this third session believing the proportions were incorrect when in fact they were correct. It was perhaps a belief the proportions were incorrect that made Abby even more reluctant to share this work with Sara in the fourth session. Keeping the big picture in mind by monitoring the work that was done was not one of the group processes Group 1 practiced.

Keeping the big picture in sight was important since the problem stretched over four sessions. Unfortunately good ideas that were generated in one session were not always considered in a timely manner in subsequent sessions. For example, at the end of Session 2

Abby and Sara observed that some of the triangles had sides that were proportional. When Session 2 ended, I was convinced they were well on their way to getting the problem completed, but Session 3 began unproductively with Jane's persistence in exploring a problem from the textbook even though Abby pointed out several times how the textbook problem differed from the Buried Treasure Problem. Eventually Eve tried to interest Abby and Jane (Sara was absent) in the work they did in Session 2 by referring to what they did "last time." It was not until about halfway through the session that Abby finally said, "the ratios," and picked up Sara's paper from the previous week and shared what Sara had written. This sharing ultimately--although not immediately--led them to determine proportions that expressed the relationship among the triangles. At the end of the third session, a correct system of two equations expressing this relationship in the pairs of triangles was set up. Eve, however, made a procedural error as she solved this system which led to the identity $0 = 0$. They were not working collaboratively so no one realized Eve's error and neither Jane nor Abby had a clear understanding of the work Eve did—even though Abby was instrumental in getting these proportions. Time did not permit them to make sense of this result in the third session, and Eve's absence in the fourth session kept them from continuing this line of reasoning. Although progress was made during this third session, in Session 4 Abby was reluctant to share with Sara the work from the third session and Jane was so sleepy that she had very little input. By not referring to the work from previous sessions in a timely manner, they wasted a lot of time.

Eve saw her role in the group as trying to keep the group on track; that is, she saw herself as helping them keep sight of the big picture. In Session 1 she questioned how finding the angle measures would be helpful. In Session 3 she was the one who kept saying "Didn't like last time..." to try to call their attention to the work they did in Session 2. Eve's persistence in focusing their attention on the work done in Session 2 helped them set up the proportions that were ultimately used to complete the problem in Session 4. Abby in her final interview also acknowledged how Eve kept them on track by asking questions.

Abby: [Eve] questioned what we were doing to make...and that way we had towe almost had to prove why, why we did what we did and then some ways realize stuff before we did it, like realize maybe something wadn't (sic) going to work, just by her asking and then by asking, too, we...it made us realize other stuff. She asked questions and by her asking questions it made us have to prove what we were doing...or when somebody had an idea, she questioned them and then they had to you know prove why they thought it was gone (sic) work and then....

Learning and understanding the problem was the goal for Sara and Eve as they worked together. Abby wanted to learn from the experience. As the constant pupil her focus was more on taking rather than giving. I believe she saw her ability to contribute as minimal and compounded by her fear of being wrong. Jane's primary goal was to solve the problem—to get the answer. Making sense of the mathematics and learning were not her goals and she used face saving strategies that limited her opportunities to learn.

Leadership factors were an influence in this group. Specifically Sara had the potential to be the charismatic intellectual that Watson and Chick (Watson & Chick, 2001) identified as beneficial to small-group learning. Her SAT-M score was 530 and her SAT-V score was 560. Additionally, she trained new employees at the restaurant where she worked. Even though Sara had the potential to be an effective leader, she did not want to be the leader because she did not want to be perceived as taking over. Nevertheless, she did not want anyone to feel left out so she frequently asked the participants for questions or suggestions which caused the group to perceive her as the leader. Her influence, however, was more negative than positive as her interaction in the group was distant, and her interest in another's thinking was somewhat superficial. Her actions had a negative impact and alienated the group rather than encouraged collaboration.

Sara's desire to include everyone was especially problematic where Jane was concerned. Jane had a very procedural approach to mathematics. If one procedure did not work, she was more than willing to commit vandalism with hopes of getting it to work. Changes

in procedures could be completely arbitrary, lacking mathematical justification. Her explanations often included illogical statements and her suggestions were often unproductive. Ironically, Sara did not want to be the leader, but in her effort to include everyone, she wound up being the leader albeit an ineffective one. Part of Sara's willingness to go along with Jane could be an example of Watson and Chick's (2001) social collaboration whereby a member agrees with another out of politeness. For Sara, I believe it was more her intent to validate all the contributions of the others so that no one would feel left out.

Abby confirmed in her final interview that she saw Sara as the leader. Even though Abby said she trusted Sara, Abby was intimidated by Sara as evidenced by Abby's relaxed demeanor during the third session when Sara was absent. Furthermore, Sara's requests for input from the others were usually superficial and vague. Sara claimed to want their input but often ignored what was said. Furthermore, she did not negotiate a common workspace which I believe led the others to perceive her as distant.

According to Watson and Chick (2001) egocentrism is a potentially negative characteristic as individuals become self-absorbed to the extent they are not open to the ideas and opinions of others. They found the factor of egocentrism to be more passive with little or no cognitive engagement. In my study I saw evidences of egocentrism in Jane and Abby. Their egocentrism was played out by using face-saving strategies and camouflaging techniques.

Both Jane and Abby used face-saving strategies but these strategies were different because their beliefs in their mathematical abilities were different. Jane admitted that she was not a strong mathematics student but she was unable to see her weaknesses stemmed from a lack of a conceptual understanding. For her, knowing formulas and procedures was sufficient; it was not necessary to know why. Jane was not reluctant to make a suggestion and even offered quite extensive justification for her assertions. It usually took Sara's expressing some doubt that would motivate Jane to use a face-saving strategy. Her strategy was usually some quick explanation such as "Cause tangent and 180 at pi is 0...because on the unit circle it's one zero

[i.e., the ordered pair (1,0)]...and if you do the tangent, it's undefined," or "Oh, hah!...just kidding...cause you have to cross multiply!"

Committing vandalism with procedures was also a way for Jane to save face. When a procedure did not work one way, Jane was completely willing to "tweak it" to make it work. This willingness to commit vandalism is further evidence of her view of mathematics as nothing more than procedures that really had no meaning for her.

Jane's face-saving strategies were more overt—offering explanations and manipulating procedures. In contrast, Abby's face-saving strategies were more covert; that is, Abby tried to conceal her lack of understanding by keeping quiet. Abby was so fearful that others would find out that she did not understand; she explained in the final interview her fear of being wrong.

Abby: I don't know...sigh....it's just this...me and the...I don't know...I don't...I don't like to be wrong...not because I don't like to be wrong but because I don't want people to know I am wrong.

Thus, to save face she would refrain from asking a question or from making a suggestion.

During the fourth session I had to insist that she share the work with Sara they did in Session 3. Even though Abby was instrumental in generating the proportions in Session 3, she lacked the confidence to share these with Sara in Session 4. I think Sara intimidated her, and she did not want to risk being wrong in front of her.

Jane and Abby did not have learning as a primary goal. As such, it was necessary for them to camouflage their lack of understanding. Their use of camouflaging techniques is further evidence of their egocentric nature. Jane and Abby sacrificed a real understanding of the task as they attempted to protect their egos. They were more concerned about how the group perceived them than they were about learning. Camouflaging techniques allowed them to appear engaged in the task; their engagement, however, was superficial. Jane would make cryptic remarks that offered little if any contribution to the task at hand. Abby would offer

supportive giggles as if she understood and approved of a remark. Abby also discreetly copied the work from those sitting near her but without understanding.

Watson and Chick (2001) identified social collaboration as collaboration that may include off-task socializing but may also involve collaboration that is influenced by social conventions such as agreeing just to be polite. In this clinical setting it was not surprising that off-task socializing was not observed. I did observe members of the group either agreeing or at least refraining from disagreeing at inappropriate times. It was more likely they engaged in these behaviors to show support rather than just out of politeness. For example, when Jane suggested they find the angle measures in Session 1, Sara discarded her own suggestion and led the group to explore Jane's. Although there were two suggestions on the floor, in an effort to have everyone participate, Sara supported Jane's suggestion. In Session 2 Jane ignored the conversation about the variable and then suggested an approach that was not mathematically correct. Rather than disagreeing with Jane, the group (led by Sara) listened to Jane's lengthy explanation. The times of agreeing out of politeness or as a show of support obviated the opportunity for effective communication and sense-making.

Abby also engaged in social collaboration by offering frequent supportive giggles. She also explained in the final interview that she believed Sara needed someone just to say that she was doing a good job. Abby seemed to believe these forms of social collaboration were effective. Abby's efforts at social collaboration were perhaps related to her own need to have her thoughts and comments affirmed. I observed that Abby's contributions were dependent upon her receiving verification from others. Thus, she believed she was offering support and verification through these forms of social collaboration.

Cognitive ability cannot be ignored as an influencing factor in the problem solving of Group 1. Abby's SAT-M and SAT-V were 470 and 510, respectively, and her grade of 48 on the MATH 1113 final exam was 21 points lower than the third lowest grade. These scores, if nothing else, served as reminders to Abby of her weak ability in mathematics and could

certainly strengthen her reluctance to share her ideas with others. Jane's highest SAT-M and SAT-V were 490 and 450, respectively, and she took the SAT four times. She made a 69 on the final exam. Although her mathematics scores indicate deficiencies in mathematics, her SAT-V perhaps explains her dislike for giving explanations in MATH 1113. Eve's SAT-M was 500, and she made 96 on the final exam. Eve's scores indicate that she was probably an over-achiever. She would not necessarily catch on to concepts quickly but with work and effort could make sense of the ideas. Such a characteristic perhaps explains Eve's need to withdraw cognitively to make sense of the ideas. Finally, as already noted Sara's SAT-M and SAT-V were 530 and 560. Her final exam score in MATH 1113 was only 74. She explained in her initial interview that after a very high score on the first MATH 1113 test, she became slack in doing the homework which affected her work in the rest of the semester.

The task itself of working collaboratively on a mathematics problem was one of the external factors that influenced the outcomes. The students came into the study with minimal experience of working mathematics problems collaboratively. They also came into the study with different obligations to the experience. Their obligation to the group experience and their obligation to themselves were often in conflict. They knew they were to work together but how to mesh that knowledge with their individual needs was problematic. For example, Eve was unable to understand the problem and could have benefited by additional reading time in the first session, but she felt the obligation to go along with the group. Sara and Eve both stated they wished the group had discussed the problem in the beginning but they did not make the request. Abby wanted to contribute but was fearful of being wrong so she wound up saying very little. If another student first said something that Abby had been thinking, then she would share her thinking. In this way, she was less likely to be wrong.

Another external factor was Sara's and Eve's absences. Of the four Sara and Eve were the strongest mathematics students. Their understanding of mathematics was more conceptual than Jane's, and they had more confidence in their mathematics ability than did Abby. Because

the group saw Sara as the leader, her absence during the third session really changed the dynamics of the group. Jane persisted in exploring a textbook problem even though Abby pointed out the limitations of this problem. Furthermore, the group considered using similar triangles in Session 2, but Sara had been most instrumental in using this approach. Even though Sara's work was available, they did not reference it until much later in the third session. It was as if **THAT** was Sara's work and not the group's work.

Sara returned for Session 4, but Eve was absent. Jane and Abby, however, were either unwilling or unable to help Sara understand the work they did in Session 3. As a result, Sara primarily worked by herself. She would ask Jane and Abby for feedback but her requests were largely ignored. Only after I insisted Abby share the work from Session 3 did she finally comply. Again, Abby's reluctance just confirms my belief that not only was she so unsure of herself but was especially so in Sara's presence. And just as Sara's work was not accessed in Session 3 to help the group begin where they left off in Session 2, Eve's work was not accessed in Session 4 to help them begin where they left off in Session 3.

Related to the idea of considering an individual's work as her own rather than the group's was the group's failure to develop a common workspace. A common workspace may have encouraged more collaboration as they would have been more likely to understand and make sense of each other's thinking. Without this common workspace, they could only assume they understood another's references. There were instances of students talking past each other. There were instances of misunderstandings. Although a common workspace may have minimized these instances, there could be no guarantee collaboration would have occurred. The students could easily have shared their work with each other or made requests to see another's work as they tried to make sense of the student's thinking.

Another factor that perhaps influenced the outcomes was their decision to maintain the same seating arrangement throughout the four sessions. Perhaps if I had suggested a different

seating arrangement the dynamics in the group would have been different and different outcomes would have been possible.

I, as researcher, certainly influenced the problem solving. In the first session my role was more of observer. When I realized the misconceptions dealing with variable representation, I assumed a more active role in the second session with my objective to help them make sense of this concept. In retrospect, I believe I should have dealt with this misconception more directly. The students were focused on solving the problem and probably viewed my questions as intrusive rather than helpful. Furthermore, I was not aware the misconception was so pronounced and assumed they could make sense of their misconception through my questions.

I also influenced the problem solving by encouraging them to share their thinking when there were periods of silence. My request to Sara to elaborate on her meaning of “it’s all proportional—to the angle” in the second session that helped her see the relationships among the similar triangles. I insisted that Abby share their previous work in Session 4 with Sara. Sara and Eve had similar problems with solving a system of two equations with two unknowns. I did not point out Eve’s mistake in Session 3 but Sara’s question in Session 4 allowed me to make sense of her misunderstanding so that I could provide the help she needed to continue. Had I pointed out Eve’s mistake in Session 3, the group probably would have completed the problem in this session.

Although the influencing factors identified by the participants included some of those I saw, their list was fairly short. Lack of familiarity with the other participants was named. Eve especially talked about how uncomfortable she was on the first day. Lack of familiarity perhaps exacerbated Abby’s fear of being wrong and contributed to her reticence.

A factor that was perhaps important to this group was the ability for the girls to be socially compatible. For example, Abby and Jane noted that the girls in the group “got along.” If group members believed it was important for the members to be compatible and that conflict

should be avoided, this may partly explain the missing questions and doubts that could have led to sense-making.

Jane and Abby alluded to their assumptions about how the problem should be solved as influencing the outcomes for them. Jane assumed the problem should be solved using trig and continued to search for a strategy to make this happen. After solving the problem, Abby and Jane both talked about how surprised they were the problem was solved so easily with ratios.

Sara observed that having no common workspace made it difficult to know what each person was doing. She suggested it would have been better to have had a single piece of paper to work on so that everyone in the group could be aware of what was being done with explanations given along the way rather than just at the end. Sara even made this observation in her reflection following the first session but never suggested the group develop a common workspace.

Sara was also constrained by the task of working together. She said that she believed she could have worked the problem herself but did not want anyone to be left out. She said it was frustrating for her, though, to ask for feedback and to get no response.

Jane named external factors that influenced the outcomes for her. One influencing factor was the time at which the sessions were scheduled. Early morning (8:00 AM) and during the summer made it difficult for her to stay focused, and she admitted she had a very difficult time staying focused during Session 4. Sara, in her final interview, even commented on how sleepy Jane appeared during this session.

CHAPTER 5

CASE STUDY OF GROUP 2

In the following sections the case study of Group 2 is presented. The description of the participants in the first section capitalized on data from the initial interviews. The second section is a scrutiny of the group's investigation of the Buried Treasure Problem. In the third section data from the reflections of the session and the final interviews have been used to present each participant's perspective and understanding of the experience of working together to solve the Buried Treasure Problem. The fourth session is my synthesis.

Participants

Group 2 consisted of Helen, Lisa, Sandy, and Tess. All four students were in my MATH 1113 course spring 2002. Lisa and Tess were in the section that met at 8:30 AM; Helen and Sandy were in the section that met at 10:00 AM. Tess is African American, and the other three students are Caucasian. Some demographics for the students are in Table 3 .

Table 3
Demographics for Group 2

Student	Age	Major	Highest SAT MATH Score	Highest SAT Verbal Score	MATH 1113 Final Exam Grade
Helen	19	Early Childhood Education	490	610	80
Lisa	20	Early Childhood Education	560	490	87
Sandy	37	Middle Grades Education	N/A	N/A	41
Tess	19	Biology (Nursing)	450	490	80

Helen

Helen is the fourth of 5 children. Her parents are both teachers in the small private school Helen attended, and her father taught her geometry and trigonometry. She described her family as being very close and doing “a lot of things together.” Helen enjoys reading and enjoys writing about her reading. She is majoring in early childhood education.

Because her father is a mathematician, she was somewhat reluctant to admit mathematics is her least favorite subject. She explained, “I’ve always struggled with it [mathematics]...I mean I’ve never been a bad math student but it’s always been a...a...I’ve always had to work harder at math than any of the other subjects and it doesn’t come easy.” Helen’s next comment points to difficulty remembering all the rules.

Helen: I can grasp the concepts and everything ...but it’s just a lot of work to get there...remembering the formulas and the steps and everything. Once I get to a certain point...all the formulas...like I can solve math...it’s just the steps, the formulas...and like I can put numbers into formulas, no problem, that’s real easy but it’s like I said the steps to get there because I always skip them...I always look for the shortcuts...I make really careless mistakes because I rush, cause I just want to get it over with.

Helen claimed to be stronger in geometry and trigonometry than in algebra. She also spoke of how much she enjoyed all the applications they did in her high school physics class. Her strength in geometry, trigonometry, and physics showed an ability to see relationships. In contrast, her weakness in algebra stemmed from her difficulty in remembering all the rules and formulas. She admitted that in Algebra 1 and Algebra 2 she “just went through the motions just to get through them...so I never really grasped the concepts.” In contrast, application problems are more meaningful to her. In the following dialogue Helen’s desire to understand mathematics conceptually was evident. Also evident was algebra as a stumbling block because she did not learn it conceptually.

LBC: What about a problem...an application problem? How do you feel when you are confronted with an application problem?

Helen: Like the boat is so far away from the cliff?

LBC: Uh huh.

Helen: I do much better with those cause I can visualize....like there is real things...it's not x and y which are unknowns. It is a boat and a cliff you know and it's much easier.

...

LBC: Do you think your understanding of algebra is that it is just a bunch of rules that you have memorized and you don't really know why they work...and you don't know when to apply what?

Helen: It's some...and it's partly I don't know when to apply what. But it's more...it's just a set of rules that are like...that have no application to me...it's like I can't see how any of this stuff applies to real life so if I can't apply it to real life then I'm just going to write it off as nonsense [inaudible]...it's good for the grade now; I'll just do it.

LBC: But it's not very relevant?

Helen: Right.

LBC: Do you see those kinds of skills as being able to help you solve say a boat problem?

Helen: [Giggle] Yes, yes. It's completely contradictory, I know! But it is just the way my mind works.

LBC: Are you more likely to do better with those skills when they are embedded in a problem-- like say a boat problem?

Helen: uh huh. Like I'll remember...like if I have the boat problem then I will start remembering things like I would have no reason to remember it...but if I just have the x's and the y's, then it is still all just x's and y's and it's still all just "OK, it's nothing," you know. But if I have the boat problem with the cliff, with the water, with the lighthouses, then it's like everything starts pulling together and you draw a little picture.

Helen had worked in groups in high school physics and did not mind pairing up with others students in the MATH 1113 class. She saw them as being “on the same team” rather than as a competition. When she worked with others, though, she was unwilling to accept an answer from someone else without working the problem herself. The group experience in MATH 1113 was positive because it gave her the opportunity to hear how somebody else thought about a problem and “made [her] think about problems differently.” Helen appeared to have learning rather than performance as a goal when she spoke of wanting to work with people who were not just after “the grade.”

Helen: You have to match people with people who will be challenged, like want to work with or be challenged to work with them...getting...not...it's...not just for the grade...but for the self-gratification of “I solved this problem and we all did it together and we pooled our knowledge.”

I asked Helen how she felt about having to explain her thinking in writing in the MATH 1113 class. She frankly admitted that she “hated it at the time” but came to realize “it was definitely the most productive part of all of last semester.” When explaining her thinking, she acknowledged she often “found all sorts of holes in my thinking...it made me follow my thought process.”

Lisa

Lisa is majoring in early childhood education and said she has always wanted to teach. Her parents divorced when she was six years old, and she has a sister, a half-brother, and a half-sister. She attributed her current like of mathematics to her mathematics teacher in her senior year of high school, claiming mathematics has “been fun since then...kind of like solving riddles.”

Lisa took college algebra at ASU and made an A. She said she is very “comfortable with algebra...I know [algebra] backwards and forwards.” Because she did not take trigonometry or precalculus in high school, MATH 1113 was difficult because “I had **never** [emphasized] seen anything like it before.” Lisa took MATH 1113 twice with me. She made a D in fall semester of

2001 and an A in spring of 2002. She talked about how her understanding changed from one semester to the other.

Lisa: And in the midst of it—not the first time I took it—I understood it and I knew what I was doing. It was really understanding it though, you know. I mean, I understood some of it [that is, the first time] but I didn't have a full grasp on it. The second time around, uhm, that was when I was really able to start analyzing.

LBC: Did you approach it differently the second time than the first time?

Lisa: I don't think so...the second time around I had a lot more fun with the word problems. I was able to pick them apart and uh, you know, so you need to do this before you can do this, and I was able to understand the steps.

Understanding is important to Lisa. She said, "I can't just listen to somebody else and say, "Ok, that's how it's done." I have to understand *why* it's done that way." She said that if a problem was done in class that she did not understand, she would go home from class and try to make sense of the work that was done in class.

Lisa: Some things I felt like...if the book did a good job explaining it, I would usually go back to the book because the book had better diagrams than what I could scribble down in class. But if I felt like there were some ways where you [LBC] would teach it differently than the book so I would go back to my notes...uhm...but then there were other things that you [LBC] would teach differently and I'd go back to the book and I would learn the way the book had done it...uhm...so it depends I guess on how well I understood it in class...if I didn't get it in class, I'd almost always go back to the book...because it would break it down, you know, and give you step by step instructions whereas my notes...I didn't do that...I would just copy down the problem.

Lisa said she does not like group work because it is hard to work with "somebody else unless they're on the same level." Furthermore, she likes to do things her own way. She said

she tried to avoid working with other people in MATH 1113 when asked to pair up. However, her perception of the benefit of group work was tied to her level of understanding.

Lisa: I think the first time I took it [that is, MATH 1113 in fall 2001] when I needed the help, it [group work] was more helpful...the second time, it wasn't that I didn't want to help people ...it was just a lot of times when you are in a class like that...people that think they don't know what they are doing, the person next to them probably doesn't know a whole lot either...and I was still learning so much that I didn't want to go and tell somebody my way of doing it and it completely throw 'em off...you know I'm not comfortable enough in that area where I can just say "Ok, this is the way I would do it"....and try and teach and show...I would rather I guess sit there and work on the problem myself than have to stop and say "Ok, this is the way I would do it" and explain it to somebody else because I just needed to work on it and really break it down piece by piece.

Lisa saw the group work during her second semester of MATH 1113 as interfering with the time she needed to make sense of the work. Furthermore, she was reluctant to explain her way to someone else because it was still so new to her. It appeared that she saw group work as beneficial when she needed help herself, but less beneficial to her when she personally did not need help.

I asked her about the requirement to explain her thinking in writing in MATH 1113. She said it was "effective in some ways" but frustrating at the same time. She said it was not that she could not explain her thinking, but it was just time-consuming to have to write it down.

I asked her to describe the kind of mathematics problems at which she believed she was best. She said she enjoys word problems because "you kind of have to look for the material you need and throw away the material you don't." Word problems were a fun challenge to her because there were times when she would "just pull out word problems and just do them because [she] wanted to see if [she] could figure them out." Although she preferred a symbolic approach, she would use graphs and tables on her calculator to help her verify her answers or

to help her approximate an answer. Lisa was confident in her ability to solve problems and claimed that “usually I know how to set something up.” Moreover, she checked her work—especially on a test--by using a variety of approaches.

Lisa: If I was taking a test I would always usually do both methods [symbolic and graphical] because I am big into checking my answers you know and then I always in math take until the last minute to do the test [that is, completely finish the test] because I will go back and plug my answers back in to make sure I got them right and every way that I can because I want to know that what I did was the correct way to do it.

LBC: So you would sometimes use two different ways to solve problems?

Lisa: Most of the time I would, you know, because the graphing problems I could just throw into the calculator and get my answers where I would have already worked them out algebraically. So then I would use the graphing method to go back and check.

Sandy

Sandy is a widowed mother of two teenagers. After graduating from high school, she served over three years in the army and was a stay-at-home mother from 1987 until 1994. For six years she worked as a paraprofessional in the public school system. She began college in 1999 with plans to become a middle grades teacher. Her husband died shortly after she began college.

Mathematics is one of Sandy’s areas of concentration for her middle grades program. She recalled how gratifying it had been for her as a paraprofessional to help students who struggled with mathematics.

Sandy: I worked with some students in the math area; particularly I remember third grade and working with some kids...and they weren’t able to comprehend, but when I sat down with them and tried to give them some new strategies and different strategies, a light bulb went on, and it was so gratifying to see that light bulb going for them.

Sandy said she enjoys mathematics but it is very challenging to her. She made a D¹ in MATH 1113, a required course in her program. Her first experience with a graphing calculator was in MATH 1113. Sandy likes classes that challenge her. She said she does not “like any class that’ll give me the grade. I like any class that’ll make me work for the grade.” She recognizes herself as a visual, hands-on kind of learner.

Sandy: I am a much better learner...if I have the visuals and not just someone in front of class just you know shooting out the information and expecting me to just retain and maintain all the information.

Sandy maintained a well-organized notebook for the course and even brought this notebook to the problem sessions for this study. She spoke of the stress she felt in test-taking situations because she could not refer to her notes. Sandy said that she often talked to herself as she worked a mathematics problem—“trying to recall different things that would be associated with that problem so that I can put that in practice.” Doing her MATH 1113 homework was not a task she looked forward to because she knew it would take a lot of her time. She said she would “dread” tackling the assignment because she would confuse herself.

Sandy: You are being taught in the classroom and when you leave the classroom the homework that we’re given is an assessment of what you’ve done in the classroom, or an enhancement of what you’ve done in the classroom. And in the classroom I’m understanding and comprehending well, but when I get put by myself and on my own...everything just goes out the door. So I look at my notes a lot, too. Trying to figure out the math problems and when I’m **done** [her emphasis] I’m quite pleased--most of the time with the results because it was a lot of time and effort put into it--most of the time I’m able to solve the problems.

¹ Sandy retook MATH 1113 with me fall 2002 and made a B in the course. Her final exam grade was 69. After one of the classes during the fall semester she told me, “Everything is so much clearer now—I never knew there were so many things I didn’t understand!”

Sandy's quote above shows that mathematics was not easy for her and yet there was such pride in successfully completing an assignment. When asked if she ever worked a problem in more than one way, she laughed and said, "Most of the time I only know one way to work the problem!" She found it very beneficial to pair up with other students in the classroom. She referred to Carl, the student with whom she usually worked, as her personal tutor because he "could break it down so that I would understand it or he would show me exactly how we came about figuring that out." She even met with Carl a couple of times outside of class for additional help.

Tess

Tess, a very quiet student, admits to being shy. Her mother has always encouraged her to do community service, and she has volunteered with organizations such as the Salvation Army. Tess said she has always loved going to school and wants to be a nurse. She initially planned to attend Georgia State in Atlanta, but during the summer after high school graduation, she decided she did not want to leave home so she enrolled at Augusta State. She plans to pursue her nursing degree at the Medical College of Georgia.

Tess believes mathematics to be her strongest subject. Her strength, however, is in memorization of rules rather than in applications. She claimed to "know the steps" and to be "best at solving equations and stuff like that...I'm not good at word problems...I hate them." I asked her why she disliked word problems so much.

Tess: I don't know...I don't think that I think about like what you should, what you have, what you should...I don't know how to really set them up from the words into an actual math problem...I don't know how to transform back into the... like numbers...

LBC: So any kind of problem that is stated in words is somewhat of a stumbling block for you?

Tess: Somewhat, somewhat. When it gets tough but simple problems I can get them...but it's like when they have, you've got to do a number of formulas and set it up this way, set it up that

way...I don't know it is like I try but I end up setting it up the wrong way or just not thinking about other things that I should have thought about.

Tess said the experience of explaining her thinking in MATH 1113 was a first for her. The experience was beneficial because she often found errors in her work as she explained her thinking. When working a problem, she preferred a symbolic approach but would "solve it graphically just to confirm [her] answer." Because she disliked word problems, I asked how she would approach them.

Tess: Oh....then....I look at it, I read it, I read it about 5 times, and then I write down all the information—what it gives me—and I use whatever formulas or whatever it is to set it up and try my best to solve it.

LBC: Ok. What do you do when you get stuck on a problem?

Tess: If I get stuck, I leave it and come back to it later. I don't know why I do that—I think it is just to give myself a break or I'll go and watch TV for maybe 30-45 minutes or I might get a snack or sometimes I just sit on the porch and I'll come back to it.

Tess seemed surprised that the strategy of leaving a problem and coming back to it later was useful to her. Although she did get help with word problems from the Math Lab (the department's tutorial service), she never came by my office for help. She generally paired up with a student, Cindy, in class and said it helped to work with other people because "some people look at it differently and does [sic] it this way and I might not know how to do it that way and they will show me or I could help them." She said she never got together outside of class with other members of the class to study but said had she been "more interactive in class" she may have joined or formed a study group. Although she and Cindy would work together in class, her shyness prevented her interaction with others in the class.

Investigation of the Buried Treasure Problem

The session began at 2:00 PM. Group 2 talked to each other casually while I got the cameras started. Although Tess had little to say, Helen especially kept the conversation going.

They laughed as they remembered things that occurred during the previous semester's MATH 1113 class. Helen and Sandy had been in one class while Lisa and Tess were in a different class. They seemed at ease with one another even though they did not all know each other.

I asked them to introduce themselves and to tell what they have been doing that summer. I then explained that I was studying how a group of students work together to understand and solve a problem. They were asked to work together and to listen to each other. I further explained that although I would be in the room and would interject comments from time to time, I planned to let them do the work themselves. I asked, "What do you need from each other to make this a good experience?"

Lisa: Communication from everybody, I think. Cause I think between the four of us we will all remember----

Helen: Hopefully [and there was laughter.]

Lisa: ...and we can put it all together.

It was interesting that Lisa pointed out the need for communication from everybody. As the session progressed, there were a number of episodes during which Lisa, Sandy, and Helen engaged in effective communication. Tess, the quieter of the four, frequently listened but without engaging in the conversation.

I asked them to write in ink and made rulers and calculators available to them. I forgot to bring a MATH 1113 textbook to this session. Sandy, however, did bring her MATH 1113 notebook.

The problem was handed out, and they began reading. Without looking to see if everyone had finished reading, Sandy observed, "Obviously we are dealing with some right angles here," and Helen gleefully responded with, "Pythagorean Theorem!" Lisa said she had not finished reading yet which caused them to return to their reading. Shortly, though, Sandy noted, "Oh, OK so we have to find a point of intersection," and Lisa and Helen both pointed out that E is the point of intersection. The problem reminded Sandy of a problem from MATH 1113,

and she reached for her MATH 1113 notebook. She said, “It was toward the end—it was toward the end of the unit where we had to do—finding distances.” The others smiled as she began flipping through the pages, and Sandy continued, “... and you’ve got your observation point—I think that is what it is called... you’ve got your angle of depression and your line of sight—can we use that?” Helen said, “Well,” as if she was not completely convinced by Sandy’s observation. At this point, Lisa and Helen looked again at the problem and Sandy continued, “And you’ve got your degree factor where—and that’s where you can use the Pythagorean Theorem.”

Helen giggled, propped her elbow on the table, rested her chin in her hand, and looked as if she were trying to think of a tactful way to disagree. Lisa also sounded doubtful.

Lisa: I don’t know—uhm.

Sandy: cause you are dealing with a lot of right angles—what I mean...

Helen: But since we don’t need to know the measure of an angle—we need to know a distance so I would tend to stray [sic] away from doing all the angle stuff—cause that adds a lot more to the problem than just finding the straight lines.

Sandy: OK.

Helen: All they want here is a distance and not an angle. They don’t want the measure of the angle from E to D or whatever—they just want the leg. If you think about it, it is like a triangle—they want the leg of...

In this opening episode Sandy’s tendency to be egocentric is evident as she was the first to initiate conversation even before the others had finished reading. Although Sandy did not specifically suggest finding angle measures, her description of the problem must have made Helen assume Sandy was suggesting they find angle measures. Helen and Lisa seemed doubtful about the suggestion, and although Sandy mouthed a couple of obligatory “uh huhs” as Helen explained her reasoning, Sandy continued to look at the work she was engaged in with

no indication that she was ready to concede. In the following episode Sandy's tenacity for her idea is obvious as Helen explained why she disagreed with this approach.

Sandy: Oh, right here...so we need to find this point right here—from here to the road.

Helen: Right.

Sandy: And we know that's a right angle there but wouldn't we still use like the tangent...

Tess: That's what I was thinking.

Sandy: Yeah...like if you know the tangent is—like in this example—the tangent is like forty degrees then you can figure it out like that. [turning her notebook and pointing to the problem]
[pause]

Helen: Yes, but we'll have to go figure out what those two angles are then because all we know is...this is how I would do it...all we know is this has ninety degrees. That means this can either be forty/fifty, forty-five/forty-five, thirty/seventy. I mean it can be any number of...so

Helen's mistake of "thirty/seventy" went unnoticed. As Helen was speaking, though, Sandy nodded her head affirmatively but appeared to be ignoring Helen's comments. In the following exchange Lisa stood out as a potential leader of the group when she interrupted by suggesting they record the measurements on the drawing, successfully curtailing the disagreement between Sandy and Helen.

Lisa: Wouldn't it be best to go ahead and put the numbers in—what we know—like go ahead and say B to D is 300 and see what we've got first...?

Helen: Right...simplify it.

Following Lisa's suggestion, each person recorded the measurements on her original drawing; no one drew a schematic representation of the situation. Helen observed they needed to find A to C and B to D, and then they would have the two main triangles completed. She did, however, express some doubt about the usefulness of this information by stating, "Not that that would help us any...it is just an observation."

Although Sandy and Lisa voiced their understanding of Helen's observation, Lisa let the group decide whether to proceed this way, again showing her leadership skills. Helen still appeared doubtful.

Lisa: Ok, so do we want to figure that out?

Helen: I don't know...

Sandy: we can use the Pythagorean... [In the final interview I found out how little Sandy knew about using this theorem.]

Helen: ...it's just an observation.

Lisa: Yeah, you would use the Pythagorean Theorem to get that.

As shown below, the real-life situation of this problem was a concern of Helen's. Specifically she was concerned how the information about the lakes affected the distance.

Helen: [addressing her question to me] Does that mean that the lakes change the distance?

LBC: No...[giving the question to the group] What do you think the lakes mean?

Sandy: It just means that's a distance they can't measure...but through this [that is, the mathematics] we can get them measured...just knowing two sides.

LBC: They could do this with paces—walking that off-- but they couldn't walk over the lakes.

Helen: Ok...I was just making sure they weren't going under [playfully gesturing with her hand to indicate going under]...and adding things that weren't really there.

Here, Helen tried to make sense of the situation by asking me how the lakes affected the distance. Although she addressed her question to me, I returned it to the group. Sandy responded to Helen in such a way that I was convinced Sandy understood the problem situation. Sandy, an older student, had served in the military and was a widow with two teenage children. Although Sandy's mathematical skills were rather weak, her response to Helen showed her ability to reason logically and confirmed how a group can benefit from the previous experiences of its group members.

Sandy's weak mathematical skills were revealed in the work she did to find the lengths of segments AB and CD with the Pythagorean Theorem. Lisa asked which of the two triangles they were using first, and Sandy suggested triangle BAD. Her work is shown in Figure 37 and although her written work revealed her weak skills, the following exchange gave no indication of this weakness.

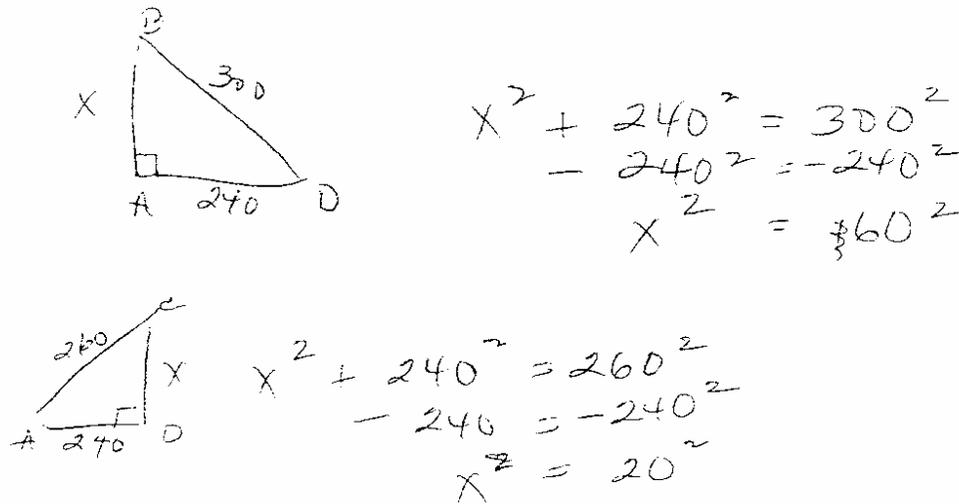


Figure 37. Sandy's use of the Pythagorean Theorem.

Sandy: I know it's a squared plus b squared equals c squared.

Helen: Yeah and c squared being the hypotenuse. So it would be...

Sandy: [to herself] x squared plus 240 squared [inaudible]

[they worked independently]

Helen: 180

Lisa: That's what I got.

LBC: 180 for...?

Lisa and Helen: for AB.

[Sandy wrote 180 on the figure although her work did not support this answer.]

Sandy: Ok.

Lisa: So now do we want to find CD?

Sandy: Uh huh. Will that help?

Helen: Can't hurt!

[worked independently again]

Tess: What are we finding?

Sandy: We are finding the length of CD.

They worked independently to find the length of \overline{CD} , and Sandy even picked up her calculator and entered data into it. When Lisa said, "So CD is 100," Sandy looked on Lisa's paper, and then recorded the value of 100 on the drawing. Although Sandy's work did not show a final value for AB and CD, it became obvious during the final interview that she lacked the manipulation skills to carry out the computations. Sandy knew she worked more slowly than the others so she accepted their answers without realizing her work would not produce the answers reported. Her behavior is an example of social collaboration that did not lead to lifting for her. (See Sandy's final interview for a discussion about using the Pythagorean Theorem.)

After getting the lengths of \overline{AB} and \overline{CD} , Helen restated the goal of the problem as an example of keeping the focus on the big picture. "We need the distance...from E to the road...so this makes its own little triangle again so if we could find...either these two legs or these two legs...cause we know that this is 100-- but that's not a right triangle so that's a moot point." Helen's observation was ignored by the others, but I asked her to identify the triangle to which she was referring.

LBC: Which triangle is not a right triangle?

Helen: CED.

[pause]

LBC: And why does that pose a problem then?

Helen: Because then you can't use the Pythagorean Theorem. Cause I was going to try to find the triangle E—and I was going to name this F, the point that we need to find to the road, where E drops down and intersects \overline{AD} —I was going to try to find EFD or whatever.

Even though the others ignored Helen, her think-aloud conversation was effective as it helped her make sense of the relationships in the problem. It was then that Sandy suggested finding the point of intersection, E. The following exchange shows Sandy's tenacity about finding the slope even though Helen tried to disagree. It also shows how Sandy's mention of "the slope of a line" prompted Lisa to consider a graphical approach to solve the problem.

Sandy: What about finding the point of intersection? [pause] I'm not sure how to do it—wouldn't we find like the slope of a line?

Helen: Yeah, what I was thinking we could do if we knew what E to D was, then we could solve...

Sandy: [interrupting] Right, and to do that we've got to find the point of intersection. Right?

Helen: Oh, yeah. I mean, yeah, I wasn't calling it a point of intersection, I'm sorry—I was just calling it a [inaudible]

Sandy: So how do we do that? Is that where we find the slope of a line?

Helen: uhm...I'm not sure.

Lisa kept looking from Sandy to Helen as they exchanged ideas. I sensed Lisa wanted to say something so I prompted her.

LBC: What are you thinking?

Lisa: What if we graphed it?

Sandy immediately said "uh huh" but apparently did not see how to coordinatize the drawing because she said, "But we've got to have some figures to graph." Lisa assumed Sandy was referring to the coordinates of points as the "figures to graph" and said, "We have 'em [the figures]." Lisa then explained how to coordinatize the figure, but her words and her hand

motions did not agree. She moved her hand vertically and said, "I mean if we put this on the x-axis, and we go up a hundred and eighty then that would be your point from zero to one eighty-- will be your line AB." Moving her hand horizontally, she said, "Your two forty. You've got some points." No one in the group noted the inconsistencies. Perhaps they were not listening carefully or perhaps they did not understand what Lisa was suggesting.

Sandy lifted Lisa to consider a graphing approach. This is a good illustration of how listening carefully to others in the group can provide lifting and is similar to Cobb's (Cobb, 1995) indirect collaboration . Although Sandy lifted Lisa, Sandy did not understand why Lisa suggested they coordinatize the points.

Sandy: ...but you still have to find the slope, don't you? Because you've got to know the angle at which it's...

Lisa: But to do the slope you've got to have points on a graph.

Sandy: OK... [she sounded doubtful]

Here, Sandy appeared to be agreeing with Lisa but without understanding, an example of social collaboration. As Sandy and Lisa talked about a graphical approach, Helen (as she explained in her final interview) did not understand how graphing would work. Egocentrism may have played a role in her suggesting a different approach rather than trying to make sense of the graphical approach. It was on her third attempt to interrupt Sandy and Lisa that Helen was able to explain her idea. Her explanation included representations that were inconsistent.

Helen: We could go algebra with it...and make like E to D x and 300 minus x would equal E to D.

No one questioned the inconsistency of her statement. Lisa even said, "Right." Helen's statement, however, was not a slip of the tongue. My discussion with her during the final interview verified she did not understand the concept of variable representation but was not lifted until her conversation with me during the final interview. Either the others were not

listening carefully to Helen's comment or they, like Helen, did not understand the concept of variable representation.

Helen then made a comment that baffled me, and I asked her for further explanation. In the exchange below, Lisa disagreed with Helen's explanation but the disagreement was not resolved so lifting did not occur for Helen.

Helen: But then you have two unknowns which kind of defeats the purpose of algebra.

LBC: Why do you think having two unknowns defeats the purpose of algebra?

Helen: Cause you are supposed to solve for...

Lisa: [interrupting] You can do it.

Helen: In its simplest form you are supposed to solve for **the** [emphasized] unknown and having two unknowns defeats the **the** unknown. But I guess they are both the same unknown.

It was not clear what Helen meant by "I guess they are both the same unknown," and no one asked her for clarification. Tess had done no talking, and Sandy made an effort to involve her by asking, "Tess, what do you see?" Tess, who admitted she is shy, gave a slight shrug of her shoulders but did not respond. After a fifteen second pause, Helen weakly suggested, "Or we can compare...if we..." but was not allowed to finish her statement because Sandy interrupted when she finally realized they did have points on a graph.

Sandy: [excitedly] We do have points on the graph...we do...because if you go up 180...

Lisa: That's what I was saying.

Sandy: ...if you go up 180

Lisa: and you go over 240, there's your point...A, B, and D.

Sandy: over 240...ok.

Lisa: Do you see?

Sandy: Yeah. So we've got to figure our points on a graph. And actually this right here can be our y-axis and this is our x-axis, right here.

Lisa: Right. That's what I was saying.

[Sandy looked toward me for approval.]

When Lisa explained about plotting points earlier, Sandy obviously did not make sense of the explanation at that time—it took time for her to construct this meaning for herself. Helen continued to be dubious of the graphing approach as the following exchange shows. Sandy apparently had some sense of the big picture although she probably did not possess the knowledge and tools to get there. She depended on her partners to fill in the gaps and to correct her when she made an error. Sandy had some trouble naming the coordinates of the points, but Helen and Lisa provided the help needed to obtain the coordinates of A, B, C, and D. Even though Helen was not convinced the graphing approach was viable, she nevertheless helped Sandy and Lisa determine the coordinates of the points. Helen's involvement showed her willingness to understand the approach that was suggested.

Helen: I don't understand how that [plotting the points] would help us solve the problem, though.

Sandy: Because that'll help us find the point of intersect [sic]. If we know how to plot this point right here—which is 100—well, its 240 and 100, that's this point [that is, C]—and if we put this one, which is 240 and 0...or 0 and 240?

Helen and Lisa: 240 and 0

Sandy: And this one is what?

Helen: over 0 and up 180

Sandy: zero, one eighty [(0,180)] and this one is zero, zero. So we've got four points and once we plot that we will find our intersect. And then we can find our point of intersect which will give us this number right here and then we can subtract and what not and figure out our right angle.

[Sandy was excited as she explained.]

Helen: I don't know.

Lisa: Do you see what she is saying? [looking at Helen]

Sandy: Do you get it? [still excited]

Helen: Keep working it out and then I'll get it.

Sandy enthusiastically explained the approach, and Lisa offered verification that it should work by asking, “Do you see what she is saying?” Sandy had the big picture in her mind although “figure out our right angle” did not make sense and even Lisa did not question her. Helen still did not see how the strategy would work but Lisa obviously agreed with Sandy’s explanation so Helen encouraged them to continue. Helen was willing to try to make sense of the approach even though at this point she did not understand how the approach would help.

Sandy laughed as she said, “So now we’ve got to plot these on our graph.” I knew that Helen was not convinced, and I had no idea of Tess’s understanding of the problem because she had made no suggestions nor asked any questions. When I asked if they understood, Tess answered, “yes ma’am,” whereas Helen admitted, “I understand what she is doing—I’m just not making the connection to where that will solve the problem for us, but I will see as it unfolds.” Helen did not ask for clarification to the approach even though she had questions, and the others chose not to elaborate. Sandy, still excited about the approach, asked Lisa if she thought it was a good approach. When Lisa agreed, Sandy’s next remark revealed her lack of expertise in executing procedures.

Sandy: So now how do we... [She laughed so her words were unintelligible. She picked up the calculator but was having trouble so I asked what she was trying to do.]

Sandy: I’m going to plot—we are going to plot the four—

Lisa: the four points that we know—

Sandy: the four points that we know and then we are going to find the point of intersection—and that’ll give us E and then we’ll be able to go from there...but I don’t know how to graph this thing.

Helen suggested going to “y equals” but Lisa pointed out they had no equation so “y =” menu would not be helpful. Tess then suggested they enter the data into a list but because the others did not hear her, I asked her to repeat what she said—an example of an outsider

influencing the problem solving. Sandy, though, did not remember how to enter data into a list so Tess reminded her of the necessary keystrokes.

They spent about 2 minutes and 15 seconds entering the data into the lists. During this time Sandy thought aloud as she entered the x-coordinates of A, B, C, and D into L_1 and the y-coordinates of these points into L_2 . Helen entered the data but questioned, "How do you change your x's and your y's?" Realizing Helen's question was how to change the minimum and maximum values on the axes, Lisa reminded her to select WINDOW. The following exchange shows a collaborative effort as they set up their windows and compared their scatter plots.

Lisa: So what are ya'll using for your window?

Sandy: I've got xmin negative 10, xmax 200.

Lisa: You are going to need to go further out on your x.

Helen: Yeah...

Lisa: You are going to need to go out to at least 240, probably 300...because you've got 240 on your x...

Sandy: OK...yeah...and I'm counting by tens.

Lisa: You may want to go up higher...I mean, you can but...

Helen: [sounding somewhat exasperated] I got one point.

Lisa: [leaning over to see Helen's calculator display] Let me see...

[Lisa clearly entered Helen's workspace. She and Helen jointly held Helen's calculator as Lisa suggested a change for Helen to make. Sandy, too, leaned over to hear their conversation.]

Lisa: Take your window out to like -50...and you've got your other one...you just don't see it...Do you see? Because your line was the same as your y-axis.

LBC: So what does your graph look like?

Lisa: I've just got two lines...two straight lines...because this right here we didn't put in...I mean these are really the only point and this point...

LBC: [asking Tess] What does yours look like?

Tess: I've just got the four dots...

Lisa: My lines are connected.

LBC: How did you get yours to connect?

Helen: You can like go and...

Lisa: Go into your second y and you just change the type...

Tess: Oh, ok.

LBC: So you told it to change it to what...under STAT PLOT?

Lisa: uh huh...and then I changed it to a line.

LBC: OK...to a CONNECTED GRAPH. [Lisa used the xyLine as the type of stat plot she constructed while the others had used a scatter plot.]

The above episode consisted of several factors that influenced the problem solving. First, previous experience was a factor as Tess suggested they enter the data into lists. Tess's suggestion led them to construct a stat plot. There was effective communication as Tess helped Sandy enter the data into the lists and as Lisa helped Helen and Sandy set up appropriate windows. Helen also admitted when she needed help with the graphing calculator which led to lifting for her. Effective communication was also a factor as questions were asked, explanations offered, and listening occurred. In the following episode sense-making factors, leadership factors, and effective communication influenced the problem solving as they used Lisa's idea of a connected graph but still realized the graphs were different.

Lisa obtained "two straight lines" using the connected graph option—her segments were CD and AC. She saw Helen's graph showed segment BD and asked, "How did you get that line?" Helen explained that she selected the connected graph option but attributed the discrepancy to "my calculator is screwed." Helen's comment showed her lack of confidence in using the calculator. Admittedly, I was stumped at that point, too; why were the graphs different? To buy myself some time, I tried to redirect their thinking, an example of an outsider's influence.

LBC: What is that you need to do? What is it that you need?

Helen: We need.....

Sandy: [pointing to the drawing] We need to show these two lines.

LBC: So how can you do that?

Sandy: That's where you've got to find the slope. And then come up with a formula that we can stick in the calculator that will show the two different slopes...we've got our points...we've got a point and an endpoint so we've got two points where we can find the slope of a line. [The others listened as Sandy explained.]

Helen: We don't even need to bother with a graph. We can just find the slope of this line and then I mean graph it into the calculator...I mean as an equation.

During the next 20 seconds Lisa, Helen, and Sandy were focused on the same goal. I overheard Helen rattle off the slope formula in answer to Lisa's question about the components of the formula. Someone also recited the slope-intercept form of the equation of a line. While they were collaborating, I saw that Tess was working independently so I called her name to encourage her to share what she was doing.

Tess: I was trying to get the equation of the line but I can't get it.

Tess's graph consisted of segments BD and AC. Because no one else had a graph similar to Tess's, I said, "Look at Tess's graph." They got very quiet as they stared at the graph. Sandy finally said, "Oh, gosh!" I put the four calculators side-by-side so they could see four different graphs had been obtained. Sandy, whose graph looked like a trapezoid, laughed as she claimed hers was very different from the others. They were perplexed by the differences. Their doubt caused them to make sense of why the graphs were different.

LBC: So what do you think is going on?

Lisa suggested, "It's the way we have our graphs set up..." Helen suggested different scales on the x-and y-axes were responsible for the differences. To test Helen's hypothesis, I suggested they set up identical windows. Believing a common window would produce uniform

graphs, they reacted differently as they realized the graphs were still different: Tess showed no emotion, and Sandy and Helen giggled as if “What now?” Lisa, convinced there had to be a logical explanation, compared her lists to Helen’s lists. She observed that although she had the same points as Helen, the order in which the points were entered was different. When she entered the points identically, the graphs were the same.

Lisa: So it’s...

Helen: ...the order...

Sandy: Oh, ok...

Lisa: the order.

Even though the order was named as the culprit, they did not make sense of why the different order would matter; that is, they did not explain or perhaps they did not know that the `xyLine` option connects the points in the order entered in the lists. Sandy suggested they use the order that gave the line BD and then find a different order to get line AC. Helen, however, explained she found the slope of line AC using the slope formula, and Lisa asked if they could now “use the equation of a line, $y = mx + b$.” Pointing out they have the slope and two points, Helen agreed they should be able to get the graph of line AC.

Sandy was behind and asked about the points being used.

Sandy: What points are we using...again?

Helen: I used A and C...

Lisa: So you got m to be five-twelfths. [copying from Helen’s paper]

An examination of Lisa’s work showed she used the coordinates of D and C rather than the coordinates of A and C as shown in Figure 38.

$$\begin{array}{c}
 (0, 180), (0, 0) \\
 \begin{array}{cc}
 x_1 & y_1 \\
 (240, 0) & (240, 100) \\
 x_2 & y_2
 \end{array}
 \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{100 - 0}{240 - 0} = \frac{100}{240} = \frac{5}{12}$$

$$m = \frac{5}{12}$$

$$y = mx + b$$

$$y = \frac{5}{12}(240) + b$$

$$100 = \frac{5}{12}(240) + b$$

$$100 = 100 + b$$

$$b = 0$$

$$y = \frac{5}{12}x + 0$$

Figure 38. Lisa finds the equation for line AC.

Although Lisa obviously realized an error somewhere, she merely marked through the work and copied $m = \frac{5}{12}$ from Helen's paper. Copying from others was not typical of Lisa; perhaps her decision to copy from Helen showed she understood her error and had confidence in finding the slope on her own if necessary. Sandy only had time to write the slope formula on her paper but had no time to perform any calculations. Not asking the others to slow down so she could find the slope of line AC is an example of social collaboration.

Helen and Lisa began to talk about finding an equation of the line, and I asked them, "So you are trying to find what now; what is your effort?"

Helen: We are trying to find the equation of AC so that we can graph it [pointing to calculator] so that therefore we can use the INTERSECT function and find E.

Lisa: [simultaneously with Helen] find E.

Helen: E...where E... the numerical like where E falls...like the coordinate points of E.

It was interesting that although Helen did not initially understand the graphing approach, she now had an understanding of where the group was heading; thus, she saw the big picture at this point. I asked Tess and Sandy if they understood Helen's explanation. Although Tess said she understood, Sandy did not answer but kept looking over at Lisa's work. At this point in the session I did not know if Sandy's reluctance to ask for help indicated that she believed she understood or if she kept quiet as a form of social collaboration. During the final interview Sandy and I talked about how the others worked faster than she and how she would get left behind.

Using the slope of $\frac{5}{12}$ and the coordinates of C (240,100), they found the y-intercept of line AC to be 0. Lisa commented that she had to write all of her steps out and thought aloud as she worked through the procedure. No one realized that surely the y-intercept would be 0 as point A was at the origin. Lisa reported the equation as $y = \frac{5}{12}x$. Helen and Tess also correctly determined the equation as indicated by their work. Just as Sandy was unable to get the value for the slope, she also did not obtain the y-intercept. As Lisa talked, Sandy looked at the work Lisa was doing; Lisa, however, did not explain to Sandy what she was doing, and Sandy did not ask. Although Sandy was unable to find the equation, she observed Lisa's work; the other three students, however, were focused on the common goal of finding the equation of line AC.

Lisa and Helen worked at the same speed to find the equation. They talked aloud as they found the equation, completing sentences for each other. Although Sandy did not do the mathematics to get the equation, once Lisa reported the equation Sandy knew she was to graph it on the calculator.

Lisa: So your equation is going to be five-twelfths?

Helen: x [Helen listened to Lisa's question carefully and helped her make the correction.]

Lisa: x plus zero...so y equals five-twelfths x. [Sandy looked at Lisa's paper.]

Sandy: So now you draw the line of...

Helen: Go to Y equals [that is, $\boxed{Y=}$ on the TI-83] and go y equals five-twelfths x

Lisa: and then you find the intersection.

Helen: And there you go! Whoopee!

After they got the graph, Lisa gave Sandy the opportunity to ask for help. Checking to be sure everyone understands is a characteristic of a leader.

Lisa: Do you understand how I got that?

Sandy: Yeah, I got that...I'm just a little slow.

In all probability Sandy did not understand how the equation was obtained. Lisa was probably aware that Sandy was behind since Sandy kept looking at Lisa's work, but Lisa did not offer help until the equation was obtained. Egocentrism and leadership factors were at play as Lisa ignored Sandy for a time but then offered help after obtaining the equation. Social collaboration perhaps kept Sandy from admitting her need of help.

Tess, however, was slow in getting the equation entered. Helen must have sensed she was having difficulty because she looked toward Tess who laughingly said, "I'm forgetting where the buttons are!" and the others laughed with her. It was not until the final interview (as described in the section on reflections and final interviews) that I realized Tess was unaware of the single keystroke on the TI-83 calculator $\boxed{X,T,\Theta,n}$ for typing x; she had always used the two keystrokes of $\boxed{\text{ALPHA}}$ and x. It was no wonder that she claimed to have forgotten where the buttons were!

We spent a few minutes discussing whether $\frac{5}{12}$ should be enclosed in parentheses when entering the equation since some members of the group used parentheses and others did not. Helen said she puts parentheses around everything and called herself a "parentheses freak." As we talked about the parentheses, Lisa was already trying to find the point of intersection. She realized that although the two lines were graphed, the point of intersection

could not be found with the calculator. Here is an instance of doubt that caused lifting as can be seen by the following logical explanation that Lisa offered the group.

Lisa: Alright now here's the problem with...you can't go back and do your intersection...because these are just points that we connected....If we do 2nd CALC and we go do the intersection, you are only going to be able to hit that pointer on that one line that we put in...we are going to have to go in and find the equation of this line right here.

Lisa realized that the intersection function on the TI-83 required the graphs to be entered as functions. Because line BD was drawn by merely connecting points B and D with a segment, the point of intersection could not yet be found. A suggestion was made to draw the triangle ABD and to list the coordinates of the vertices A, B, and D as (0,0), (0,180), and (240,0), respectively. Each student drew this triangle; no one drew the axes but merely wrote the coordinates adjacent to the vertices. While they were drawing, Sandy remarked, "So, we've got to find the slope again. Right?" This was another instance of Sandy's thinking aloud as she worked.

Getting a negative number for the slope of line BD caused Helen to doubt her work. I observed her giving careful consideration as to how to enter the coordinates of B and D into the slope formula. Ultimately she realized "you are going to end up with a negative number anyway." Wanting Helen to realize the significance of the negative slope, I asked, "Is that [the negative slope] a problem?" Helen hesitated before answering but said, "uhm, no" as if she did not realize the significance of the negative slope. Sandy, however, quickly realized what the negative slope meant.

Sandy: It should be negative though...because it's sloping down.

Helen: Exactly.

Lisa: Yeah.

Sandy: Oh, I did retain some knowledge!

Helen: Yippee!

Helen's doubt about the negative slope motivated her to determine the calculations with the slope formula would always yield a negative slope in this situation. Helen was fixated on the symbolic manipulations but did not associate the negative slope with the graph. Helen has little confidence in a graphing approach so not making the connection from the symbolic to the graphical was not unusual. My question about the reasonableness of the negative slope lifted Sandy to make sense of the slope, showing she made the connection between the symbolic and the graphical. Although Helen claimed to understand about the negative slope, I was not fully convinced.

Recall that Sandy did not find the slope of line AC so I was not sure she remembered how to calculate the slope. However, her work [shown in Figure 39] to calculate the slope of line BD indicated she remembered the procedure although she first incorrectly simplified the ratio $\frac{180}{-240}$ to $\frac{3}{-8}$ but later corrected it as explained below.

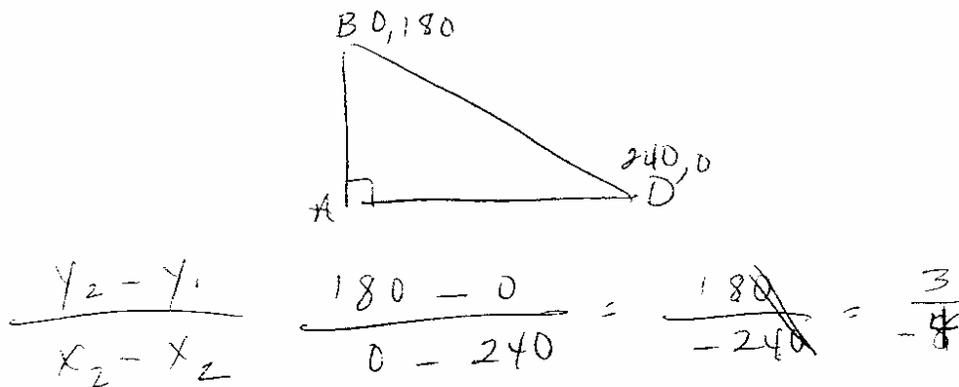


Figure 39. Sandy's calculation of the slope of BD.

Tess and Helen had $\frac{-180}{240}$ as the slope of BD and recorded $\frac{3}{-8}$ when Sandy reported this

value. It was ironic that Helen accepted Sandy's $\frac{3}{-8}$ because in the final interview Helen claimed she did not trust Sandy. Lisa was listening to their conversation because she asked "How did you get three-eighths [that is, negative three-eighths]...I got three-fourths." Sandy admitted Lisa was right, and Helen said, "I just trusted what she [Sandy] said." Because Lisa worked out the slope herself while listening to the others, she realized her value was different from theirs and was able to help the others correct their work. Lisa's behavior showed how leadership skills and careful monitoring can influence the outcome.

Helen explained they could use the coordinates of either point B or point D for x and y to find the y-intercept. She did not realize point B, whose coordinates they knew, would be the y-intercept; perhaps their representation as a triangle rather than as points in the coordinate plane camouflaged this information. Sandy then asked, "Which one would you advise?" Helen said she was going to use the point (0,180) "just because then you can multiply that times zero." Helen saw Sandy was still having trouble so she leaned across the table, pointed to Sandy's paper, and explained:

Helen: You just substitute...directly substitute.

Sandy: Oh, ok, that's right, duh!

As they worked Sandy, thinking aloud, said "three-fourths times zero", and then Helen gleefully announced, "So b is 180!"

On Sandy's paper I noticed she had written

$$y = -\frac{3}{4}(240) +$$

$$180 = -\frac{3}{4}(0) + b$$

$$180 = b$$

I asked her about the 240 and she responded, “Oh, that was a big boo-boo,” and then marked through the equation. As I reflected on the episode I wondered if Sandy realized she could have found the value of the y-intercept using the point (240, 0) as she initially started. Helen’s explanation for using (0,180) so they could multiply by zero may not have lifted Sandy to understand either point could be used.

Once they found the equation of the line to be $y = -\frac{3}{4}x + 180$ I asked if the equation

made sense. My hope was they would connect the symbolic to the graphical.

LBC: Does that make sense with what the graph looks like...or what you are looking at on the sheet...does that line make sense?

Helen: Well, after we graph it we can answer that question [with laughter in her voice].

LBC: What does the 180 indicate?

Sandy: That is the....180 is...your slope.

Helen: No, no, no. 180 is your intersect [sic]...

Lisa: Yeah, your intercept.

Helen: your y-intercept.

Sandy: The point of intersect [sic]...that’s right.

LBC: Should it be?

Helen: It’s your y...

Lisa: It’s your y-intercept...

LBC: Should it be 180...on the sheets that you have?

Helen: Where this is...no, it should be like...

Helen looked at the original drawing and “drew” along line AD as she tried to make sense.

Helen: [whispering] 240...shouldn’t it be...if it’s the y? [She looked at Lisa as she asked this.]

Lisa: [looking at Helen] I’m lost...I don’t understand what you are asking.

[Helen put her hand over her eyes as if embarrassed to admit she has trouble remembering which axis is x and which is y.]

Helen: Ok, this is the x...[pointing to line AB].

Lisa: your y-axis

Helen: your y-axis?...then yes, it's 180—

Lisa: I'm confused as to the question.

Helen: Ok, she asked what the 180 stood for in our equation...and I said that was the y-intercept...

Lisa: Right.

Helen: And then I got confused on which was the x-axis and which was the y-axis...but uhm, yes, it should be 180 because if you see up here B is zero comma one-eighty [that is, (0,180)] with it intersecting the y-axis at 180... [Helen wrote on her paper and the others watched as she wrote and explained; Sandy nodded her head in agreement.]

Lisa: Right.

Helen: So therefore our 180 is correct. And that was all she was asking—and I totally confused people with my not knowing x and y [that is, confusion over the axes].

Lisa: Right, because your y-intercept is...you substitute zero in for x which would give you zero which would be 180.

Sandy admitted that she did not understand how the 180 was found. Helen leaned across the table and pointed to Sandy's paper as she explained how the 180 was determined.

Sandy: I don't know how we figured it out.

Helen: Ok, in the whole grand scheme of $y = mx + b$, b stands for the y-intercept...

Sandy: the y-intercept, uh huh.

Helen: Right. So therefore it does make sense that this is 180 because this point [pointing to the drawing] is zero comma 180 [that is, (0,180)].

Sandy: uh huh, I gotcha!

LBC: So what about the y-intercept of the other line?

Lisa and Helen: It's zero.

Helen: And we found it to be zero.

In the episode above effective communication was a factor as they listened carefully and responded to each other's questions. They were making sense of the connection between the symbolic and the graphical. Helen's trouble with graphing was evident again, but she was lifted as she admitted confusing the axes. Helen caused Sandy to be lifted as she leaned over into Sandy's workspace to explain why 180 as y-intercept made sense.

At this point in the session, they have determined the functions to find the point of intersection. The earlier conversation about parentheses around fractions must have lifted Sandy because she explained "you put parentheses around negative three-fourths" as she entered the equation for line BD. Lisa easily found the point of intersection and said, "You get a crazy decimal." Sandy, Helen, and Tess, however, had trouble getting the intersection point as the following dialogue shows.

Sandy: How can you get it to go straight across? [Doubt was a factor when Sandy got the graph of BD to be a horizontal line.]

Helen: And this is where my calculator has trouble. It doesn't do this INTERSECT thing at all very well. [Helen doubted her competence with the calculator.]

LBC: That's what you mentioned the other day [that is, in the initial interview].

Helen: Yes. And I just don't know if I just don't know how to use INTERSECT very well...

Lisa, leaning into Helen's workspace to see her calculator screen, told Helen, "Hit ENTER, hit ENTER, and hit ENTER again." When Helen got the point of intersection, she shrugged her shoulder and said, "And of course it's going to work this time."

In the first interview Helen told me that sometimes her INTERSECT function worked and sometimes it did not so she was not necessarily surprised that she got the point of intersection

at this time. I knew I needed to help her understand why the function did not work all the time so I asked Lisa, “What were you doing with pressing ENTER?”

Lisa: You were going from the first line, to the second line, and then where the curves come together.

Helen: Ohhhh....

Lisa: So you have to move your cursor...well, really you don't have to if you only have two lines but if you were to put in three lines you have to flip your cursor in between what lines and what angles you actually wanted the calculator to intersect.

Helen: Oh, good to know now that I've gotten through the precal class!

Tess and Sandy listened but did not contribute to the conversation about using the INTERSECT function. When Lisa and Helen finished, Sandy again mentioned the trouble she had with getting the graph of line BD.

Sandy: [leaning toward Lisa] So how did you put in the y_2 ?

Lisa: [looking at Sandy's calculator] Oh, use your x...cause that's just going to add negative three-fourths to one-eighty and [unintelligible].

Lisa realized Sandy omitted the independent variable and entered $-\frac{3}{4} + 180$ as y_2 .

As Sandy and Lisa worked, Helen showed her calculator to Tess and said, “I took out mine and turned off the other one.” Tess apparently still had a STAT PLOT turned on.

This brief interaction with Helen may have encouraged Tess. Tess had been very quiet during the entire session but when Helen asked of the group whether they would round the coordinates, Lisa said she rounded and Tess said, “I rounded, yeah, I'm just guessing...I just did sixty-four point two nine [that is, 64.29]...cause all we want to know is that distance.” Tess mumbled to herself and gestured with her hands as if she were trying to make sense of the problem.

Frequently Helen would interject “fun” statements which made the atmosphere pleasant. She looked at me and said, “You know at this point I would just give up on the treasure...be like it's not worth it!”

I identified several factors that helped them find the coordinates of point E. Sandy got a horizontal line and knew this graph was not reasonable. She did not make sense of her work but instead waited for Lisa to help her, an indication that she saw Lisa as a leader. Helen again doubted her competence with the graphing calculator. It appeared that after an entire semester of having trouble with using INTERSECT, Lisa's explanation lifted her. Tess also had to make sense of her graph. These issues were reconciled with effective communication that included entering each other's workspace. Furthermore, Lisa continued to take a leadership role. Her role was effective because she was adept at using the calculator and was able to recognize the help the others needed.

They found the coordinates of point E but still needed to make sense of what the coordinates indicated. Lisa and Tess understood how the coordinates (154.29, 64.29) related to the map, but Sandy and Helen had difficulty interpreting the meaning of these coordinates in the context of the problem.

Sandy: The point of intersection is one fifty-four...

Helen: ...point two nine. [Sandy began the statement and Helen finished it.]

Lisa: Right. I kind of made another point...I drew a line down and made a point F [drawing segment EF].

Helen: Yeah, that's what I did.

Sandy: But you can also do...so that's this distance from here to here is one fifty-four twenty nine and this distance from here to here is...

Sandy pointed to segments on her paper as she talked. According to later comments, she must have been pointing to \overline{BE} and \overline{ED} , respectively. Helen looked at where Sandy was

pointing and listened to her explanation. Sandy's comments must have influenced Helen to focus on a triangle.

Helen: Exactly. So now we have this...

Sandy: would it be sixty-four?

Helen: [finishing her statement] we have this other triangle. [In the 2nd interview Helen told me she was thinking about triangle EFD.]

Tess and Lisa listened to Sandy and Helen try to make sense of the coordinates of E. Lisa jumped in to help them understand the meaning of the coordinates. The dialogue below shows that Lisa's explanation lifted Helen.

Lisa: We don't need to do that. We just need to know the distance from...

Helen: but if...

Lisa: [looking at Helen]...from here to here...the distance would be y.

Tess: Yes, that's what I'm saying.

Lisa: So the answer to this question is 64.29—assuming we did everything correctly...you don't need to go any further.

Sandy: No...that's this distance right here. E to D is 64.29...

Lisa: Ok, look at it...if you've got this on a coordinate plane...

[Lisa pointed to the drawing on Sandy's paper. Helen and Tess watched closely as Lisa explained.]

Lisa: ...this point E...you're going to go over 154.29 and up 64.29...

Helen listened thoughtfully. When Lisa said, "you're going to go over 154.29 and up 64.29," Helen nodded her head in agreement—she had made sense of Lisa's explanation.

Lisa: [continuing]...the location of the treasure is from E to the road which is here so it's just the distance from E to the road which is going to be your y. [Lisa not only talked as she explained but she also pointed out the segments to which she was referring. The paper she

was using was nearly in the center of the table, creating a common workspace. When she finished explaining, she also looked at Helen as if assessing her understanding.]

Helen: Yep!

Lisa: So it's 64.29.

Helen: Do you see that?

Sandy: I see what she's saying.

[Sandy did not really sound convinced. Unfortunately, I interrupted the discussion with my next question.]

LBC: Alright now...you said initially that the distance from B to E was 154.29...is that true? Is the length of segment BE 154.29?

Sandy: Actually...well...

Helen: No, the length of segment AF...is 154.29.

[Helen leaned across the table to point to Sandy's paper as she explained.]

Helen: The length from here to here [A to F] is 154.29 because you go over 154.29 and then go up...you don't come down and then go.

Sandy: Ok, ok.

LBC: So how does the distance from B to E compare to the distance from A to F?

[silence]

Lisa: From what...from B to E compares to A to F?

LBC: Uh huh...how does the distance from B to E compare to the length of segment AF?

[brief pause]

Helen: It's bigger.

Sandy: Because you've got the angle.

Lisa: And we could check that because we could go back in...the segment FD

is going to be 85.71...and we know that because if the whole thing is 240 you can subtract 154.29 and then you could go and solve that right triangle and get x [what is x?] and then

subtract 300 from x [sic] and you could go back and you would have definite numbers that would prove that.

Helen: Which is where I was headed.

LBC: So you could prove the distance from B to E is longer?

Lisa: Prove it is longer...or disprove it.

I was hoping to encourage them to verify \overline{BE} was longer by looking at the relationships of the segments in the drawing rather than being dependent upon a symbolic approach. My next question was not phrased very eloquently.

LBC: Does it make sense that the distance from B to E—just looking at it—should be more than the distance from A to F?

Sandy nodded her head in agreement, and Lisa tried to explain intuitively how she could tell BE was longer. In quizzing her I came to believe she knew that the shortest distance between parallel lines is the length of a perpendicular segment. Because BE is not perpendicular to the parallel lines AB and EF and AF is perpendicular to these parallel lines, she concluded BE must be longer. As Lisa and I had the following exchange, the other group members listened and nodded their heads in agreement.

Lisa: Because this is a slanted line and if these points are the same...then the shortest distance is a straight line...so if it's a diagonal, it's going to be longer...to get to that same point.

LBC: [trying to make sense of Lisa's explanation] Because these two segments BA and EF are parallel to each other?

Lisa: Right.

LBC: So the distance from A to F...

Lisa: is smaller than B to E.

LBC: Because BE is...

Lisa: not a straight line. [She meant horizontal line.]

LBC: it's not perpendicular to parallel lines.

They were ready to put some closure on this problem. Sandy first asked if the answer is 64.29, and then asked, "Are we done?" I ignored her question and asked, "So do you feel good about it?" I heard giggles and yeahs and heard Helen say, "I feel smarter." I then asked, "Do you have a question about it?" Hearing no response, I decided to see if they could think of a way to find the point of intersection without using the graph. Helen, Sandy and Lisa tossed out some ideas as the exchange below shows.

LBC: Could you figure it out without using the INTERSECT key on the calculator?

Helen: We could have but it would have taken a lot longer.

LBC: What could you have done? Once you found these equations of those lines, what could you have done...could you have figured out the distance without using the INTERSECT key?

Helen: Is there an equation for....?

Sandy: Isn't that where you'd probably use like the tangent? But we would need to use degrees then.

Lisa: I don't remember how to do that.

Helen: I know...I know there is a method...

Lisa: Would you... I'm thinking the distance formula but that's not...I don't remember.

I sensed they were ready to be finished with the problem so I decided not to push them anymore. They had found a solution and seemed content to put closure on the problem. They did want verification the solution was correct.

Sandy: Did we do it right?

LBC: Ok, alright, good.

Helen: Are you going to reveal the answer?

I turned off the digital recorder at this time although the video camera was still capturing data. Sandy leaned back in her chair and smiled as she again asked, "Did we do it right?" I responded by telling them they had used a right way—implying their approach was not the only

way. My implication perhaps challenged Lisa and Helen to reconsider finding the coordinates of point E without using the INTERSECT function. Their consideration showed they had learning as a goal rather than just obtaining an answer to a problem.

Helen: Is there an intersect formula?

Lisa: To find where two lines intersect, how would you do that without using the calculator?

Helen: yeah.

LBC: You think about it.

Helen: I am going to be thinking about this all day.

[I turned the digital recorder back on.]

LBC: Alright, write those two equations down on your paper.

Helen and Tess correctly wrote the equations $y = -\frac{3}{4}x + 180$ and $y = \frac{5}{12}x$. Sandy incorrectly wrote $y = \frac{3}{8}x$ as her second equation, and Lisa omitted the variable x in her second equation and wrote $y = \frac{5}{12}$. I later detected and pointed out Sandy's and Lisa's errors so outsider influence was a factor. A common workspace may have helped them detect the errors. I asked Lisa to remind the group of her earlier question.

LBC: So, Lisa, you asked what question now? What did you ask me?

Lisa: If you had two lines, without using the calculator, how could you find out where they intersected?

LBC: Alright, anybody with a comment?

Lisa: Would you substitute the five-twelfths in from your line two [that is, the second equation] in for y in line one [that is, the first equation] and then solve for x to find out where they intersect... because five-twelfths is a straight line? [Earlier Lisa used "straight" to mean

“horizontal.” Because her second equation was $y = \frac{5}{12}$, this perhaps explains “five-twelfths is a straight line.”]

Helen: Oh.

Lisa: I mean, I’m not positive.

Sandy: Because it is something that you can interchange...you can substitute.

Lisa: Let me try that. [Lisa wrote on her paper $\frac{5}{12} = -\frac{3}{4}x + 180$.]

I pointed out to Sandy that she had written “three-eighths x” instead of “five-twelfths x”. As she made the correction, she remarked, “That’s where all of my errors come from.” As an example of social collaboration, Helen smiled and said, “Mine, too.”

Lisa and Sandy were writing, but Helen and Tess were not. I asked a question to see what plan was being considered. When Lisa began explaining, I pointed out she had written $y = \frac{5}{12}$ as the second equation, and Tess told her, “It is $\frac{5}{12}x$ “. Lisa made the correction but then said, “Never mind, “ as if her suggestion was no longer viable. Helen, however, was apparently lifted by Lisa’s earlier suggestion of using substitution of one equation into another. Lisa was ready to abandon the approach, but Helen saw potential although she had trouble with the interpretations of the variable y.

Helen: Can you make them equal to each other? I mean cause y is...not that the y’s are the same but could you set them...?

LBC: Alright, should the y’s be the same? What are you looking for?

Sandy: Where they intersect.

Helen: Well, the y is the line itself and they shouldn’t be the same because we have one going this way and one going this way [indicating the different directions with her arms]. They are not the same line—they are not equal to each other.

The others were not listening to Helen, but I realized Helen did not see the equations as expressing the relationship between the x-coordinate and y-coordinate of points on the line. Instead she only saw the equations as representations of the lines. No one said anything so I tried to help Helen make sense of y.

LBC: Well, the lines aren't equal to each other...the lines aren't the same line, but what do you know at point E?

Sandy and Helen: That's where they intersect.

LBC: So at point E would the y-value of this line [pointing to \overline{AC}] be the same as the y-value of this line [BD]?

Helen: Oh, it would...Could you set them equal?

LBC: What do you think?

Sandy: Yes, you can because they are going to be equal.

LBC: And you are trying to find out where they are equal. Does that make sense?

While Sandy and Helen were trying to understand why setting the two expressions for y would make sense, Lisa just listened. But when Sandy and Helen realized they could equate the two expressions, Lisa's facial expression changed—she smiled and began working. Sandy's and Helen's conversation was effective not only for them but for Lisa as well. Making sense of what the intersection implied for the two lines, lifted them to understand why the two expressions for y could be equated. They worked independently, but all four of them symbolically found x and y to be 154.29 and 64.29, respectively.

In the following exchange Helen expressed her preference for the symbolic approach. Evident was her lack of an understanding of the connection between the two approaches. The two approaches were not different; the only difference was the “tool” used to find the point of intersection. Sandy and Lisa tried to help her understand.

Helen: [giggling] That's a lot easier way!

[10 second pause]

LBC: What do you mean “That’s a lot easier way”?

Helen: It’s more direct.

LBC: Easier than what?

Helen: Easier than what we did. [flipping back through her work]

Sandy: Actually not.

Lisa: But we still had to do what we did to get to...

Sandy: Right...we had to...

Lisa: We had to have the equations...

Sandy: We had to have the equations to do this so actually the calculator was the easier way.

Lisa: What we just did replaced the INTERSECT on the calculator—that’s all it replaced.

Helen: Well, I would have been more comfortable using this on a test...my INTERSECT button never worked for me and I never liked that stuff so I always did it the long way.

LBC: [to Helen] So you feel more comfortable doing the symbolic approach?

Helen: I don’t like being too calculator dependent.

The following comments in which the group talked about the session came after the video camera was turned off.

LBC: Do you think you could have done it on your own?

Sandy: I would have had to do a lot of looking through my notes and figuring things out.

Lisa: I think I would have had to sit there for ten or fifteen minutes and kind of digest everything—I probably would have re-written the triangle without the lakes in it because I think the lakes distracted me...

Helen: Uh huh

Lisa: but I think I would have been able to do it on my own.

Helen: Yeah, I would have done a lot of excess work that didn’t need to be done.

Tess: Yeah, I think I would have—I would have probably did tangent and cotangent and all that—just stuff that didn't need to be done.

LBC: Why would you have gone with the trig?

Tess: I guess when I see the triangles, that's what I automatically think about—I think about tangent, cosecant, secant...I don't know.

Helen: See, I would never have gone with the trig in this problem because they didn't give us any degrees or any angles—they didn't ask for angles—

Lisa: But you can use trig to find the lengths of...

Helen: I know but since they didn't provide any of it I just wouldn't have made the effort.

LBC: So what would you have done?

Helen: I would have done the algebraically [sic] but ...

LBC: Like ya'll did it?

Helen: Best case scenario, yes [giggling]. It would have taken me a little while to get there, though.

Sandy: I think I probably could have come up with this answer but the only thing is I probably wouldn't have made sense of it enough to know that's the answer and you don't have to go any further.

LBC: So when you got to 64.29 you wouldn't have known that that was the answer?

Helen: Yeah, cause she was ready to go on—she was like, "Ok, let's go find [unintelligible]"

Sandy: yeah [agreeing with Helen]

LBC: Alright, anybody else?

I then gave them the reflection questions and asked them to answer them as if a "Dear Diary"—told them I was interested in how they thought about the problem—what the experience meant for them. I told them they could take the work from the session home with them to trigger their thinking but asked them not to change anything and not to lose it.

Reflections and Final Interviews

Helen

In her reflection following the Buried Treasure Problem Helen wrote that she and Lisa were more vocal and “sort of ran away with the problem, not waiting for the other two to catch up.” She regretted they did not “incorporate Sandy and Tess more into it.” Undeniably Tess was very quiet so Helen’s regret about leaving Tess out was understandable. Because Sandy made a number of suggestions throughout the session and appeared engaged in the work, I was unsure of why Helen believed Sandy was left out. It became clear to me during the final interview, however, that Helen put little stock in Sandy’s suggestions. Evidence of her lack of faith in Sandy’s suggestions include the following eight comments that were scattered throughout Helen’s final interview.

“She [Sandy] is a big fan of making things more complicated than they are.”

“Sandy kept throwing out, ‘Well, let’s go find all these angles,’ and I was like the angles have nothing to do with the problem.”

“She [Sandy] is talking about the point of intersection and at that point I just didn’t ...I didn’t trust her...”

“Cause she [Sandy] tends to throw out like big things and then not necessarily know what she is talking about...not that she is stupid but she doesn’t know like all the steps to what she is talking about sometimes and I was like...cause I had seen her in class...and I had seen her like make significant goofs and that was one thing that kind of prejudiced me against trusting what she did.”

“She [Sandy] just pulls all these things out of her hat like...like the intersection, the slope...she just thinks everything ties together.”

“I was very frustrated cause she [Sandy] just rattles on and on and on and she doesn’t...she doesn’t necessarily know...I feel bad she doesn’t know what she is talking about.”

“I was bad...I kind of tuned her [Sandy] out...if she had a problem and she was discussing with somebody else I kind of tuned it out cause I didn't want her to confuse me any more...cause that's what ended up happening when she started rambling off.”

“Sandy just rattled things off...and I don't know where she got them from.”

Frankly, I was surprised at the above comments from Helen about Sandy because I observed Helen leaning across the table several times during the session to help Sandy. It was not until the end of the final interview that Helen acknowledged that she tried not to let Sandy know how she felt.

LBC: And even though you tell me that you were frustrated with Sandy, that didn't come across [in the session].

Helen: I tried my hardest not to let her know it because I knew that she was trying...you know she was making an effort to help the group and where it might not have been helping me--it might have been frustrating to me--it might have been helping Tess or Lisa and I was not above...I was not going to say “Oh, you are wrong, shut up.” I was like, you know, maybe she will stumble upon something and we will just be able to go with it.

Helen's effort to camouflage her feelings may be an example of social collaboration. I believe, however, it was more an example of being a good leader whereby she controlled her true feelings for the betterment of the group. It was unfortunate that Helen allowed her observations of Sandy in the MATH 1113 class to influence her perception of Sandy's suggestions during the problem session. Sandy did make some useful suggestions, but Helen said in the final interview that she “can't remember being significantly helped by much that she said.”

Although Helen was quick to criticize Sandy for tossing out ideas, she felt vindicated in doing the same thing herself. For example, during the session she suggested they find the

length of \overline{DE} but realized triangle CED was not a right triangle. During the final interview, I asked if finding the length of \overline{DE} would have been helpful.

Helen: Probably not...at this point I still had no idea where we were going so I was just throwing out things...no one else was throwing out anything better or worse or anything so...See I tend to talk out...talk myself through in my head...and when I started talking out loud you realize how dumb you sound...but then it's just like it moves things along...like it gets people thinking...like maybe we should focus on...

It was also interesting that when Helen watched herself on the video, she commented that it is “confusing listening to me cause they’ll listen to me say one thing but I’ll write what I mean on a paper but I’ll say out loud something else.” Ironically, she chastised Sandy but merely acknowledged similar behavior in herself but without the chastisement. This observation is an example of egocentrism.

Helen was successful in convincing Sandy and Tess that finding the angle measures was unnecessary. She wrote in her reflection that her plan was to use the known lengths to find the distance algebraically. When a graphing approach was suggested, she admitted she was dubious. She wrote, “When they first suggested graphing it, I had no faith that it would actually work because I couldn’t make it logically connect with any possible answer.” We watched the clip where Helen suggested they could “go algebra with it” by letting E to D be x and D to E be $300 - x$. Helen did not pick up on the contradiction, and I chose to question her instead about her claim that having two unknowns defeats the purpose of algebra. The following exchange shows Helen’s misconception about variable representation as well as her attempt to save face.

Helen: It was like I was saying...since I didn’t understand why she wanted to graph it...I didn’t understand so I was like well maybe—she didn’t elaborate, she just was like well we could do that—and I was saying we could go 300 minus x equals ED...but that to me is pointless because you end up with 2 unknowns...

LBC: Alright, now where is the other unknown?

Helen: x —now granted they are the same unknown.

LBC: Alright, now help me now.

Helen: 300 minus x ...

LBC: ...is going to be what?

Helen: E to D...that length. But to me that's pointless because....

LBC: Alright, now that is one unknown.

Helen: No, the way I would write the equation would be 300 minus x equal ED, the segment ED...in that way.

LBC: Ok so you are saying...alright what do you know about the relationship between those two?

Helen: [confidently] x and ED are the same thing.

[I am trying hard to follow her.]

LBC: x and ED?

Helen: And...

LBC: Alright now think about it now. Suppose x were 100.

Helen: Then ED is going to be 200 but that's...it just clicked in my head...ok.....yeah, I mean if x is 200...I mean if x is 100 then ED is going to be 200 and that is going to be...

LBC: Alright, so what would x be?

Helen:..... x would be the 100. I mean... x would be the length of ED.

LBC: Alright, but you are telling me that ED is 300 minus x ...

Helen: Ok, that's not logically....

LBC: Ok, think now.

Helen: [sighing]...I don't know...I skipped a step or two and it just doesn't...it sounds good but it doesn't make sense...

LBC: Ok....so what is 300?

Helen: 300 is the length of segment BD. [Pause] So if you were just going to do this geometrically or whatever, BD minus ED...is not going to equal x, is it?

LBC: What would it equal?

Helen: It's going to equal BE.

LBC: uh huh. So what is x? What would x be? What does x represent?

Helen: x represents BE...the length of BE...I'm glad none of them caught on to that!

LBC: Do you see what I am saying?

Helen: [confidently] Yes, I do.

LBC: Now, this is...OK...you can say 300 minus x is the length of ED...but then you...

Helen: ...you have to define x as BE...

LBC: Yeah...does that help you at all?

Helen: No...because you still have 2 unknowns...and I don't believe that algebra should have more than one unknown. [She was referring to BE and ED as the two unknowns.]

LBC: Why?

Helen: Because...the basic purpose of algebra is to solve for THE unknown and having more than one defeats the purpose of THE unknown...

Helen appeared to be lifted in her understanding of the connection between the symbolic and the geometric representations of the segment length. The discussion about having two unknowns, however, was not pursued. Instead we watched the clip of Sandy and Lisa suggesting they use graphing to find the point of intersection. Helen confessed she went along with the approach but was still unconvinced the approach would work. In her reflection she wrote, "I started to ignore them when they suggested we graph the problem because I didn't understand, and I honestly didn't think it'd work so I wanted to find another way to solve the problem. I realized pretty quickly that I was the one that needed to be paying attention!"

We then watched the clip where each student obtained a different scatter plot and I had them show their plots to each other. Helen said, "I felt so stupid right there cause they were all

looking off my calculator.” I asked her why she felt stupid and she explained, “Stupid as in I don’t like being put in that position where everybody is trusting what I do...” These comments of Helen’s again point to her egocentrism. Although she claimed during the interview feelings of stupidity, these feelings were not apparent to me during the session. Camouflaging her true feelings could be a form of social collaboration or a form of leadership.

We watched the clip where Helen and Lisa talked about finding the coordinates of point E. The conversation from the problem session is shown in Figure 40.

Helen: *We are trying to find the equation of AC so that we can graph it [pointing to calculator] so that therefore we can use the INTERSECT function and find E.*

Lisa: *[simultaneously with Helen] find E.*

Helen: *E...where E...the numerical like where E falls...like the coordinate points of E.*

Figure 40. The conversation from Session 1.

As we watched, Helen noticed Lisa agreed with her and said that at this point she is feeling confident. I became aware of how important it was for Helen to have verification of her ideas.

LBC: So do you find yourself working better when somebody is on the same page as you?

Helen: Yes, definitely...and yeah...and it isn’t going to sound interesting but it felt good cause she [Lisa] was definitely one of thestronger ones...and to have her agree with me ...it was like OK you know...we must be doing something right.

Helen recognized Lisa’s mathematics ability and was encouraged by Lisa’s verification. Interestingly, Helen later said it would not be her choice to work with Lisa:

Because Lisa and I both had strong personalities and I personally would not choose to work with her only because of the fact that we both had the strong personalities and I know we would have clashed had we had any more time together...but I trusted her math the most.

Although Helen appreciated the verification she received from Lisa, it appeared she was somewhat intimidated by Lisa and yet I, as an observer, was unaware of this intimidation during the problem session. Again, I believe Helen's ability to control her feelings for the good of the group showed leadership.

We looked at some of the clips where Helen had trouble during the problem session. One of these clips showed her confusion with the x- and y-axes. During the session, I assumed differentiating between the axes was an ongoing problem for Helen, but she offered the following explanation for her confusion.

Helen: We didn't have the axes drawn... they were imaginary axes...and so I got flipped around like which one is x and which one is y...

LBC: Do you get the x- and y-axes typically mixed up?

Helen: No, no. I don't know why I did then...I just had a stupid moment and uhm I don't know what I was thinking about the y-axis and...how it shouldn't be 180.

Although she claimed differentiation between the axes was not problematic for her, I was not convinced particularly since we frequently had graphs where the axes were not visible. Helen's denial may illustrate her egocentrism.

Helen also had trouble finding the intersection of two functions with the calculator. She believed her calculator was broken because sometimes she would obtain the correct point of intersection and at other times she would not. During the problem session, Lisa helped her use the INTERSECT function on the calculator and we watched this clip. She explained to me that never did she use INTERSECT in high school and in MATH 1113 she just "winged it" because sometimes it would work and sometimes it would not. I found out that her confusion was related to the term *curve*.

Helen: It had to do with the curve...cause I didn't realize that the curves could be lines or actual curves....

LBC: See *curve* is just a generic term for a graph.

Helen: I wasn't thinking of it as a generic term...I was thinking of it as a **curve**...and I was like...when it said first curve, second curve and I had lines, I was like I don't know what you're talking about so I would just hit enter and normally I'd get it on the lines...but like if we had two...like...if we had more than one...like things entered...I was like my calculator doesn't work...and I was like....

LBC: So if you've got three functions that you are graphing [I entered 3 functions in the calculator]...and you want to find the intersection of this line and this line, what would you do?

Helen: I would go.....[Helen was still not comfortable with these commands...they were not automatic but she was able to figure out what to do.]

Helen: I always thought you had to get it as close to the intersection point as possible so I was probably always hitting first curve as y1, second curve as y1, and it was like wait...we can't do this... [We spent some time practicing with finding the point of intersection.]

LBC: Why did you never ask me...the whole semester?

Helen: I don't know...cause I seriously thought it was my calculator.

LBC: You really thought the calculator was broken?

Helen: I seriously thought it was the calculator.

Helen's explanation about the term *curve* was enlightening for me as a teacher. Because lines were not curvy, Helen could not make sense of the calculator's questions related to first and second curve. During the problem session, Helen learned how to use the INTERSECT function. It was only during the interview, however, that I learned the source of Helen's trouble, and Helen learned the use of the term *curve*.

I reminded Helen that at the conclusion of the problem session she had asked about the existence of an "intersect formula" to find the point of intersection of two lines. We watched the clip where the group found the intersection symbolically. Although Helen suggested they set the two expressions for y equal to each other to find the point of intersection, my conversation with her during the interview showed that her understanding of the relationship between the symbolic

representation and the graphical representation of functions was incomplete. The following dialogue illustrates her misconception and how she began to make sense of the connection between the two representations.

Helen: Cause they both equal y.

LBC: Why?

Helen: Cause they are both equal....they don't equal the same...they are both equal to y, cause they both equal a line so if you set them equal to each other you can solve for x.

[Helen saw y only as a line but not as the second coordinate of a point on a line. She concluded she could equate the two expressions for y because each of them was a line.]

LBC: What do the x and the y represent in that equation right there?

Helen: The y's represent the line itself. And the x represents the uhm.....I don't know what x represents.

LBC: What does that equation y equals $\frac{5}{12}x$ have to do withline AC?

Helen: That is the equation of line AC, meaning that if you graph it that's what you will get.

LBC: Alright. What do the points on that line have to do with that equation?

Helen: For every time you put in...if you have x and y, you...whatever your y is...then when you multiply the x by the $\frac{5}{12}$ you will get the y.

LBC: Ok. So for example...point E is on line AC. And we found E to have coordinates of (154.29, 64.29). So what do these coordinates have to do with line...

Helen: [interrupting] If you were to multiply 154.29 times $\frac{5}{12}$, then you would get the 64.29.

LBC: And what about this other equation...this one?

Helen: Same thing [said confidently]. If you substitute...if you substitute the point (3,4) or whatever...then if you multiply the 3 times the negative $\frac{3}{4}$ then add 180, that will give you 4.

[I missed a real opportunity here—Helen obviously did not see that the point needs to be on the line in order to satisfy the equation—she arbitrarily chose (3,4) which is not on the line so multiplying 3 by $-\frac{3}{4}$ and adding 180 does not give 4.]

LBC: Ok...so what if I plug...what if I substitute 154.29 in for x here, in that second equation, what would I get?

Helen: I don't know...cause...Oh, you would get uhm the same thing cause you are finding the point of intersect so you would get that answer...you would get, if you plug in 154.29 and then you would get 64.29...so that's why....oh, ok, gotcha [She indicated a connection being made.]

LBC: And what you are doing is finding the point of intersection...

Helen: Right.

LBC:so you are wanting the y-values to be the same.

Helen: Gotcha [said confidently].

LBC: Does that make sense?

Helen: Yep! I don't know if it made sense that day, but it makes sense today.

LBC: Cause I don't think it did here [referring to the tape].

[We continued to watch the tape where they talked about finding the point of intersection.]

LBC: So you [Helen] are thinking the y is the line and actually the y....

Helen: ...is the point...

LBC: ...the coordinate ...

Helen: Right....and I wasn't differentiating...the y from the line.

[We watched the clip where Sandy claimed the y-values were equal.]

LBC: Are you convinced? [that is, with Sandy's claim that the y-values were equal]

Helen: No [said without hesitation]. Cause I'm still thinking that...I mean...I'm thinking line...I wasn't completely convinced, no.

Again, it was the one-on-one conversation with me during the interview that helped Helen deal with these misconceptions. Although in the problem session she used a procedure

to find the coordinates of point E, she still had a misconception about the relationship between the symbolic and the graphical representations of lines. The effective communication we had in the interview gave Helen the opportunity to be lifted and gave me the opportunity to further my awareness of the misconceptions students have.

Helen identified several factors that influenced the group problem solving. One was the lack of familiarity with the others in the group. Because she did not know the other students, Helen said it was difficult in the beginning for her to trust their math skills. Furthermore, she was unsure how the others would react.

Helen: I wish...we would have been more familiar with each other...cause I think the first 20 minutes was just...kind of like we all put our feelers out and we were like OK....you know...trying to gauge not their intelligence but like their personalities—how are they going react if we step on their toes and say “Wrong” or how are they going to react if...and if I don’t know people I’m not going to...even if I know I am right, I’m not going to assert the fact that I am right.

For Helen, the task of working in a group prevented her from sitting back and thinking about the problem. She said the pace of the group work interfered with her ability to think about the problem. I asked if they should have taken a few minutes to think about the problem independently.

LBC: So would taking 10 minutes or so to think about it by yourself...?

Helen: That wouldn’t have worked in the group because I would have felt pressure to think faster, to do things...cause I feel in a group that I am working for the group, not for myself. And if I am working for myself, then I think more logically...I don’t feel the pressure to be right...and I definitely felt the pressure to be right... and I didn’t like it when I was wrong...because this is just a blow to the ego.

LBC: Did you ever talk to yourself whenever you had a blow to the ego?

Helen: Yeah...I had to be like..."Ok, just take a breath....you know, they are not doing this to make you feel stupid...they are just pointing it out...the same way that you would point out to them..." And I was also very hesitant to point out to people when they were wrong because I didn't want them to feel like I was pulling almighty super smart person on them...you know...cause I hate it when people do it to me ...I hate it when people make me feel like I'm stupid because they know what they are doing so therefore I try my hardest not to this to other people...

Helen had the need to preserve her ego, and her egocentrism was a factor that influenced the problem solving. She did not like for others to point out her mistakes and as a result she claimed reluctance to point out the errors of others. It was perhaps her reluctance to point out the mistakes of others that kept her frustration with Sandy under control. Although Abby in Group 1 was fearful of being wrong and protected her ego by keeping quiet, Helen's need to be right was often satisfied by having to explain her position more fully. Helen was a talker and often found herself saying and doing things her group might question. At these times she would not back down which meant discussion could occur, often times characterized by effective communication.

Most beneficial to Helen in the group situation was the fact that her group members supported her weaknesses. She observed they "balanced each other out." Thus, Helen recognized that the prior experiences and knowledge of others can influence the problem solving of the group.

Lisa

Lisa wrote in her reflection that she did not think the Buried Treasure Problem was difficult although it required her "to think and really analyze." She also wrote, "I think that this process of group solving takes me longer to do one problem because I have to stop and explain my thinking to others; however, the explaining helped me to see the problem more clearly." Lisa was not a big fan of group work and yet she did recognize the benefits of it. Although not a big

fan of group work, she did not withdraw from the group but instead listened to the suggestions of the others and gave consideration to those suggestions. Often times she would withhold her own ideas while investigating the suggestions of the others. The final interview provided several examples of this occurrence.

We watched the clip where Sandy talked about an angle of depression. Lisa said she did not understand Sandy's comment about the angle of depression and said "those words were scary to me." As we watched the clip where Sandy talked about angle measures, Lisa said, "You can tell that I don't necessarily agree with the way she [Sandy] is trying to solve it cause I am not saying anything." When I pointed out that Lisa suggested they write the given lengths of the segments on the picture, her response showed that she was open to the suggestions of others but was also willing to take a leadership role if she thought the suggestion would not be useful.

Lisa: I was following and I think I wanted to give her [Sandy] an opportunity to explain how she was thinking about it ...because if she had a reasonable way of doing the problem, I didn't want to just completely just not listen to it...but uhm I didn't want to have her also to go and explain this long detailed explanation of something that she probably really didn't know how to do...and waste time when I thought this was the way to get started on it...I figured that that would help. So I kind of gave her a few minutes and then I said why don't we look at this.

We watched the clip where Lisa asked, "What if we graphed it?" This question was subsequent to Sandy's comments about finding a point of intersection and the slope of a line, and Lisa admitted how Sandy's comments led her to consider a graph.

Lisa: I don't think it was the point of intersection [that made Lisa think of doing a graph]...I think it was the slopes...when she [Sandy] said something about slopes...I don't remember what it was...when she said point of intersection and then slope and when she said slope I'm thinking "Hum, slope might work and in order to find slope we have to find a graph and I think that's

when I started to think graph, graph, graphthis is a graph, this is the x-axis, this is the y-axis, there is no reason why we can't graph it.

Sandy's comment lifted Lisa to suggest the graphing approach the group ultimately decided upon. Although Lisa did not particularly enjoy group work, she realized the benefit of listening to others.

Lisa: There are a lot of times that I just listen and pay attention...because I think there are a lot of times where somebody will say something and I don't think they know what they are talking about but then saying that will trigger something in my head that will make me think "OK, well, that might not necessarily be the way to do it, but this might be."

Lisa talked about how she made sense of the graphing situation.

Lisa: I just looked at this and taking away all lakes and all the trees and just looking at the triangles I just saw these lines on a graph...I mean once I started thinking graph, all I could see was this A to B being y-axis and this A to D being an x-axis....

We watched the clip where Sandy exclaimed, "We do have points on the graph." I asked Lisa if that was the first time she realized they had points on the graph, and she said, "No...as soon as I saw this, I knew I had points on a graph." She said she had already begun thinking about the coordinates of the points and was glad when Sandy noticed the points. Sandy's observation gave verification to Lisa's thoughts.

Lisa: Right...and I remember thinking when she said "We do have points on a graph"... I'm going duhhhhh....in my head I'm thinking "Ok, finally somebody is seeing what I'm seeing." Because Helen kept thinking that...I just felt like she was thinking that I had no idea what I was talking about, I didn't understand what I was doing and that a graph wasn't going to help us. And I kept thinking...Ok...I guess trying to understand how she was trying to solve it because...I would rather I think almost rather than try and explain myself to Helen who didn't want to go that route...I would rather just try and understand how she was doing it and go her route but her route wasn't making any sense to me. And I felt like I was confident enough to be able I guess

to voice my opinion and explain how I thought it would work...so it was like, "Ok, now that she [Sandy] understands, maybe we can get [it]."

Although Lisa believed the graphing approach would work, it was not until Sandy saw the "points on a graph" was she comfortable pursuing the approach. If Sandy had not been so persistent, they perhaps would have pursued a different approach especially since Lisa believed Helen doubted her and felt obligated to consider Helen's approach. Once they decided to pursue graphing, however, Lisa felt an obligation to consider Tess's suggestion of constructing a scatter plot. Lisa said she would not have constructed a scatter plot but would have gotten the equations of the lines. However, when Tess suggested the scatter plot, Lisa went along with the suggestion because she said, "Tess had an idea...she wasn't really saying much." Although Lisa believed the scatter plot would not be useful—and ultimately it was not—she did not question its use because "I wanted to give everybody a shot and try...I didn't want to just say, 'This is my way and this is the way we're gonna do it and if it doesn't work...oh well!'" There were times when Lisa provided leadership; at other times social collaboration superseded her leadership skills as she thought it important to consider the ideas of others. A strong leader would have suggested they consider the different options to determine the most appropriate before beginning implementation. Even though Lisa did not believe the scatter plot would help, she said "at that point I didn't think it was going to hurt us...but I think that we spent a lot of unnecessary time trying to figure out why we weren't all getting the same thing."

Lisa also gave some insight into why students are reluctant to speak up in a group situation. I asked her why she thought Tess did not speak up.

Lisa: I don't know...I don't know...and I tend to do the same thing...I just sit there and mull it over in my head and wait for some brilliant idea and I don't know if that is how she [Tess] is or ...I don't know, I guess if it were me I wouldn't want to say "Ok, wait."...you know I would feel stupid...everybody else is understanding what is going on...everybody else kinda knows what

they are doing and if I didn't have any idea what we were doing, I don't know if I would say anything either....cause I wouldn't want to slow everybody else down.

Even though her explanation was not surprising, it shows how the need to protect one's self can supersede a search for understanding. I saw this as egocentrism.

Lisa never doubted that she would have been able to solve the problem herself. As pointed out earlier, Sandy's mention of slope helped Lisa consider a way to get the graphs. Knowing Lisa was not fond of group work, I asked her if the group was of any help in solving this problem.

Lisa: I think the group helped me to the point by explaining myself to them I was really able to really analyze it myself because I knew what I was doing and I kind of understood why I was doing it this way but by having to explain it to them and trying to get everybody else to understand what it was that I was doing I was able to kind of analyze it and think, "Ok, maybe that is not the best way to go or maybe this is." I guess I was able to understand a little bit more...cause I was teaching it kinda.

Lisa realized that communication with the other group members was beneficial to her because her explanations helped her make sense of the problem. At the same time the communication with the others also caused her to doubt herself when they would question her ideas; however, I do not believe she realized the doubt probably contributed to her making better sense of the problem. She talked about how she doubted herself:

Lisa: I think sometimes they would ask questions that kind of made me doubt myself...uhm....or a lot of times when I think of a way to do something I still sit there and kind of look at it...and if I say out loud, "Let's make a graph," immediately everybody wants to know why or...and I don't know if right away I can give a "Why answer"....because sometimes a graph looks good but I could have solved this whole thing and then realized a graph wasn't what I needed to be doing this whole time...trig would have worked, you know...so...

Lisa is a student who likes to have the big picture in mind. Her preference for sorting through the details on her own rather than trying to explain herself to her group members is explained by the following comment: "I don't want to take everybody on this side trip that I think is going to work and then it not work and everybody look at me and go, 'Ok, well we just wasted 20 minutes, trying to figure that out and it's not working.'" I believe this reluctance to have the group explore her ideas could be a form of egocentrism. On the other hand, it may simply reflect Lisa's learning style. She confidently spoke of how she may not understand a concept in class but could take her notes and her textbook and "go home and look at the book and basically try and teach myself." A group situation actually limited her ability to think, especially when she had no idea of how to solve the problem. She needed time to withdraw from the group so that she could make sense of the problem herself. One of the last comments she made in the final interview summed up how she views group work. Apparent in this comment is how Lisa feels constrained by a group situation.

I don't mind doing group work if I know what I'm already doing...I guess if I feel smart in a subject , I don't mind doing group work because I feel that I can actually contribute something...if I don't feel like I am contributing and I'm just lost from the get-go, I hate group work cause I just sit there...and....cause I have to figure it out myself...I can't sit there and even if they had figured those problems out...to be honest with you if they are sitting there telling me how to do it I still don't know if I would have....I have to be able to figure it out myself [said adamantly]....to be able to understand anything ...no matter how much teaching in math...that's probably why I have to do everything twice...because I have to teach myself...you know I can have the worst teacher in the world...and as long as I have a good book, I'm Ok...cause I teach myself anyway...I have to or I'm just not going to get it...I cannot sit there and you know it makes sense to me watching somebody work out a problem...but to go back and do it myself...I'm never

going to be able to do that; I have to sit there and I have to know how to do it myself...that's just how I work...I guess how I learn ...I have to do it myself.

Lisa and Helen both spoke of the tension between the two of them and yet I did not observe this tension myself. Lisa said, "I think Helen and I are both the kind of people where we have a way to solve it...there were a lot times where I would make a suggestion and Helen just didn't want to hear it...she wanted to do it her way...and there were a lot of times when she would say something and I didn't want to hear it." Although Lisa and Helen had learning as a goal, I think their egocentric natures prevented them from always understanding the other's point of view. Lisa even said they "buted heads a little bit." To their credit, they handled the tension well. Lisa noted, however, that for problem solving to be effective in a group it is important to listen to everybody.

Lisa: I think listen to everybody...I mean don't throw away...don't just not pay attention to what somebody else is saying...don't discredit anybody...you know cause I mean even the D student can have an idea you know that might not necessarily be the right way to do it but it might help you figure it out...you know it might help you...I mean I'm a fan of if you don't know how to do it, try something and if it is the complete wrong way to do it you are going to see it and that might give you another avenue to go down as to how to solve it.

Sandy

Sandy was instrumental in the group's decision to solve the problem graphically. In fact, it was her persistence about finding the slope that led Lisa to coordinatize the drawing. Sandy had a sense of the big picture that could be used to solve the problem but she lacked the tools. She wrote in her reflection, "I had an idea how the problem needed to be solved, but I was not sure of the formulas to use."

It was difficult for Sandy to keep up with the others because she said she is a slower worker.

Sandy: Being a slower worker it was difficult for me to catch up, or to stay with them at times...and then I was playing catch-up and then, of course, you miss something when you are trying to play catch-up...and so you don't have that full understanding.

Although she asked some questions, there were others that went unasked because she did not want to "hinder the group by asking so many questions." She sensed Lisa's and Helen's frustration with some of the questions that she did ask. It was interesting that she observed Helen was frustrated but also believed Helen "was still willing to work it out so maybe I could understand the error of my way." It was to Helen's credit that Sandy did not sense the level of frustration that Helen claimed she felt; Helen was able to mask her frustration well.

Sandy believed she would have been more comfortable asking questions if the group members had known each other better. Although they talked briefly before the session, she said she "didn't feel comfortable enough to ask them questions that kind of loomed in my mind." Sandy knew there were questions she had but "reserved to ask them because I know that we were like, 'Let's hurry up and get this done, too,' and so I didn't want to be the one to hold up the group because I have questions...and so I didn't want to inconvenience anybody." Moreover, there were questions that Sandy should have had for which she was unaware. In other words, there were misconceptions and incomplete understanding that Sandy did not know existed for her. It became apparent that Sandy knew pieces of ideas but had little understanding of how the pieces fit together. An episode in the interview that illustrated her limited understanding was our discussion of how to find the length of \overline{AB} . In the session Sandy first talked about an angle of depression and Helen tried to steer them away from finding angle measures.

LBC: Helen suggested that you not find angle measures...she said because you are looking for the lengths of sides....how did you....what were you thinking about when these suggestions were coming at you?

Sandy: Well, I wasn't thinking of the angle itself but I was thinking of the measure of the...like the height of the angle and we already knew A to D was 240 and so I was looking at trying to find the hypotenuse and the side...that's what I was meaning but not the angle itself because I didn't want to know...well, if I said that it was probably because I was thinking of like the tangent, SOHCAHTOA, where you can find uhm you know the hypotenuse where we had the uh like adjacent sides and you could find the other measurements according to SOHCAHTOA...cause we had two sides already.

Sandy recalled the mnemonic, SOHCAHTOA, used in MATH 1113 to remember the trigonometric ratios of the acute angle of a right triangle. I was curious as to how this information would be used.

LBC: Ok, so if you knew that from B to D is 300 and you knew that AD was 240, then...

Sandy: You could find the A to B...

LBC: By doing what?

Sandy: The SOHCAHTOA thing.

LBC: And how would you use SOHCAHTOA to do that?

Sandy: Ok, if you have let's say D was the...like uhm sine of D if you are looking for beta of D...then you know that...let's see SOHCAHTOA...

LBC: Alright now feel free to write it down here.

[Sandy was repeating the mnemonic, trying to remember it to write it down.]

Sandy: I forgot it...SOH...CAH...TOA...is that right? [Sandy first wrote SOACAHTOA but then corrected herself.]

LBC: Uh huh [I told her it was correct.]

Sandy: So if took the sine of B then I'd have to have opposite, which I didn't have...I had adjacent and the hypotenuse so I have cotangent...right?

LBC: Cosine [I corrected her.]

Sandy: And then I could find out this angle by putting...

LBC: Alright find out which angle now?

Sandy: The...the...not angle...the side AB...I could find that...no, I'm taking that too far...[unintelligible]...

LBC: Alright, what are you thinking, why?

Sandy: Because all you have to know is to find a right angle, it would be uhm...like a side plus b side... a squared plus b squared equals c squared... c squared is your hypotenuse...and so I just went too far.

LBC: Ok, so you would use...would you be using a trig ratio to do that?

Sandy: No...no...I messed up on that.

LBC: Ok.

Sandy: No, you would just be using the right angle.

LBC: The Pythagorean Theorem?

Sandy: Is that what that is?

LBC: Which is what you were doing.

Sandy: [Giggling] Yeah...exactly what we were doing. So that's the Pythagorean Theorem? I just didn't know the name.

During the interview I assumed Sandy was unfamiliar with the name “Pythagorean Theorem.” During analysis I realized she assumed SOHCAHTOA was called the Pythagorean Theorem. One reason I decided this was because Sandy used the term “Pythagorean Theorem” in the session—granted, she did not know how to use it correctly but after hearing Helen suggest they use the theorem to find the sides, Sandy subsequently made a similar suggestion. Furthermore, when I looked at Sandy's final exam from MATH 1113, there were two problems that required the use of the Pythagorean Theorem. Sandy's work on these two problems is shown in Figure 41.

7. Find the exact value of the expression $\csc \theta + \tan \theta$ for the angle θ shown. (4 pts)

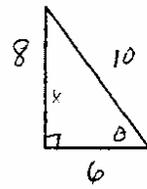
SOHCAHTOA

$\sin \theta = \frac{8}{10} = \csc \theta = \frac{10}{8}$

$\tan \theta = \frac{8}{6}$

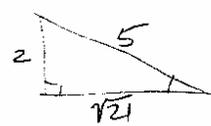
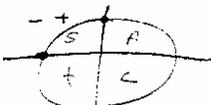
$\csc \theta + \tan \theta = \frac{10}{8} + \frac{8}{6} = \frac{5}{4} + \frac{4}{3} = \frac{15}{12} + \frac{16}{12} = \frac{31}{12}$

$\frac{10}{8} + \frac{8}{6} = \frac{5}{4} + \frac{4}{3} = \frac{15}{12} + \frac{16}{12} = \frac{31}{12}$



$6^2 + x^2 = 10^2$
 $36 + x^2 = 100$
 $-36 \quad = -36$
 $x^2 = 64$
 $x = 8$

SOHCAHTOA

11. Suppose $\sin(\theta) = \frac{2}{5}$ and $\frac{\pi}{2} < \theta < \pi$

a. What is the value of $\cos(\theta)$? (4 pts)

$\cos \theta = -\frac{\sqrt{21}}{5}$

+4

$2^2 + x^2 = 5^2$
 $4 + x^2 = 25$
 $-4 \quad = -4$
 $x^2 = 21$
 $x = \sqrt{21}$

Figure 41. Sandy's work on questions 7 and 11 of the final exam in MATH 1113.

On the exam she correctly used the Pythagorean Theorem to find the length of a side of a right triangle. The problems also required using a trigonometric ratio, and she wrote SOHCAHTOA to help her. It was not surprising, then, that Sandy assumed I was claiming SOHCAHTOA was the Pythagorean Theorem. This conversation was an example of ineffective communication between the two of us.

Although Sandy correctly used the Pythagorean Theorem on the exam, her work from the problem session was incorrect (see Figure 42). I asked her to explain how she obtained these statements.

....

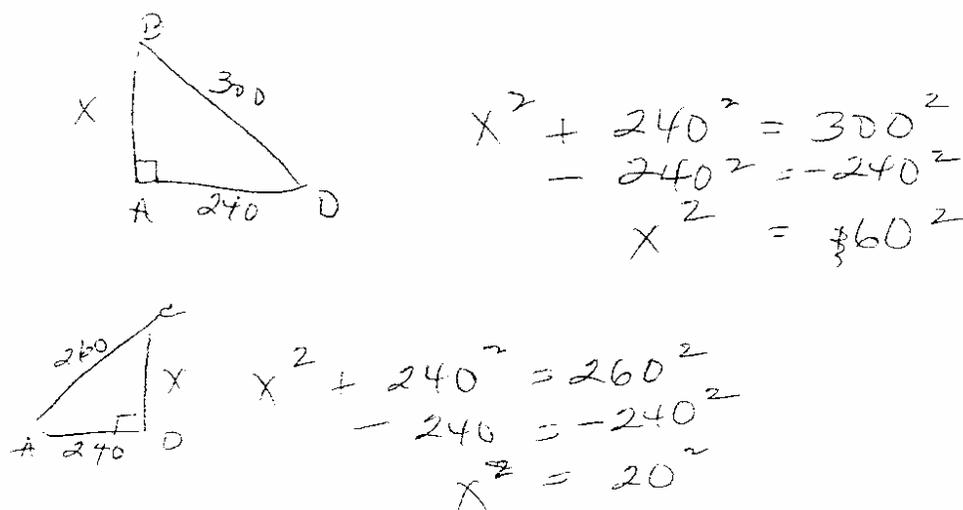


Figure 42. Sandy's work with the Pythagorean Theorem.

LBC: So do you see how the square of x plus the square of 240 is the square of 300—then you get the square of x is equaled to 60 squared—now how did you get 60 squared?

Sandy: Well, you had to subtract 240 squared from both sides and then your x squared which is your side that is your unknown...you were able to get that 60 squared.

LBC: Alright, so what would x be?

Sandy: 60...no 180.

LBC: How do you get 180?

Sandy: Let me see what I got...180...[she looked at her work]....because 60 squared is 180.

LBC: OK...[I handed her a calculator.]

Sandy: Oh, no that's not it [when she calculated 60 squared on the calculator]—how did we get 180? Oh, I don't know how we got that....I know it was right...

LBC: Look right here at this one...where you've got the square of x plus the square of 240 is the square of 260 and you get x squared is 20 squared.

Sandy: Did I get that wrong, too?

LBC: Now that's CD.....

Sandy: That doesn't make sense either...CD we get 100...

LBC: Uh huh...

Sandy: I don't know where I got that from...

LBC: How do you think you should do that?

Sandy: I don't have my notebook with me....[she laughed when she said this].

LBC: Alright, you do want to subtract 240 squared from 300 squared...but....consider this....suppose you had to do...let's change it a little bit...suppose we had to do....5 squared...alright, let's make a different triangle...suppose I tell you this length is 5 and this length is 3 [the triangle is right with a hypotenuse of 5]....and I want to find out this [that is, the other leg of the right triangle]...so I've got one leg that is 3 and I want to know the other leg and I've got the hypotenuse is 5...now how could I figure that out?

Sandy claimed dependence on her notebook and yet she did not have her notebook during the final exam. She and I worked through several problems with smaller numbers. She could easily see that if she had $x^2 = 16$, then x would be 4. She also answered that if $x^2 = 81$, then x would be 9. Through these examples she finally concluded how to handle the calculations and yet seemed surprised.

Sandy: So right here I need to do 240 squared and then figure that out and 300 squared and figure that out...ok, 240 squared is 57600 equals 300 squared...do we really have to do it like that?

LBC: uh huh

Sandy: equals ninety thousand...and then you subtract 57600 from 90000 and that's how you come up with.....there is an easy way to do this... minus five seven six oh oh....and you came up with...so x squared equals thirty two thousand four hundred...

LBC: And how would you find out what x is?

Sandy: You would square...take...square this number right here.....mult..no, divided by

Sandy had a difficult time remembering how to use the square root to find the value. She explained that x would be 4 when $x^2 = 16$ because 16 is a perfect square. She gave a similar explanation for $x^2 = 81$. I asked how she knew x would not be 7, and she said because "seven times seven is forty-nine. (Similarly, the exam question #7 had the perfect square of 64 but on exam question #11 Sandy merely wrote $x = \sqrt{21}$.) Finally, however, she exclaimed, "Oh! The square root sign...duhhh...I'm really stupid!" Still she had trouble calculating the square root of 32400 because she entered $32400\sqrt{\quad}$ on her calculator and got an error message. I concluded, then, that Sandy understood the concept of a square of a number but her trouble was in performing the calculations. She even wrote in her reflection that she wished she had "brushed up on [her] calculator skills" prior to the session.

I finally asked her if she just accepted from the others the answers of 180 and 100 and without hesitation or embarrassment she admitted this was the case. This lack of verification prevented her from being lifted. She said the others were working so fast and she was unable to keep up. However, she was unaware she had misconceptions as indicated by the following exchange.

LBC: Did you not want to say, "Time out." Why did you not stop them?

Sandy: Well, actually it didn't even dawn on me...I mean...if it did it just was for like a brief second and then I just passed it off...but you know, I don't know...

LBC: Were you not concerned that maybe you really...were you aware of the fact that maybe you weren't getting all the little things filled in?

[Sandy did not appear to realize the seriousness of her errors.]

Sandy: Well, no...I mean I understood the Pythagorean Theorem...I just went about you know squaring them at the wrong time.....so I did understand that part...I understood what they were doing...but it was just a matter of actually doing the math myself and actually seeing that my answers matched up with theirs.

LBC: Alright, what if you had been given this on a test, could you have figured it out? If you had just been given this much where you had to figure out the length of that side?

Sandy: Oh, I would have blown it...especially using big numbers like this...I would definitely have blown it...make a mistake like that...

LBC: Ok....ok....so do you see....how it might have helped had you said, “Whoa, gang, show me how you are getting the 180, show me how you got the 100?”

Sandy: Well, not really because I mean I went on the basis of the Pythagorean Theorem...now I went about figuring it the wrong way...but...you know I knew that...well, I knew that they were right...I was on the right track and they were on the same track as I was...and so I didn't question their answer because I know there was two of them that you know had stated their answers...so I didn't question it...I didn't think twice about you know...I was just ...my focus was on just trying to solve this problem...and not realizing that there in itself was a mistake, on my behalf.

Sandy knew how to use the Pythagorean Theorem, and I believe it was her lack of calculator skills and the largeness of the numbers that gave her trouble. Social collaboration factored into Sandy's acceptance of Lisa's and Helen's answers. She did not ask for verification because Lisa's and Helen's answers agreed, and her goal was on solving the problem. It appeared that because she understood what was being done even though she could not do it herself, it was not necessary for her to have a complete understanding—her goal, again, was on solving the problem.

Sandy and I talked about her suggestion of finding the slope of the two lines so the point of intersection could be found. She admitted, “I had a great brainstorm but I didn’t know how to do it!” This is further evidence of Sandy’s having the big picture in mind but lacking the tools to carry it out. Eventually we watched the clip where Sandy said, “And then come up with a formula that we can stick in the calculator that will show the two different slopes.” I was unsure what she meant by “formula.”

LBC: Alright , when you say “Come up with a formula” what do you mean?

Sandy: Because with your y equals you have to have a formula....

LBC: Ahhhhh....is that what you are thinking about? [I realized she was talking about finding an equation of the line.]

Sandy: Uh huh...

LBC: And to be able to find a formula, what do you need?

Sandy: You need to know the slope of a line....OK....find that slope...

Sandy again stressed the group was moving too fast for her as they found the equations of the lines. In the session as well as in the interview, she could explain the meaning of the slope. She was able to explain a slope of $-\frac{3}{4}$ meant the line BD was sloping down. The following exchange showed how Sandy made sense of the y-intercept of 180.

LBC: Now you got for line BD, this equation right here [that is, $y = -\frac{3}{4}x + 180$].....and I said does that make sense as far as the graph is concerned.....What does the 180 tell you in that equation?

Sandy: Oh, actually we had to find out.....your 180 is your y.

LBC: Ok, what y?

Sandy: That point up there [pointing to the y-intercept]...and you are trying to find here....we found the $-\frac{3}{4}$ and that's the slope and then you have to find b...and so we put in for y cause you can't have 2 variables...we put y in for....we put 180 in for y....and y is a point on BD....I mean 180 is a point on BD...

LBC: Alright, 180 is a point?

Sandy: Yes, is a y...is a y point...and that's where we got 180 from...and then we found out after you did this, 180 equals b cause $\frac{3}{4}$ times zero is...zero is the x-value on there...and so when you multiply that out you end up with b equals 180.

LBC: And what does that tell you as far as the graph is concerned...when you get b to be 180, what does that mean?

Sandy: That means you can fill in the...slope formula...where $y = mx + b$.

LBC: Alright...what does the 180 mean as far as the graph is concerned? When you got b to be 180, what does that tell you?

Sandy: That is.....I'm not sure.

LBC: Alright....keep going then.

Sandy: [remembering] 180 is actually like the start at the top, going down.....I think.....am I right?

LBC: Ok, so you got b to be 180 and then what did you do?

Sandy: And then we plugged in...put the 180 in where b was...so we end up with

$$y = -\frac{3}{4}x + 180.$$

LBC: What does the negative three-fourths tell you?

Sandy: That's your slope.

LBC: And does it make sense that it should be negative?

Sandy: Yes, because it is sloping down.

LBC: Now, you got $y = \frac{5}{12}x$ for...

Sandy: ...the line that's sloped down...[Sandy later realized this mistake.]

LBC: Alright, does it make sense that...what is the $\frac{5}{12}$?

Sandy: Actually I don't know how I even got that... $\frac{5}{12}$ is the slope going up... [Sandy realized her earlier mistake.]

LBC: Does it make sense that it should be positive?

Sandy: uh huh....

LBC: Alright, I think at this point the others had calculated this [that is, $y = \frac{5}{12}x$] before they

did this one.....and then you were behind here and just wrote that down....you got $\frac{5}{12}x$, this is

negative three fourths x plus 180...this would be....now the equation of a line is y equals mx

plus bwhat's b here [that is, in $y = \frac{5}{12}x$]?

Sandy: zero.

LBC: Now what does a b of zero mean...do you have any idea?

Sandy: No, I don't.

LBC: OK...Just like here you've got $y = -\frac{3}{4}x + b$,, $-\frac{3}{4}$ is your slope...your line is sloping

downwards so you would expect a negative slopethis slope is positive and it goes up...

Sandy: uh huh...

LBC: This 180 is your y -intercept....which means that's the point on the y -axis where the line crosses...

Sandy: Oh....OK.....that makes sense...because the point where it crosses on the y-axis here is zero....and that's why you don't have to have plus zero...it's just extra.....I get it.... [She sounded very confident.]

LBC: And this one says it is going to cross at....

Sandy: 180

LBC: ... at 180 on the y-axis, and it does.So what happens is....when you are solving this thing algebraically and you get a y-intercept of 180, you look to see if that makes sense on the graph...if it crosses there.

Sandy: Right....Oh, OK.....[Sandy really sounded as if this was new information for her...she was making sense of this meaning of b.]

After getting the two equations, Sandy used her calculator to find the coordinates of point E, the point of intersection of lines AC and BD. I asked her to explain the meaning of these coordinates. First she explained the x-value is 154.29 and the y-value is 64.29 and the lines cross at that point. When asked to explain how the coordinates are helpful with the map, she offered the following explanation.

Sandy: Well, it helps us because it tells us that it is 64.29 feet away from right here...because I'm looking at my y [perhaps y-axis?], my y is actually this line going up and where does it intersect the point of intersection on the line is at 64.29.

LBC: Alright, why is it not 154.29? Where is 154.29 on this graph...what does that mean?

Sandy: That's your x-value...that's how many paces from this point here to here...from this point here to here.....

LBC: So it is 154.29....

Sandy: Well actually, no....from here to here....

LBC: From A to F?

Sandy: Oh, you are going to go from here to here, that's 154.29 paces, and if you go from here to here it is 64.29 paces...and that's where your map should be.

LBC: Alright.....[Sandy stated these distances correctly.]

Sandy: So if you walk over 154.29...from A to F...and then from F to E you should land right on the treasure map.

LBC: OK.

Sandy: You know what? That did not click....I still left confused because I was...I didn't realize...it just sunk in now... [It was obvious that Sandy understood.]

LBC: So even then you didn't realize exactly what you'd found?

Sandy: Yeah, I was unsure...like I was really confused but I didn't want to hold everybody up...because of my confusion...I knew that if I would go home and think about it that it would come out...and that's what I do a lot of times.

It was not until the final interview that Sandy was able to make sense of the relationship of the coordinates of point E to the map. Confusion for Sandy was common. As she pointed out, she would have to go home and think about the problem because she did not want to hold the others up. Not only did she leave the problem session with questions, I believe it was common for her to leave mathematics class with questions. As a result, there were gaps in her understanding. It was the one-on-one communication with me during the interview that helped her deal with some of her misconceptions.

Tess

Tess also said that working with people whom she did not know well made her uncomfortable. She admitted that she is by nature a shy person and yet was surprised at herself for being as quiet as she was during the session. She said, "Cause I have opened up a lot, you know, broken out of my shyness in the past year or two, maybe three, but I didn't think I was gonna sink back down to the bottom of the barrel [she laughed when she said this]."

Although the lack of familiarity with the other group members made it difficult for her to interact, she was also overwhelmed to find the problem was a dreaded word problem.

Tess: When I read it the first time, it was like...word problem...and it was like I just stayed on that word problem when in fact I didn't have to...it was like all we needed was the...what they gave us...the numbers: the B to D, A to C, and A to D.

LBC: What do you mean—"You stayed on that word problem"?

Tess: I stayed on...I don't like word problems and when I saw it, it was like I think just stayed on the idea, "Oh, I don't like word problems, oh, I don't like word problems!"

LBC: Was that going through your mind?

Tess: yeah, yeah, that was going through my mind...and then it was like, well I think we got half way through it... was like... you know, it didn't... I mean...none of this...well you know it related but it was like we didn't really need it...it was just there, just telling us about it.

LBC: So what were you thinking about when she [Sandy] was thinking about the angle of depression...

Tess: I was like, "Oh, we've got to do so much stuff to get the answer when was she was naming all the stuff, "...I was like, "Oh, my goodness, we've got to do all this stuff," and I was looking, I was like....uhh....

Working a dreaded word problem with people whom she did not know raised Tess's anxiety level. Anxiety may have contributed to the frequent long pauses or no responses following questions I asked her during the final interview. Although she said she understood the graphical approach to solving the problem, she still admitted there were times when she had questions. Her reason for not asking those questions showed her egocentrism.

LBC: Did you have some questions that you wanted to ask and just didn't?

Tess: uhm....yeah, I think I did...

LBC: Why did you not ask them?

Tess: uhm.....because I wasn't sure of what the exact words or ...uhm....the name...I wanted to have a name...you know for what...a name of...I'm just going to say a name of the

formula so I wouldn't be trying to, you know, explain and they would be like, "I don't know what she is talking about."

Tess kept quiet rather than risk a conflict with the others. Of course, this meant that she contributed very little as well. She was frustrated with herself for her lack of contribution and said that she would behave differently in future group sessions.

Tess: I'm not going to sit and say nothing...I'm going to say something even if it is wrong.....I don't know.....[long pause].....I just don't want them to think that "you know she's not doing anything" so I am going to speak up more.

Tess had little to say during the session but did understand how to find the point graphically. In fact, she said she "was thinking about plotting the points...around the same time she [Lisa] was but I didn't say anything. I was like 'if we plot 'em, how will they connect?'" Again, this showed that Tess avoided the risk of being wrong.

Although she understood how to find the point of intersection graphically, she admitted that she would never have thought to "set the two formulas equal to each other." That is, finding the coordinates of point E symbolically would not have occurred to her. She recognized how working in the group allowed her to consider different approaches.

A lifting experience for Tess came when she learned the shortcut key on the TI-83 calculator. Since middle school, she had used ALPHA x [that is, the ALPHA and the STO keys] to type x rather than using the shortcut key x^2 . During the problem session while trying to type x, she was pressing 2nd rather than ALPHA. Although she exclaimed during the session "I'm looking for where those buttons are!", it was not until the interview that I realized she was unaware of the shortcut key.

Synthesis

Group 2, like Group 1, had very little prior experience working mathematics problems in a small group. Unlike Group 1, however, Group 2 worked together to solve this problem. It was perhaps prior experiences outside of the school environment that influenced the group to be

more collaborative. For example, Sandy was older, a widowed mother of two, and formerly served in the military. Helen was one of five children in a close family so compromise and collaboration were not uncommon for her. These personal experiences helped Sandy and Helen develop the interpersonal skills that are useful in working with others.

Prior knowledge of the participants was an influencing factor. Even more importantly was the willingness of the participants to admit their weaknesses so that others in the group could provide help. Lisa was very adept at using the graphing calculator and was able to answer questions when calculator issues arose. Tess, too, was fairly confident with the calculator. When Helen claimed the INTERSECT function on her calculator never worked, Lisa explained how to use this function correctly. When Sandy had trouble constructing a scatter plot, Tess showed her how to enter the data into lists. Not all issues with the calculator were resolved during the session. For example, Tess had trouble remembering how to type the variable X on the calculator when entering an equation. She was unaware of the shortcut key for typing this variable and was slow in finding the two keys for typing the variable. In the final interview, I realized she was using ALPHA X to type the variable and was able to help her learn the shortcut.

Sandy had trouble with symbolic manipulation but also admitted when she needed help. It was not uncommon for one member of the group to lean over into another's personal space to provide help. Although there was not a single workspace for writing, the group frequently created a common workspace as they shared their written work, shared their calculator displays, and entered each other's personal space.

It was Sandy's suggestion of finding the slope that helped Lisa realize they could use a graphing approach by finding the equations of the lines. Sandy could not remember the formulas and needed help from the group in doing this. Furthermore, once the answer was found, Sandy needed the group to help her interpret the answer in the context of the problem. It

was Lisa and Helen who could explain the meaning of the coordinates of the point of intersection.

The communication in this group was effective. They listened to each other and tried to make sense of the other group members' suggestions. This characteristic was evident early on; after reading the problem, the group discussed their options and developed a sense of the overall solution strategy they would use. Negotiating this strategy resulted from their sense-making perspective and their willingness to communicate effectively. For example, Helen did not agree with finding the angle measurements and convinced the group that approach would not be useful by explaining they were not looking for angle measures but instead needed the lengths of segments. As another example, when Sandy and Lisa recommended the graphing approach, Helen did not understand their recommendation and really preferred a symbolic approach. Nevertheless, Helen was open-minded enough to listen and follow along. Not only did she later understand the graphing approach but she also was helpful in finding the equations of the lines. Thus, there was an overall sense-making perspective. They did not always agree, but doubts and questions were dealt with by trying to make sense.

Their sense-making perspective was also evident when they searched for an explanation to an unexpected result. For example, when they constructed a scatter plot of points A, B, C, and D and determined the graphs were different, they did not abandon the approach but made sense of why the graphs were different. Further exploration led to lifting as they realized the limitations of the calculator in dealing with a graph constructed as a scatter plot versus one that is constructed as a function.

The participants were respectful of each other and did a good job of keeping their personal feelings under control. In the final interview, Helen was especially vocal about her feelings toward Sandy and Lisa. She said she was distrustful of Sandy's mathematics and made a number of statements in the final interview that reflected her lack of confidence in Sandy's mathematics ability. Helen explained that although she trusted Lisa's mathematics the

most, she would not have chosen to work with her because they both had strong personalities and would have clashed had they spent more time together. To Helen's credit these negative feelings were not obvious to me during the problem sessions.

Sandy said in the final interview that she believed some of the participants became frustrated with her questions from time to time. Even though Sandy may have sensed their frustration, she did not appear intimidated and continued to make suggestions. She did, however, refrain from asking all the questions she had as became clear in the final interview. Part of her failure to ask questions was probably the result of her experiences with learning mathematics as she fully expected to have lingering questions.

Further evidence of their respect for each other occurred when Lisa allowed Tess to pursue the scatter plot idea even though Lisa said she would have approached the problem differently. Her reason for allowing Tess to pursue this idea was because Tess's participation had been minimal.

Cognitive disagreement was not necessarily avoided in this group and yet there were times they could have pursued it further. For example, when Helen said that having two unknowns defeats the purpose of algebra, Lisa clearly disagreed with her statement and said, "You can do it." Helen did a poor job of defending her position so the issue was dropped. As a result, lifting did not occur for Helen. Lisa did not see the issue as relevant to the current problem so she chose to drop it.

Helen had a misconception with variable representation. Specifically, Helen believed the equation $300 - x = ED$ implied x and ED are the same thing. Because the group chose to use a graphical approach, her misconception did not have significant implications during the session and was consequently overlooked. During the final interview, however, her misconception became apparent, providing me with the opportunity to help her confront her misconception so that lifting could occur.

Another misconception for Helen became apparent at the end of the session when she had trouble making sense of using a system of two equations with two unknowns to find the point of intersection. Specifically she had trouble understanding the meaning of this system in relation to the graph. The work in the session was not sufficient to help her understand *why* the

expressions for y in $y = -\frac{3}{4}x + 180$ and in $y = \frac{5}{12}x$ could be equated to find the

point of intersection. Although she accepted the procedure, she said it was not until the final

interview that she understood $-\frac{3}{4}x + 180$ and $\frac{5}{12}x$ were the y -coordinates associated with an

x -coordinate on the line. Thus, the seed was planted in the session for Helen to make sense of this concept but full understanding did not occur until the final interview.

Egocentrism was a factor for Helen. My conversation with her during the final interviews revealed egocentrism was a factor and yet she camouflaged its influence through well-designed face-saving strategies. For example, in the session she set the two y -values equal to each other but without really understanding the logic behind such a procedure. Helen also made amusing remarks that were probably face-saving strategies. Perhaps because her father taught mathematics she felt the need to appear mathematically literate.

Egocentrism may also explain why Lisa waited until Sandy saw the points on the graph before she pursued the idea with the group. Lisa admitted she is reluctant to explain her idea to another unless she is sure it will work. Tess also admitted there were times when she had questions but did not ask them because she was afraid she could not explain herself well.

Although egocentrism was a factor to some degree, for the most part the students were not afraid to admit their mistakes. They saw the group as providing help and they were not embarrassed to ask for that help. As a result lifting could occur.

Sandy had difficulty finding the length of a side of a right triangle with the Pythagorean Theorem. She was not lifted during the session because she merely accepted the work of the

others. Acceptance of their work was partly due to her not wanting to hold the others up by asking questions and partly because she did not know that she did not understand. It was not until the final interview that I helped her work through her trouble related to this procedure.

Cognitive ability was an influencing factor in this group particularly in creating a “charismatic intellectual.” Lisa’s SAT-M of 560 and Helen’s SAT-V of 610 made them good candidates for the role of “charismatic intellectual.” Helen truly had the gift of “gab.” As one of five children in a very close family, she had plenty of opportunities to use her verbal skills. Her SAT-M of 490 showed her math ability was lacking somewhat but Lisa’s math ability meshed with Helen’s verbal ability to provide the leadership that was beneficial in Group 2. Tess made 450 and 490 on the SAT-M and SAT-V, respectively. Her score of 80 on the final exam in MATH 1113 probably indicates that she, too, is an over-achiever and ideas will not be immediately obvious to her. SAT-M and SAT-V were not available for Sandy but her MATH 1113 final exam score of 41 indicates her understanding of the mathematical concepts in MATH 1113 was weak.

Tess, a shy person, claimed in the first interview that she “is not good at word problems.” As a result, she was very uncomfortable when she was given the Buried Treasure Problem. The task of solving a word problem influenced her ability to make sense of this problem.

The students were comfortable asking me questions. Generally I answered them with a question. They listened to my responses and used them to make sense of the problem. I also perceived my role as encouraging them to clarify their explanations.

The participant’s pointed out several factors that influenced the outcomes of the session. Helen and Sandy mentioned the lack of familiarity with the group members as a factor that made working together difficult. They also said the pace of the problem solving interfered with their ability to make sense of the problem. An obligation to keep up with the others made it difficult to understand the problem fully. Tess said that she could not “run off and do [her] own thing” because they were to work as a team. Lisa admitted that she did not like group work and

yet believed the process of explaining “helped more than anything.” Lisa also observed how important it was to listen carefully to every idea; for example, Sandy’s comment about slope helped Lisa formulate a workable plan.

CHAPTER 6

ANALYSIS OF THE CASE STUDIES

As Watson and Chick (2001) found with their study, cognitive, social or interpersonal, and external factors influenced the outcomes of the problem solving. I began the study by explaining my definition of problem solving was not limited to getting an answer but instead referred to the processes students use as they attempt to solve a problem. My goal then was to identify the factors that influenced these processes. I attempted to describe the twists and turns the students took in their pursuit of the problem and to understand what factors influenced those twists and turns. The Buried Treasure Problem was the only problem used. Group 1 took four sessions to solve the problem while Group 2 solved the problem in one session only. The approaches they used were totally different.

One of my research questions was to identify the factors influencing the problem solving of a small group of students as they work together to solve a mathematics problem. Because the factors interact, identifying the factors and how they influenced the problem solving was rather cumbersome. Nevertheless, to provide some degree of organization to my findings I have listed each of the factors as a subheading; however, discussion of one factor may well touch upon another factor. My difficulty in delineating the different factors merely confirms the complexity of small-group learning. Watson and Chick (2001) alluded to a similar dilemma by acknowledging there are cognitive aspects associated with social factors and acknowledging external factors influence both cognitive and social factors.

In the following sections I will identify the factors influencing the problem solving in the two groups and describe how these factors influenced their problem solving. Each group composes a case in this study but within each group are the individual students. Patton (2002) explained how a case may have nested cases, and I saw these individual students as cases

nested within the case of the group. As a result, my analysis of the influencing factors required that I look at the individuals to gain an understanding of the factors influencing the group. Watson and Chick's framework (2001) was especially useful in helping me identify the factors influencing the individuals. Because an understanding of the cognitive factors depends so much on the reader's understanding of the social factors, the social factors will be investigated first, followed by the cognitive factors, and finally the external factors. Part of the analysis also includes the participants' perspectives of the influencing factors. Only after looking at the influencing factors at these levels could I then make sense of the influencing factors for the group.

Social or Interpersonal Factors

Leadership Factors

Watson and Chick pointed out the benefit of a "charismatic intellectual" as a participant in a collaborative situation. In Group 1 Sara could easily have been the charismatic intellectual because she certainly possessed the characteristics to fulfill this leadership role. Unfortunately she was unwilling to assume the leadership role so Group 1 floundered. Group 2 did not have a single leader but really had their charismatic intellectual created by the characteristics of both Lisa and Helen. Lisa provided the intellectual aspect while Helen, the more vocal of the two, was clearly the charismatic partner. Although unrecognized by the two of them, they really complemented each other in the group situation. It was the meshing of their strengths that created the leadership for Group 2.

Egocentrism

Egocentrism was an influencing factor in both groups. The egocentrism was related to pride and ego-preserving. Perhaps it was the age of these students as well as their not knowing each other well that contributed to this need to preserve one's ego. Some students found it difficult to interact. The students were not well-acquainted with each other which influenced the level of interaction for some of them. Some students were not confident in their abilities and

were fearful of being wrong; as a result, keeping quiet was a way to preserve their egos. Cryptic remarks and other face-saving strategies were used to camouflage mistakes rather than confronting these mistakes. Egocentrism was especially evident in Group 1. As a result, understanding, sense-making, and the opportunity to make valuable contributions would be sacrificed in order to protect the ego. Egocentrism had less of an effect in Group 2. The final interviews with the participants in Group 2 confirmed its existence, but they were able to keep it under control so that its existence was less obvious and less influential.

The difference in influence of egocentrism in the two groups may be a result of the participants' perspective of the task of working together. Group 2 showed more collaboration. Working the problem as a group was their goal. For Group 1, however, a sense of collaboration was not developed. Their personal obligations to the experience superseded the obligation to work together.

Social Collaboration

Social collaboration occurred in both groups and occurred as both a positive influence and a negative influence. Social collaboration as a positive influence in Group 1 was virtually nonexistent and was limited to the occasions where Abby showed support of another's contribution by giving a supportive giggle or saying, "Yeah." Abby's superficialities served no purpose other than to help her believe she was contributing. Missing in this group were opportunities to build team spirit through genuine supportive comments. On the other hand, Group 2 had occasions where one participant identified with another by expressing understanding of her struggles. This identification made the atmosphere more relaxed and promoted team spirit.

In both groups, instances of social collaboration whereby the influence was negative occurred. There were times when a participant would agree with another even though an error was made. This was usually the result of either not listening carefully or agreeing just to appear knowledgeable. This agreement had the potential to lead to false assumptions which sacrificed

progress. In Group 1 the effects were more pronounced because no one carefully monitored the work that was done. In Group 2, the group members monitored each other's work so that potential errors resulting from misdirected social collaboration were more likely addressed.

At other times social collaboration occurred when agreement was expressed just to avoid a conflict. This was especially true in Group 1 where Jane was concerned. Eve and Sara would give a casual "uh huh" without really listening to Jane's comment. Sometimes the casual "uh huh" was enough for Jane to continue her recitation of her work—usually vandalized mathematics. Jane's lengthy recitations often impeded the group's progress by diverting their focus. In Group 2 Sandy frequently agreed with the other participants when clearly she did not understand what was being said. Sandy would agree to avoid delaying the progress of the group. Unfortunately this social collaboration meant that Sandy was not lifted since she did not seek help from the others.

Cognitive Factors

Cognitive Ability

Cognitive ability was an influencing factor. The data used to give information about cognitive ability were the SAT-M, the SAT-V, and the final exam grades in the MATH 1113 course. An examination of this data helped explain some of the behaviors of the students. For example, Abby's low SAT-M and low exam score could contribute to her reluctance to share her ideas as she had little confidence in her mathematical understanding. The scores of two students (Eve and Tess) indicated they were over-achievers. It was not surprising, then, that in the group they were rather quiet but nevertheless engaged in the problem as they tried to make sense of the problem.

Cognitive ability interacted with leadership factors. Sara's scores revealed she had the cognitive ability to be the leader; moreover, her experience in training her coworkers made her a perfect candidate to be the leader Group 1 needed. It was a conscious decision not to be the leader that made her an ineffective leader. The scores of Lisa and Helen in Group 2 revealed

their strengths to create the “charismatic intellectual” with Lisa providing the intellect and Helen providing the verbal ability needed to lead.

Previous Experience

The participants’ prior experiences and preexisting knowledge certainly impacted the outcomes of the study. In this study on small-group learning when the four participants in each group brought their individual experiences and knowledge together, the potential existed for the group to accomplish a task perhaps unattainable by the individual participants. I will first discuss the prior experiences that influenced the problem solving and then discuss the preexisting knowledge that was influential.

Their lack of experience with solving a mathematics problem in a group situation was especially influential. The little experience some had with learning in a small group was not very beneficial in this study. The directions for this study merely asked them to work together to solve the problem, and as a result, they had different ideas about the meaning of “solving a problem together.”

Their preexisting knowledge about mathematics was an influencing factor. According to Good, McCaslin, and Reys (1992a) one of the advantages of group work is that subject-matter knowledge is increased when students pool their understanding, content knowledge, and problem-solving skills. In this study, the knowledge possessed by one participant did lift another participant so that the problem solving in the group was influenced. Sometimes the dispenser of the information was unaware of her contribution. For example, it was a rather casual comment or question by a student in each group that determined the approach each group would take. In Group 1, it was Eve’s question about the measure of an angle common to two overlapping triangles which initiated Sara’s consideration of similar triangles. In Group 2 it was Sandy’s comment about the slope which gave Lisa a way to get the graphs. Interestingly, although Sara and Lisa were both helped by these somewhat innocuous comments, neither student conveyed to the other group members how the comments helped them develop a plan. In fact, both

students put their ideas on hold while other suggestions were explored; their decision may be the result of leadership factors, egocentrism, or social collaboration.

There were times when one participant's knowledge was not shared with the others, especially in Group 1. This group had communication problems as will be discussed later. In Group 2, calculator issues, finding equations of lines, and the connection between the points on a graph and the equation were examples of preexisting knowledge which were shared with the members of the group and helped the group make progress.

Other knowledge that influenced the problem solving came in the way of misconceptions. Admittedly, I was surprised to discover the extent of the misconceptions which existed in both groups. One misconception present in both groups was that of variable representation. This misconception existed for at least Sara, Eve, and Jane in Group 1. The discussion about this concept in the second session lifted Eve but lifting did not occur for the other two students until their final interviews. In Group 2 Helen was the only student for whom I was certain a misconception about variable representation existed. Although I cannot be certain this misconception did not exist for any of the other Group 2 participants, the misconception did not become evident in the approach Group 2 used. Helen's misconception about this concept was first identified by me as a "slip of the tongue," but in the final interview the extent of her misconception was realized. Her misconception did not influence the problem solving of the group because of the approach Group 2 decided upon. It was not until Helen's final interview that lifting occurred for her.

Group 1 used the concept of proportionality to solve the problem. Abby and Eve saw that one triangle was a smaller version of another, but during Session 4 I was convinced Sara was confused about the concept of proportionality. Not until the final interview did I realize the concept of proportionality was not her main stumbling block but instead she still possessed a misconception about variable representation. The lengthy discussion we had in Session 2

about variable representation did not lift Sara. Furthermore, my failure to identify Sara's real misconception in Session 4 delayed the group's progress.

The concept of proportionality was definitely problematic for Jane in Group 1 and lifting never occurred for her. Other problems with the mathematics existed for her for which lifting never occurred because she was close-minded to any help. Her view of mathematics was so procedural that she never recognized her misconceptions and misunderstandings. Furthermore, committing vandalism to the mathematics to obtain an answer seemed perfectly logical to her. Jane did not like to explain her thinking in MATH 1113 and did not recognize its benefit. During the problem sessions, she tended to work alone. Her lack of sense-making behavior was so like that described by Schoenfeld (1994) for students who are accustomed to learning mathematics as received knowledge rather than learning it as something that fits together meaningfully and to be shared with others.

The concept that an equation such as $y = -\frac{3}{4}x + 180$ expressed a relationship between the coordinates of the points on the line represented by this equation was not clear for Helen in Group 2. Social collaboration or lack of sense-making allowed her to proceed without fully understanding this concept. It was not until the final interview that lifting occurred for Helen.

Students in both groups had problems implementing procedures but were reluctant to admit these difficulties. For example, finding the side of a right triangle with the Pythagorean Theorem was not automatic for Jane in Group 1 and for Sandy in Group 2. Jane knew she had trouble and kept working until she was able to get the answer. Egocentrism kept her from admitting her difficulty. Sandy, however, may not have realized the trouble she had with this procedure. Furthermore, she believed she would be able to figure it out if needed and did not want to hold up the group by asking for help. Not wanting to hold up the group could show social collaboration.

Another procedure that was problematic for Group 1 was that of finding the measure of an acute angle in a right triangle given a trigonometric value of this angle. None of the four students could remember how to obtain the angle measure. The opportunity to recall how to perform this procedure, however, was influenced by poor communication which included misunderstandings, no common workspace, and social collaboration. A need for this procedure never surfaced in Group 2 because Group 2 decided not to find the angle measures. When Group 2 needed to find a way to graph the lines, however, they worked together. They shared calculator displays (creating a common workspace), and they listened to the explanations of others, asked questions, and engaged in cognitive disagreement. Thus, a noticeable difference in the two groups was the degree of collaboration used. The students in Group 2 worked together—or at least no one was working independently of the others. The students in Group 1 would cognitively withdraw from the group with no explanation of their intentions. Collaboration in Group 2 was enhanced with more effective communication and the creation of a quasi-common workspace.

Sense-Making Perspective

A sense-making perspective influenced the problem solving. Group 2 had more of a sense-making perspective than Group 1. Group 2 was interested in making sense of the problem and in making sense of the thinking of the other participants. Group 2 listened to one another's suggestions and were more likely to ask questions when clarification was needed. In contrast, the participants in Group 1 showed little interest in the comments and suggestions of the others. Because they listened halfheartedly, misunderstandings were quite common which often led to unproductive investigations. Furthermore, Group 2 was more receptive of my suggestions and questions. They used my comments to help them make sense of the problem whereas Group 1 was less likely to use my comments.

The disparate personal goals of the participants of Group 1 created an environment whereby sense-making was not valued. A strong leader with a sense-making perspective was

needed to push this group to question, to clarify, to look at different options, and to monitor their work.

Communication

A group with a sense-making perspective is more likely to engage in communication to convey their ideas to the other group members. Such communication I have termed “effective communication” and have used the research by Sfard and Kieran (2001) in which they claim communication is effective unless there is evidence to suggest otherwise. Because Group 1 lacked the sense-making perspective, their communication was not effective and influenced the problem solving within the group. There were times when they clearly could not understand what was being said and yet no clarification was requested. There were times when they talked past each other. There were times when they did not listen. Cognitive disagreement rarely occurred in Group 1 which meant understanding was delayed or avoided. On the other hand, the communication in Group 2 could generally be classified as effective. The communication was used to make sense of the problem and to understand one another’s thinking. There were times when cognitive disagreement was a part of the sense-making endeavor. When questions were asked, detailed explanations were often given. There were times when I asked them to elaborate further, and they complied with more extensive explanations.

The Big Picture

The development of a qualitative approach and keeping sight of this approach influenced the problem solving. A noticeable difference in the two groups was the way they decided on the approach to solve the problem. After reading the problem Group 2 first discussed different options. They decided upon a qualitative approach for solving the problem before they began implementation. Even though Helen did not understand initially how the graphing approach would work, she was open to the suggestion and willing to see how it would play out. Group 2 also did a better job of monitoring the work that was done and making sense of the mathematics when they encountered problems. Lisa stood out as the participant who kept the big picture in

mind. In contrast, Group 1 was likely to implement a procedure without discussing how the information gained would be useful. Furthermore, if one procedure was unsuccessful, they would search for another rather than trying to make sense of their work. Thus, most of their time was spent implementing one procedure after another with little discussion of how the results of these procedures would help in the solution of the problem. They knew they needed to find a distance, but the qualitative approach for finding that distance was not made clear. Sara had the idea of using proportionality to obtain the solution but did not communicate this approach to the others. Leadership factors perhaps prevented her from explaining her thinking on the approach she believed to be useful. Sara's failure to explain this approach together with her absence in the third session meant the group had difficulty making tangible progress in the third session. Eve was absent in the fourth session and again progress was slow when the group failed to use the work generated in the third session. Sara's and Eve's absences certainly broke the continuity of the group work but most influential was the inability of the other group members to take up the task where the group left off in the previous session. Because they were reluctant to refer to an absent member's work, they may not have viewed the work as belonging to the group. In summary, Group 1 never discussed a qualitative approach and also did a poor job of referring to work generated in a previous session to help them in subsequent sessions.

Goal Focus

Their personal goals for the experience also influenced the problem solving. I believe their lack of experience with working in a small group contributed to the influence of their personal goals. For some students the goal was to get an answer to the problem; other students wanted to be sure no one felt left out; others tried to make sense of the problem; others were there "to soak up" as much information as possible. And then there was my goal to enhance their ability to think mathematically as they worked on this problem collaboratively. These unspoken and disparate goals especially hindered the progress in Group 1.

Although Group 2 also had minimal small-group, problem-solving experience, they seemed to develop a common goal of working together to solve the problem. It was perhaps the following observations that contributed to their more collaborative experience: (a) Three of these four students planned to teach so explaining was perhaps a part of their nature; (b) Sandy, a 37-year old widow and mother of two teen-agers, served in the military; (c) Helen was the fourth of five children in a family that spent much time engaged in activities together. These personal experiences I believe contributed to a more collaborative experience.

External Factors

Task Factors

The problem itself—a word problem—created confusion in the mind of at least one of the participants. Her dislike of word problems made it difficult for her to make sense of the problem.

The task of working together was influential. The opportunity for a participant to explain her thinking to others could help her make sense of the mathematics. Sara and Lisa shared in the final interviews how the task of explaining helped them clarify their thoughts. For those students whose egos needed preserving, the task of working together meant that risks had to be taken if they exposed their thinking to the group.

For some students the pace in the sessions was too quick so they were unable to make sense of the situation. The need to keep up with the group caused them to sacrifice understanding at the time but only because they believed they could figure out the work later on. (Lifting did not occur for some students, however, until the final interview.) Both Sandy and Helen in Group 2 allowed the group to move on even though questions still existed for them. In Group 2 Helen admitted she did not understand the graphing approach when Lisa and Sandy first talked about it, but she asked them to continue and she would follow along. Eventually, she was able to make sense of this approach. Sandy apparently was accustomed to being overwhelmed by mathematics; for her, it was routine to take good notes and then try to make

sense of it on her own. Other participants did not ask questions because it was more important to preserve the ego rather than to gain understanding.

The students were not required to write up a solution to the problem. Therefore, once a solution was obtained there appeared to be a lack of appreciation for the mathematical thinking and reasoning which was needed to obtain that solution. This was true for at least Jane and Abby in Group 1 and for at least Helen in Group 2. Of the four students in Group 1, it was Jane and Abby whom I believed had the poorest understanding of the problem (as determined by the final interviews). Nevertheless, they said at the conclusion of Session 4 they were surprised the problem was so easily solved with ratios. In Group 2, where the point of intersection was found using a graphical approach and a symbolic approach, Helen claimed the symbolic approach was “easier” but did not recognize the work to get the equations would have been the same. I concluded these students left the experience without recognizing the mathematical thinking required to lead them to a solution because they were not asked to reflect on the experience by preparing a write-up of their solution.

Outsider

I, as outsider, was an influencing factor in both groups. My influence in Group 1 could have been more helpful had the participants been more receptive to my questions and suggestions. Some of them, I believe, saw my interaction as a hindrance rather than as a help and chose to ignore my comments. In contrast, Group 2 paid attention to my questions and comments and seemed to appreciate my help. My involvement often pushed them to examine their thinking as I challenged them to explain and clarify their comments. Perhaps the sense-making perspective influenced how the groups responded not only to my comments but to the comments of one another.

The misconceptions about fundamental mathematical concepts present in both groups was surprising to me. According to Schoenfeld (1983), however, evidence of these misconceptions was typical. In fact, he claims misconceptions will typically surface only when

students are asked to show us what they know. Consequently, the evidence of these misconceptions confirms the benefit of small-group problem solving as a vehicle in identifying the misconceptions students possess. When I observed these misconceptions in the sessions, I felt compelled to help the students confront them. Rather than directly pointing out the misconceptions, I asked questions hoping they would recognize their misconceptions and make the necessary connections and modifications to their learning. My efforts did not always produce the results I anticipated so knowing when and how to intercede when misconceptions surface as students work together is certainly an area I see for further research.

My goal for the session was for them to work together; thus, when there were periods of silence in Group 1, I would remind them to talk to one another and share their thinking. My intrusion, however, may have interfered with the need to reflect quietly on the problem. It was unfortunate that when students did withdraw cognitively from the group, they did not communicate to me and to the others their intent. Some of the students observed it would have been helpful for these students to explain what they were independently working on rather than just withdrawing with no explanation. In contrast, there was not a single time in the session for Group 2 where I reminded them to talk to one another. Group 2 worked together and shared their thoughts without prodding from me.

Logistical Factors

The students sat around a circular table and each had her own pencil and paper to use. There was no common workspace upon which all the work was done and shared with the other participants. The seating arrangement made it difficult to see the work of the other participants unless a deliberate effort was made to share this work. Sara, in Group 1, recognized the need for a common workspace following Session 1 but never suggested its creation. The participants in Group 2 did not have a common workspace, but frequently created a quasi-common workspace by leaning into each other's personal space. Such movement rarely occurred in Group 1.

The Buried Treasure Problem stretched over four sessions in Group 1. Sara missed Session 3, and Eve missed Session 4. These students were the strongest mathematically of the four students, and their absences had a noticeable impact on the problem solving.

Influencing Factors from the Participants' Perspectives

The participants needed to feel comfortable with the other members of their group and cited lack of familiarity with the other participants in the group as a negative influence in this study. Making assumptions about the problem, having no common workspace, and forgetting mathematical ideas were also cited as negative influences. Some participants wished they had been given more time to reflect on the problem before beginning work.

The factors they believed had a positive influence on the problem solving were a willingness to share ideas, to work together, and to support one another's comments. They also noted the problem was solved because information forgotten or unknown by one participant could be supplied by another.

Factors Influencing the Group

A Community of Learners

Schoenfeld (1992) said that in order to understand how students develop their mathematical perspectives, the mathematical communities in which the students live and the practices that underlie those communities must be considered. The small groups in this study provided one kind of mathematical community to be examined. Ideally, the group would function as community of learners. To be a *community of learners*, however, the group must first function as a community. The good of the group must supersede the good of the individual. Therefore, personal agendas are sacrificed for the group to function as a unit. The development of a community is enhanced as students get to know one another and become comfortable in one another's presence.

Students' mathematical perspectives, their mathematical thinking, their problem-solving skills depend upon their functioning as a *community of learners*, a community whereby learning

is the primary goal. Groups composed of students who have learning and sense-making as their primary goals are able to experience and benefit from opportunities for cognitive lifting. These groups support one another. They are not afraid to disagree or to ask for further explanation or justification of ideas. Social collaboration is important but to be a community of learners, social collaboration cannot be superficial. The individuals need their ideas verified but a mere “rubber stamping” of these ideas is of no benefit. In this study, Group 2 came closer to being a community of learners.

Interaction

Schoenfeld (1992) also said understanding the role of interactions is the key to understanding learning. Interacting involves a degree of risk-taking that factors such as egocentrism can inhibit. Ineffective communication consisting of misunderstandings, talking past each other, and incoherent speech can also hinder the group’s ability to interact in a productive way. Productive interactions will not occur when each individual works on his or her own work without sharing or discussing ideas with the others. No common workspace also minimizes the chance for interactions to be productive.

Several researchers (Cohen, 1994; Goos, 2000; Goos et al., 2002; Lumpe, 1995) have observed interaction characterized by elaborated exchange of ideas, cognitive disagreement, and effective communication are necessary for conceptual learning to occur. These researchers have also noted the importance of ill-structured tasks designed to foster this type of interaction. Even though both groups in this study participated in the same task, their level of interaction was noticeably different, making their productivity in the sessions noticeably different. The task, I believe, was ill-structured enough. My attempts at getting Group 1 to talk to each other and to share their thinking may well have constrained their interaction. Cohen observed that when the task directions are too prescriptive and when teachers do not delegate authority to the group, interaction may be constrained.

Status Factors

One of the benefits of small-group learning is the opportunity for the strengths of individuals to support the weaknesses of others. Good, McCaslin, and Reys (1992a) claim subject-matter knowledge is increased when students work together because a student working alone may not know how to approach a problem but when students work together they can pool their understanding, content knowledge, and problem-solving skills to increase the likelihood of obtaining a solution. When students allow status to become an influencing factor, this benefit of small-group learning is inhibited. Expecting every individual to contribute equally would be ludicrous; however, students who make assumptions about the strengths and weaknesses of their peers may overlook valuable contributions. In both groups in this study, significant contributions were made by students who were not necessarily the strongest in ability.

The role of a leader falls in this category of status factors. An effective leader can promote the development of a community of learners and can encourage the members to interact. An effective leader can also help monitor the situation and help keep the group on track. There are times when a suggestion could take the group on a “wild goose chase, “ and an effective leader can pull the group together to consider the appropriateness of the suggestion without alienating the contributor. It is not enough for the group to perceive someone as a leader—the individual must be willing to lead to be effective. Group 1 suffered because Sara was unwilling to be the leader the others perceived her to be.

Accountability

Smith and Waller (1997) noted how important it is for the members of a group to be committed to a common goal and to hold each other personally and individually accountable to the effort. Part of the problem in Group 1 was the issue of accountability. The students knew the work was not for a grade, it was summer time, and they had other obligations and interests. As a result, the members were not wholly committed to the effort. The unexpected absences of Sara and Eve provide evidence to their lack of commitment. Further evidence was their lack of

sense-making behavior. A required write-up may have encouraged the group to be more accountable.

In contrast, Group 2 tried to understand and make sense of the suggestions of the others. The requirements for Group 2 were no different from those for Group 1 and yet Group 2 held themselves and each other accountable.

CHAPTER 7

SUMMARY, IMPLICATIONS, AND RECOMMENDATIONS

This chapter includes a reflective look at the purpose, rationale, and framework of the study. Included, too, are the conclusions I drew, the implications for the use of small-group learning in mathematics, and recommendations for future research.

Purpose and Rationale Revisited

The purpose of my study was to identify factors influencing the problem solving of a small group of students as they worked together to solve a mathematics problem and to describe how these factors influenced the problem solving. My interest in such a study was a result of my having assigned problems for students to work in small groups outside of class. This study afforded me the opportunity to observe students solve such a problem. Although my interaction in the two groups for this study was more than that in my MATH 1113 classes, my interaction allowed me to have a better sense of how my students think and learn in a group. It is this knowledge which will help me as I plan group work for my classes.

The study generated volumes of data and making sense of this data was overwhelming. Resnick (1988) warned the investigation of shared problem solving is difficult because the problem sessions are longer, and it is more difficult to collect and transcribe the data. He also pointed out that because the density of episodes worthy of detailed examination and qualitative analysis is low, shared problem solving looks inefficient according to the standards of traditional pedagogy. Resnick acknowledged that detailed examination of such episodes can nevertheless be enlightening. I must admit that even though I tried to examine the episodes closely, there may be influencing factors I missed and outcomes I did not see. Because I wanted to present a

case study of these students as they worked on a mathematics problem, it was important to present thorough continuity to the sessions. These records of the episodes helped to identify and document factors important to the processes students were encountering during group problem solving.

My purpose was to investigate the factors influencing the problem solving, and I chose a problem similar to that which would be assigned as a project in my MATH 1113 course. Out of this investigation came not only the importance of interactions but the difficulty some groups have in fostering interactions that can lead to productive problem solving. The investigation also highlighted several misconceptions students have about mathematical ideas.

Davidson and Kroll (1991) claimed that an area of needed research is that of the interactions that take place during cooperative group work to identify how academic, social, or psychological effects are produced. Webb (1985) pointed out that in order to obtain the detailed information about peer interaction that is needed, verbatim audio or video recordings may be required. My study not only adds to the body of needed research claimed by Davidson and Kroll but does so in the manner suggested by Webb. The interactions of the students as they solved the Buried Treasure Problem were both audio- and video-recorded. These recordings were then transcribed. The video recordings were important because they allowed me the opportunity to examine and identify the nonverbal communication and interactions.

Researchers claim that studies are needed that will help us understand how students learn and think as they work together on tasks (Bossert, 1988-1989; Davidson & Kroll, 1991; Good et al., 1992a). Mere on-site observations cannot capture the level of understanding which comes from protracted and frequent observations afforded by video recordings. Thus, the video recordings allowed me to revisit the sessions to have a better understanding of how the students learned and thought as they worked together.

The individual interviews and personal reflections gave another way to interpret the interactions to understand how the students learned and thought. Because my study used

individual interviews and reflections subsequent to the problem sessions, I had the opportunity to determine how an individual's learning was shaped through the interactions with the group. Furthermore, the final interviews were conducted as the student and I watched clips of the problem sessions to stimulate their recall. Using stimulated recall is supported by Bossert (1988-1989) who recommended the technique of stimulated recall together with observations of participants to allow researchers to document how peer interactions in cooperative groups shape the thinking and processing skills of group members.

My study looked at the factors that influenced the problem solving of a small group of students as they worked together on a mathematics problem. The study includes descriptive episodes replete with quotes, showing how these students thought and learned together. I believe my study will contribute to the body of literature that focuses on how students think and learn as they work together in mathematics.

Framework Revisited

I used the framework developed by Watson and Chick (2001) as a starting point to identify the factors that influenced the problem solving of the group of students. Like Watson and Chick I found influencing factors that could be classified as cognitive, social, and external. Watson and Chick's framework was helpful because their study concerned itself not with the overall success of collaborative group work in mathematics but instead with identifying the factors associated with outcomes considered productive or not productive while the students worked together. My definition of problem solving mimicked Schoenfeld's (1994) as the means by which one learns to think mathematically where

learning to think mathematically means (a) developing a mathematical point of view— valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure—mathematical sense-making.

(p.60)

This definition allowed me to look at the processes that occurred as the students worked on the problem and to be less focused on whether or not the final answer to the problem was obtained. Furthermore, this definition of problem solving allowed me as the researcher to interact with the students as I tried to make sense of their mathematical thinking.

Although this definition allowed me to show less concern for the final answer and more concern for the processes used to solve the problem, one may question my wisdom in allowing Group 1 to use four sessions to obtain a solution. There were several times they were very close to a solution, but the factors identified in this study influenced the direction they would take. Thus, it was the protracted look at their four sessions that helped me observe factors which may have been missed otherwise.

Even though Watson and Chick's study gave me a starting point to look at the influencing factors, I had to make adjustments in their framework. The students in their study were third, sixth, and ninth graders whereas the students in my study were older and not well-acquainted. This lack of familiarity with each other made my participants more wary of interacting. Furthermore, some of my participants had personal agendas which inhibited their willingness to interact. I found the influencing factors of goal focus and ego-preserving (a component of egocentrism) to be significant.

Watson and Chick identified cognitive disagreement, doubt, misunderstanding, and tenacity of ideas as four influencing cognitive factors. I chose to integrate these into the influencing factor I called *sense-making perspective*. Related to a sense-making perspective but treated separately in my study was the influencing factor of communication. Communication was not a factor identified by Watson and Chick but I found it helpful to use this factor to recognize the instances of students talking past each other, making illogical statements, engaging in incoherent speech, and not listening.

Only three of Watson and Chick's social factors were identified; these were leadership factors, egocentrism, and social collaboration. As stated above, I saw ego-preserving, a

component of egocentrism, as an important factor to consider. Again, the age of the students and their lack of familiarity with one another may have made this factor significant. Ego-preserving was strong in Group 1 and inhibited the interaction. In Group 2, ego-preserving was less influential because this group behaved as a community of learners where the good of the group prevailed over the good of the individual.

Watson and Chick's framework did not include group processes that influence problem solving so several other studies were helpful in identifying the group processes (Cohen, 1994; Goos, 2000; Goos et al., 2002; Lumpe, 1995).

My study extends the work of Watson and Chick by examining the work of older students. Their framework, as stated in the title of their study, assumed collaboration. For the students in my study, interaction (and therefore collaboration) did not always occur. I also extended their work by using final interviews to gain the participants' perspectives on the influencing factors whereas Watson and Chick did not consider the participants' perspectives.

Conclusions

Davidson (1990) pointed out that small-group learning involves more than just putting students together in small groups and giving them some task to do. It also involves giving deliberate thought and attention to various aspects of the group process. This study afforded me the opportunity to scrutinize the group process to identify the factors that influenced the group problem solving so that I can help my students maximize their opportunities through the group activities in my classroom. Watson and Chick's (2001) research helped me identify social, cognitive, and external factors that influenced the problem solving in the two groups I studied. Although the factors were grouped into these three categories, they did not exist in isolation but instead frequently interacted. I also extended the work of Watson and Chick by identifying group processes that were influencing factors.

Social or Interpersonal Factors

I identified leadership factors, social collaboration, and egocentrism as the influencing factors in this category. Group 1 had an individual who had the qualities to be a charismatic intellectual, but she was unwilling to be the leader. Exacerbating the problem was the other group members' perceptions of her as the leader. They expected more direction from her than she gave. The influence of leadership was seen in Group 2 through the talents of two individuals. Although one was the more charismatic and the other was stronger mathematically, both possessed charisma and mathematical ability which helped move the group along in their investigation of the problem.

Social collaboration was seen in Group 2 to contribute to a more relaxed atmosphere. The social collaboration in Group 1 was more superficial where the students would agree just to indicate some level of involvement when they really had little idea of what the other person said. Group 1 lacked a sense-making perspective so social collaboration often led them to express approval for incorrect mathematical ideas.

Egocentrism was especially influential as students felt the need to preserve their egos. There was a real need to preserve the ego and the participants used face-saving strategies, camouflaging techniques, excessive talking, and no talking as ways to preserve their egos. Although the final interviews revealed the presence of egocentrism in Group 2, it was a testament to the group that during the session its influence was controlled. Control of their egocentrism perhaps indicated their commitment to the experience of working this problem collaboratively.

Cognitive Factors

Cognitive ability, previous experience, a sense-making perspective, communication, the big picture, and goal focus were the cognitive factors I identified. The social factors interacted with these factors.

The students who participated in my study had little experience working with other students to solve a mathematics problem. As a result, they entered the study with different personal goals for the experience, and these different goals influenced the way they worked together. An advantage of group work often cited is the opportunity for students to pool their resources so that the likelihood of solving the problem is increased. This advantage was seen in both Group 1 and Group 2. The lack of effective communication in Group 1 pre-empted the opportunity for the entire group to be helped. Rather, a single group member may have benefited from a comment or suggestion made by another but she failed to share her epiphany with the others. In Group 2 where more effective communication together with more collaboration existed, the opportunity for the entire group to benefit was more likely.

Significant, too, was the observation that an epiphany often came in the way of an innocuous comment or question. The student making the comment or question was not necessarily aware of her contribution. The contribution became useful only when the receiver listened carefully and made sense of it in the context of her own thinking.

The students had misconceptions about fundamental ideas in mathematics. Good, McCaslin and Reys (1992a) warned that when students work in small groups, the potential for misconceptions to be reinforced exists. I observed the misconceptions and tried to help the students deal with these misconceptions during the problem sessions. Although some students were lifted as a result of my intervention, for other students lifting came only during the final interviews. For other students, lifting never occurred. Without my intervention, these misconceptions may not have been dealt with at all and could have been reinforced as they sought an inappropriate way to deal with the misconception. On the other hand, my intervention may have constrained the interaction that possibly could have led to conceptual understanding. This observation is not unlike that addressed by Cohen (1994).

Students who have a sense-making perspective use effective communication not only to express their ideas but also to understand the thinking of the other participants. In Group 1

where the sense-making perspective was missing, there were evidences of misunderstandings, talking past each other, illogical comments, and lack of attending to the comments of another. Consequently, the group never discussed a qualitative approach for solving the problem and wound up implementing procedures without discussing how the results would be beneficial. Group 1 did a poor job of monitoring their work and did not keep sight of the big picture. Work performed in previous sessions was not used in a timely manner. Although Group 2 had only one session, they developed a qualitative approach and monitored the work during the session.

It was not surprising that cognitive ability was influential. Cognitive ability was an influencing factor in the leadership of both groups. A consideration of cognitive ability also helped me identify two students as over-achievers which led me to believe they needed time to themselves to make sense of the mathematics. The one student whose SAT-M and final exam score were both low was very reluctant to share her ideas with the group. She was the student who needed verification of her ideas before she would share them.

External Factors

There was outsider influence from me, the researcher. Task factors and logistical factors also influenced the problem solving.

Because Group 1 lacked the sense-making perspective, my questions and comments were frequently ignored. My questions and suggestions may have constrained their interaction so that sense-making could be developed. My intervention in Group 2, however, was more positive. They listened to my suggestions and used them to help them understand the problem.

The task of working together influenced the problem solving. Some students benefited from the opportunity to explain whereas others were uncomfortable and reluctant to share their thinking. The task did not include a write-up component which meant students missed the opportunity to make the connections that led them to a solution. Hence, some students left the experience with a distorted view of the mathematical thinking they engaged in to get a solution.

One of the logistical factors that influenced the problem solving was having no common workspace which minimized the opportunity for working together and for understanding fully the comments of another. Although Group 2 would often create its own common workspace, this rarely happened in Group 1. Also the absences of the students in Group 1 influenced the problem solving. The absences meant the continuity was broken. Furthermore, the students interacted differently in the presence or absence of certain individuals.

Group Factors

One group functioned more as a community of learners. In this group the good of the group prevailed over the good of the individual and, as a result, the group participated in productive interactions. The interaction in this group was characterized by sense-making, effective communication, and elaborated discussions. In this group, status factors were less of an issue as the participants listened to one another and treated each other with respect.

Implications

Observing the students work on the Buried Treasure Problem allowed me to draw several implications about the use of small-group learning in mathematics. An implication for the classroom teacher is that students can have significant misconceptions about mathematical concepts and yet can and do function quite well in the classroom. Furthermore, these misconceptions might go unresolved during the group work without teacher intervention.

This study did not have students create a write-up of their solution. However, having students write up a report explaining their thinking in a problem may not be sufficient. A report explaining the solution to a problem may not be sufficient evidence of understanding as students cannot always judge their level of understanding, partly because they do not always know what it means "to understand mathematics." Omissions from the write-up may signify misconceptions of which the students are not even aware. Details may be excluded because the student does not possess the knowledge and understanding to supply these details. Although the student may possess some understanding of the problem, misunderstandings

about major concepts in mathematics may be glossed over. The teacher plays an integral role in teasing out these misconceptions and forcing the students to confront them. A caveat is that some students will be resistant to confront their misconceptions because by the time they reach precalculus, they assume they “know mathematics.” The teacher must tread carefully as the need to preserve the ego is very strong and elicits techniques to camouflage the misconceptions and lack of understanding.

Two factors identified in the study imply recommendations a teacher planning a group activity may want to address. One is that students should be required to have a common workspace. A recorder could be appointed and work can be written on a single piece of paper, newsprint, or a white board. Regardless of the medium, the work needs to be accessible to all the participants. A second recommendation is a description of what is meant by “solving a problem together.” The teacher could acknowledge the different personal goals students could have but then share the goal for the task. Although sense-making and effective communication take time to develop, the teacher must be ever aware of opportunities to instill these skills in the students. My study confirmed the conclusion by Sfard and Kieran (2001) that we must teach the art of communicating in order for conversation to be effective and conducive for learning.

Limitations of the Study

The study was designed to identify the factors influencing the problem solving of a small group of students as they worked together to solve a mathematics problem and to describe how these factors influenced the problem solving. A limitation of the study was the interaction I had with the participants. My influence interacted with the other factors so that a different study without researcher intervention may identify different factors. Another limitation was my bias of the student’s capabilities since these students were in my MATH 1113 class the previous semester.

Another limitation was the incongruity in data collection for the two groups. Group 1 participated in four problem sessions but only investigated the Buried Treasure Problem. Group

2 participated in three problem sessions but successfully investigated the Buried Treasure Problem in the first session. Group 2 participated in two additional sessions where they investigated a second problem, but this data was not used for the study. Furthermore, not every student in Group 1 was present for the four sessions.

Recommendations for Future Research

In this study I, as the researcher, interacted to some extent with the participants. My interaction consisted of my efforts to understand their thinking and to help them deal with misconceptions that were problematic. I found my intervention to be influential and further research could be conducted to determine the role of the teacher as students work in small groups.

In particular, knowing when and how to deal with obvious misconceptions about mathematical ideas is an area for further research. Should the students be allowed to struggle until they resolve the issue? How might the teacher intervene without interrupting the interaction among the participants?

Missing in my study was the write-up as a component. Research is needed to determine the learning processes that occur as a function of the write-up component. Do group processes differ when a write-up is a part of the data collection? Does the write-up promote more of a sense-making perspective with students seeking understanding and dealing with their misconceptions? Is there a difference in the ownership of a solution if each student submits a write-up versus a write-up submitted by the group?

Communication was an influencing factor and additional research is needed to determine the role of communication in small-group learning. What can we learn about student understanding of mathematics as we listen to their communication with one another? How does communication with their peers influence their understanding of mathematics? What strategies can be used to engage students in effective communication?

Additional research is also needed to determine how to encourage students to engage in productive interactions and how to promote a sense of accountability in students as they work in a group.

Research similar to my study could be conducted with the following variations: (a) a wider variety of tasks could be used, (b) the time between the problem sessions could be reduced, and (c) group interviews could be conducted instead of or in conjunction with the individual interviews.

Concluding Remarks

Conducting this study has been a revelation for me. My eyes have been opened to the misconceptions students have. In this study, students whom I considered strong mathematics students had misconceptions about fundamental mathematics concepts. As a teacher and educator of future teachers, I must be diligent in my efforts to engage students in mathematical sense-making so their opportunities to confront these misconceptions can be realized.

The writing assignments I ask students to do in the future where they must explain their thinking must be more carefully examined. Omissions in these explanations cannot be attributed to negligence and not taking the time to be thorough. Errors cannot be attributed to carelessness. Their omissions and their errors may signify a lack of understanding for which they are not even aware.

Finally, in the past I have assigned problems for students to work on together outside of class with little or no interaction from me. In the future it may be important to require group meetings with me at some point during the assignment so that misconceptions and problems can be addressed. My role as teacher is not only to assign tasks appropriate for small-group learning but also to be sure students confront their misconceptions, modify their thinking, and make the connections the group task has the potential for triggering.

REFERENCES

- American Mathematical Association of Two-Year Colleges. (1995). *Crossroads in mathematics: Standards for introductory college mathematics before calculus*. Memphis, TN: American Mathematical Association of Two-Year Colleges.
- Bean, J. C. (1996). *Engaging Ideas: The Professor's Guide to Integrating Writing, Critical Thinking, and Active Learning in the Classroom*. San Francisco, CA: Jossey-Bass.
- Bossert, S. T. (1988-1989). Cooperative activities in the classroom. In E. Z. Rothkopf (Ed.), *Review of research in education* (Vol. 15, pp. 225-250). Washington, DC: American Educational Research Association.
- Cobb, P. (1995). Mathematical learning and small-group interaction: Four case studies. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 25-129). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Cohen, E. G. (1994). Restructuring the classroom: Conditions for productive small groups. *Review of Educational Research*, 64(1), 1-35.
- Davidson, N. (1985). Small-group learning and teaching in mathematics: A selective review of the research. In R. Slavin, S. Sharan, S. Kagan, R. Hertz-Lazarowitz, C. Webb & R. Schmuck (Eds.), *Learning to cooperate, cooperating to learn* (pp. 211-230). New York: Plenum Press.
- Davidson, N. (1990). Small-group cooperative learning in mathematics. In T. J. Cooney (Ed.), *Teaching and learning mathematics in the 1990s: 1990 Yearbook* (pp. 52-61). Reston, VA: National Council of Teachers of Mathematics.
- Davidson, N., & Kroll, D. L. (1991). An overview of research on cooperative learning related to mathematics. *Journal for Research in Mathematics Education*, 22(5), 362-365.
- Dees, R. L. (1991). The role of cooperative learning in increasing problem-solving ability in a college remedial course. *Journal for Research in Mathematics Education*, 22(5), 409-421.
- Fogarty, R. (1990). *Designs for cooperative interaction*. Palantine, IL: IRI/Skylight Publishing.
- Froelich, G. W. (1991). *Connecting mathematics*. Reston, VA: The National Council of Teachers of Mathematics, Inc.
- Good, T. L., McCaslin, M. M., & Reys, B. J. (1992a). Investigating work groups to promote problem solving in mathematics. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 3, pp. 115-160). Greenwich, CT: Jai Press.
- Good, T. L., Mulryan, C., & McCaslin, M. (1992b). Grouping for instruction in mathematics: A call for programmatic research on small-group processes. In D. A. Grouws (Ed.),

- Handbook of research on mathematics teaching and learning* (pp. 165-196). New York: Simon & Schuster Macmillan.
- Goos, M. (1998). "I don't know if I'm doing it right or I'm doing it wrong!": Unresolved uncertainty in the collaborative learning of mathematics. Paper presented at the Teaching Mathematics in New Times, Proceedings of the Twenty first Annual conference of the Mathematics Education Research Group of Australasia Incorporated, Gold Coast, Australia.
- Goos, M. (2000). *Collaborative problem solving in senior secondary mathematics classrooms*. Paper presented at the Mathematics Education Beyond 2000: Proceedings of the Twenty-third Annual Conference of the Mathematics Education Research Group of Australasia Incorporated, Fremantel, Western Australia.
- Goos, M., & Galbraith, P. (1996). Do it this way! Metacognitive strategies in collaborative mathematical problem solving. *Educational Studies in Mathematics*, 30, 229-260.
- Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, 49, 193-223.
- Goos, M., Renshaw, P., & Galbraith, P. (1998). 'We're supposed to be discussing stuff!' Processes of resistance and inclusion in secondary mathematics classrooms. *Australian Senior Mathematics Journal*, 12(2), 20-31.
- Haller, C. R., Gallagher, V. J., Weldon, T. L., & Felder, R. M. (2000). Dynamics of peer education in cooperative learning workgroups. *Journal of Engineering Education*, 89(3), 285-293.
- Johnson, D. W., & Johnson, R. T. (1985). The internal dynamics of cooperative learning groups. In R. Slavin, S. Sharan, S. Kagan, R. Hertz-Lazarowitz, C. Webb & R. Schmuck (Eds.), *Learning to cooperate, cooperating to learn* (pp. 103-124). New York: Plenum Press.
- Johnson, D. W., Johnson, R. T., & Smith, K. A. (1991). *Cooperative learning: Increasing college faculty instructional productivity*. Washington, DC: The George Washington University.
- Johnson, R. (n.d.). *Cooperative learning Q & A*. Retrieved July 4, 2004, from <http://www.co-operation.org/pages/ganda.html>
- Johnson, R. T., & Johnson, D. W. (n.d.). *An overview of cooperative learning*. Retrieved July 10, 2002, from <http://www.clcrc.com/pages/overviewpaper.html>
- Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Kroll, D. L. (1988). *Cooperative mathematical problem solving and metacognition: A case study of three pairs of women*. Unpublished Doctoral Dissertation, Indiana University.
- Lambdin, D. V. (1993). Monitoring moves and roles in cooperative mathematical problem solving. *Focus on Learning Problems in Mathematics*, 15(2&3), 48-64.

- Lester, F. K. (1985). Methodological considerations in research on mathematical problem-solving instruction. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41-69). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Lou, Y., Abrami, P. C., Spence, J. C., Poulsen, C., Chambers, B., & d'Apollonia, S. (1996). Within-class grouping: A meta-analysis. *Review of Educational Research*, 66(4), 423-458.
- Lumpe, A. T. (1995). Peer interaction in science concept development and problem solving. *School Science and Mathematics*, 95(6), 302-309.
- Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass Publishers.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- National Research Council. (1989). *Everybody counts: A report to the nation on the nature of mathematics education*. Washington, DC: National Academy Press.
- Noddings, N. (1989). Theoretical and practical concerns about small groups in mathematics. *The Elementary School Journal*, 89(5), 606-623.
- Olkin, I., & Schoenfeld, A. H. (1994). A discussion of Bruce Reznick's chapter. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 39-51). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Panitz, T. (1996). *A definition of collaborative vs cooperative learning*. Retrieved April 4, 2004, from <http://www.city.londonmet.ac.uk/deliberations/collab.learning/panitz2.html>
- Patton, M. Q. (2002). *Qualitative Research and Evaluation Methods*. Thousand Oaks, CA: Sage Publications, Inc.
- Philipp, R. A., Martin, W. O., & Richgels, G. W. (1993). Curricular implications of graphical representations of functions. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 239-278). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Resnick, L. B. (1988). Treating mathematics as an ill-structured discipline. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 32-60). Reston, VA: The National Council of Teachers of Mathematics.
- Schoenfeld, A. H. (1983). *Theoretical and pragmatic issues in the design of mathematical "problem solving" instruction*. Paper presented at the Annual Meeting of the American Educational Research Association, Montreal, Canada.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

- Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-215). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Simon & Schuster Macmillan.
- Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53-70). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, 8(1), 42-76.
- Smith, K. A., & Waller, A. A. (1997). Cooperative learning for new college teachers. In W. E. Campbell & K. A. Smith (Eds.), *New Paradigms for College Teaching* (pp. 185-209). Edina, MN: Interaction Book Co.
- Stanic, G. M. A., & Kilpatrick, J. (Eds.). (1988). *Historical perspectives on problem solving in the mathematics curriculum* (Vol. 3). Reston, VA: National Council of Teachers of Mathematics.
- Stewart, J., Redlin, L., & Watson, S. (1998). *Precalculus: Mathematics for Calculus* (3rd ed.). Pacific Grove, CA: Brooks/Cole.
- Watson, J. M., & Chick, H. L. (2001). Factors influencing the outcomes of collaborative mathematical problem solving: An introduction. *Mathematical Thinking and Learning*, 3(2&3), 125-173.
- Webb, N. M. (1985). Student interaction and learning in small groups: A research summary. In R. Slavin, S. Sharan, S. Kagan, R. Hertz-Lazarowitz, C. Webb & R. Schmuck (Eds.), *Learning to cooperate, cooperating to learn* (pp. 147-172). New York: Plenum Press.
- Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education*, 22(5), 390-408.

APPENDIX A
REFLECTIONS QUESTIONNAIRE

Name _____

Date _____

Reflections Questionnaire following problem session # _____

Please think back over the problem session and respond to the questions below as thoroughly and honestly as possible. Your responses will be kept confidential. Your reflection should include responses to the following questions but should not be limited by them.

- State all that you understand about the problem.
- What is still unclear or confusing to you?
- What aided your understanding? What hindered your understanding?
- Think about the questions that occurred to you during the session.
What are some questions that you had during the session?
Did you ask the questions? Why or why not?
Did asking the questions help you? Why or why not?
Did you ask questions that did not get answered? Why were they not answered?
- What suggestions did you make to the group?
- What suggestions did you think about but did not make? Why did you not suggest them?
- Was it difficult to stay focused on the problem? Why or why not?
What did you do to keep yourself focused on the problem?
- What do you wish you had done differently?
- What do you wish the group had done differently?

APPENDIX B
EMAIL AUTOBIOGRAPHY ASSIGNMENT

MATH 1113
EMAIL AUTOBIOGRAPHY ASSIGNMENT

This assignment is designed to help me get to know you and to provide me with your email address. Please send it to me at lcrawfor@aug.edu by Tuesday, January 15 at 1:00 PM.

Please put your name, the course number and section, and your phone numbers (for home and work) on the first three lines.

Then comment on the following items:

- A bit about yourself—your background, your interests or hobbies, your hopes and aspirations.
- Your high school and what mathematics courses you took. Did any of them involve some sort of calculator or computer technology? And if so, to what extent was it used?
- A few words about how you react to mathematics, what (if anything) it means to you. Has your previous study of mathematics in school seemed meaningful or useful? Any comments along these lines will interest me.
- Tell me a bit about yourself as a learner of mathematics—what do you think are your strong points? your weak points? What is the ideal situation for you to learn mathematics?
- Your classification (fr, soph, jr, sr, joint enroll, post bac), your expected major, and anything else you'd like me to know about you personally.

APPENDIX C
NOTEBOOK GUIDELINES

NOTEBOOK GUIDELINES

Your math notebook

1. Use a loose-leaf notebook. You will find this more flexible than a spiral notebook when you need to rearrange papers.
2. Use dividers to separate your notebook into the following sections:
 - a) daily assignments
 - b) notes
 - c) handouts
 - d) homework
 - e) tests, quizzes, and corrections
3. Write the date on each class assignment, each handout, and each homework assignment. Homework from the book should also have the page and problem numbers.
4. Homework, daily assignments, tests, and quizzes should be done in **pencil**.
5. Each day's homework should be done on a new sheet of paper.
6. Make daily corrections of homework.

Taking notes in class

1. Date your notes.
2. Begin each day's notes on a new side of paper.
3. Use color to emphasize important ideas and points.
4. Use abbreviations.
5. Review and revise your notes as soon as possible after writing them.

BRING YOUR NOTEBOOK TO CLASS EACH DAY.

APPENDIX D
NOTEBOOK CHECK

MATH 1113
Notebook Check

Name _____

Date _____	Very Good	Good	Needs Improvement
Thoroughness— Complete notes and homework			
Neatness			
Organization			

Date _____	Very Good	Good	Needs Improvement
Thoroughness— Complete notes and homework			
Neatness			
Organization			

Date _____	Very Good	Good	Needs Improvement
Thoroughness— Complete notes and homework			
Neatness			
Organization			

APPENDIX E
INITIAL INTERVIEW GUIDE

Initial Interview Guide

Name_____

Date_____

The purpose of this interview is to get some background information about you and about the experiences you have had solving math problems.

Please answer the questions as completely and honestly as you can. Some questions may seem irrelevant and repetitive; however, I would appreciate your answering them anyway. You certainly have the right to refuse to answer any questions you choose.

Do you have any questions before we get started?

Are you ready to begin?

1. Tell me a little bit about yourself—

about your family

where you grew up

brothers and sisters

close family

2. Tell me about your precollege schooling—

where you went to school

did you enjoy school—why or why not?

what activities you participated in during school

3. Tell me about your life now—

Do you have a job?

Do you live with your parents or live on your own?

Are you married?

What hobbies do you enjoy?

4. Tell me about your college career—

Why did you decide to go to college?

Why ASU?

What is your major?

What is your favorite subject?

What is your least favorite subject?

Where does mathematics rank in your least to most favorite subjects? Why?

Why do you feel this way about mathematics? Have you always felt this way about mathematics?

5. Tell me about some classes you have had that you consider particularly good. Why do you think of these as good classes?

Tell me about some classes you have had that you consider bad? Why do you think of these as bad classes?

Describe for me the kind of classroom setup you prefer:

lecture	class discussion	small group work
individual work	lab	

Why do you prefer this setup?

6. What kinds of math problems are you best at? Why?

What kinds of math problems are you worst at? Why? What can you do to get better at these?

7. Suppose you have some math problems to solve for class. Describe for me the circumstances under which you would solve these problems.

When would you begin them?

Where would you work on them?

Would you work with the radio on, the TV on, would you require quiet?

Would you take frequent breaks as you solve them?

8. I want us now to talk about how you would go about solving a math problem.

Would you refer to the textbook? other books? your notes?

This past semester you were asked to do your work on loose leaf notebook paper and to maintain a notebook. Is this the way you have always done math work? What have you done in the past? Which do you prefer? Why?

Do you first work your math problems on scratch paper and then recopy your work?

Do you ever talk to yourself as you work a problem? Aloud or just silently? What do you say? Does it help?

Do you ever draw pictures?

Do you ever work a problem in more than one way?

How much do you use a calculator? a computer?

9. If you miss a problem, what is generally the reason you missed it? Why do you think this happens? What can you do about it?

10. What do you do when you see an unfamiliar problem? Why?

11. What are some things you do when you get stuck on a problem?

12. Do you ask other people for help when solving a problem? Who? When? Why? How does this help you?

13. Do you usually work alone or with someone else?

if together—how do you choose the other person?

ability

personality

friend

Why together? How does that help?

if alone—why not with someone else?

14. Frequently in class you were asked to pair up with other students to work on a problem.

Describe to me what would happen when you were asked to do this.

Other than the pairing up that we did in class, have you ever worked in a small group to solve a problem?

Why not? or What happened?

Have you ever gotten together outside of class with members of your math class to work on homework or to study? Why not? **OR** What happened?

APPENDIX F
FINAL INTERVIEW GUIDE

Final Interview Guide

1. How well-matched do you think the group was for working together?
2. What made solving the problem with the group difficult? easy?
3. What could the group have done to make the problem solving more effective?
4. What could you have done to make the problem solving more effective?
5. What effect do you think the sessions have had on your attitude toward mathematics or mathematical problem solving?
6. What were the advantages of working with the group on these problems?
Disadvantages?
7. What effect do you think the sessions have had on your ability to solve problems?
8. As a result of participating in these sessions, what are some things you might do differently now in solving problems?
9. Were strategies ever suggested that you would not have thought of? Examples?

Did you suggest strategies that others did not think of? Examples?

Do you think you might use these suggested strategies in the future? Do you think others will use these strategies?
10. How did you feel when members disagreed or questioned a suggested approach?
11. What suggestions would you make to another group about solving a problem together?

APPENDIX G

EXAMPLE OF GUIDE USED FOR STIMULATED RECALL

Session 2 Up 1

Page 1

Abby

Eve

Jane

Sara

8:21:50			Why dep on last semester?	
8:24:30 - 8:26:30		asks Did we find a blast time? - more involved		Does anybody have any questions? Why ask this?
8:27			reaches for book - looking for problem in book	
8:28:00		asks more clarification		
8:29 - 8:32	ask what is X? Do you feel involved?		Jane not listening	Sara asks question & no one responds
		ask about X + why need -	Says what trying to find	question Sara about throwing X in there
8:32:25			keeps referring to book -	Any other suggestion -
8:35		Asks Sara to clarify		Still confused on X

Why is Eve more involved?
Also Abby is more involved -

GPI session 2

SK - Get them to see

Abby Eve Jane Sara

8:56-8:57:32				I am trying to get them to see ratios are equal.
9:05:49			Still asking about degree mode	
9:07:40	What's Abby thinking?			asks about using ratios -
9:10-9:18	What is meant by small vesicle?			ASK Sara about work at bottom of P.4
9:13		do you know this?		How did you feel when no one paying at bank
9:14				looks like ^{the} _{a possible}
9:15-9:16	Sara will ask a dumb question	Why is Eve asking above?		ratio I mean not prop.
9:16	back down			

did Jane have a hard time staying focused?