DEPARTMENT OF MATHEMATICS EDUCATION

Dissertation Research Prospectus

Doctor of Philosophy

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04/07/02

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TEACHER KNOWLEDGE OF STUDENT THINKING AND INSTRUCTIONAL PRACTICES IN ALGEBRA

RATIONALE

Background

Beginning is a promise for an ending. More important than that how you begin is not only a promise for an end but also how it does end. I do not claim that my beginning is a "good" one. I can only say that it is a significant one for me. The starting point for any reason might be simplified as either personal desire, the desire of the needy others, or a combination of both.

I have spent 20 years of my life as a formal student in an educational institute. Although I have been through a lot of problems, and difficulties, mathematics has always been my favorite. I am not so sure there was any effect of my mathematics teachers in this. Actually, they were the least favorite on my list. I cannot explain exactly why this was so, but I can say this: they were all alike in the sense that I never felt like that they were actually helping me with my difficulties and understanding. Grades were never a problem. I was an A student. I guess that was enough for my parents and teachers. What was true for me was also true for my classmates. I am not quite sure how many hands were raised when the teacher asked if there was anything not understood although almost all of the class did not understand what was going on. It seems to me that the main problem was the lack of communication. Whenever I (we) asked about something that he/she was trying to teach, the answers were simply a repetition of what was written on blackboard like a broken tape player. I do not want to be unfair but I was not learning my mathematics but the teacher's mathematics.

My love for mathematics eventually led me to a mathematics education department. I expected to become a teacher, who would not make the same mistakes that my teachers did. Well, I have not had much of a chance to test this except on several part time teaching occasions. Although I consciously tried to avoid presenting mathematics in a traditional way, I am sure I did so and unconsciously imitated my teachers.

As a young researcher, I would like to do something to reveal the mystery behind what is going on in a mathematics classroom in terms of teacher knowledge, goals, and beliefs in their actions (or instructional practices) and the role of student thinking in that context. So, as a core topic in mathematics, algebra has become my target. I investigated student difficulties in elementary algebra in my MSc research several years ago. One of the interesting but quite expected results was teachers' unawareness of student difficulties. In fact, it seemed that students and teachers were talking in different languages. Since this result was drawn from some questionnaires, I need to conduct some further research to understand why students have difficulties in algebra and whether teachers are aware of them. More importantly, this is going to be a personal understanding of my life spent in mathematics classrooms.

Not everybody is as excited as Girolamo Cardano was about algebra. For many people, algebra is a challenge and a gatekeeper. Successful completion of an algebra course is considered not only a prerequisite for further study in mathematics and other subjects but also a door opening to many jobs and later opportunities. Ever since the invention of symbolic algebra by Vieta, algebra has provided the power to be able to operate with concepts at abstract levels and then apply them. This power created a special stature for algebra, not only in mathematics itself, but also in other disciplines requiring mathematical abstraction or modeling. Considering that the role of algebra in mathematics is more than being a branch of mathematics in its own right, the difficulties encountered in teaching and learning of algebra leads to impediments in the learning of higher level mathematics. So far personal experiences and research have shown that the road to algebra is never as smooth as one may wish. Algebraic tasks are somehow difficult to learn and teach (Booth, 1984, 1988; English & Halford, 1995; Herscovics, 1989, Kieran, 1989, 1992). There are several variables that might make algebra difficult to comprehend: the content of algebra, the way algebra is taught, or students' inappropriate approaches (Kieran, 1992). Within either of these, the teacher might have a central role to prevent, lessen, or deal with the difficulties student encounter. This study, therefore, is designed to address our emerging knowledge and understanding of teacher perception of algebra (within mathematics and schooling), beliefs and knowledge of student difficulties in algebra, and the influence of those variables on their instructional practices

Statement of the Problem

I propose to conduct an interpretative study, in which I will observe and interview four in-service Algebra-I teachers to investigate and understand the relationship between student thinking and professional knowledge, beliefs, and goals of teachers and how this affects their instructional decisions/practices. To achieve this end, I plan to guide the study with the following research questions:

- How is student thinking manifested/perceived in teacher beliefs?
- [What is the nature of teachers' beliefs about student thinking in algebra?]

- How is student thinking manifested/perceived in professional knowledge of teachers?
- [What is the nature of teachers' professional knowledge about student thinking in algebra?]
- How is student thinking manifested/perceived in instructional goals of teachers?
- [To what extend are teachers' instructional goals informed by student thinking?]
- How do instructional decisions reflect teachers' beliefs, knowledge, and instructional goals?
- How do specific contexts affect teachers' beliefs, professional knowledge, and instructional goals (and so the instructional decisions)?
- [What are the general and specific factors affecting teachers' beliefs, professional knowledge, and instructional goals (and so the instructional decisions)?]

Relevant Literature

On Learning of Algebra and Student Difficulties

Traditionally, the algebra curriculum has been organized around the concept of equation and methods for solving equations. Instruction focused on the mastery symbolic manipulation skills. Within this tradition, students often develop what Skemp (1987) calls an "instrumental understanding" of algebra. He explains, "It is what I have in the past described as 'rules without reasons,' without realizing that for many pupils *and their teachers* the possession of such a rule, and the ability to use it, was what they meant by 'understanding' "(p. 153).

Many educators and students have observed that students often leave an algebra course with a feeling they have been taught some abstract system that has no meaning. Herscovics (1989) states that students have been taught the syntax of a language without the semantics. In other words, they know the rules in the grammar but they do not understand the words. Kieran (1992), on the other hand, approaches the problem epistemologically by proposing that the problem with modern algebra is that we impose symbolic algebra on students without taking them through the stages of rhetorical and syncopated algebra[1]. In the same token, Rojano, (1996) states that "This lesson from history has implications for the teaching in the sense that the potential of dominating algebraic syntax will not be appreciated by students until they have experienced the limits of the scope of their previous knowledge and skills and start using the basic elements of algebraic syntax" (p. 62). Based on the theory that the development of algebraic understanding in the individual follows the same steps observed in the historical development of algebra, Anna Sfard (1995) outlines a framework for how people learn algebra. In other words, the basic premise is that the historical development of algebra from rhetorical to symbolic must be reproduced in the individual to achieve understanding of algebra.

Lee (1996) suggests that mathematics is a culture and algebra is a "mini-culture" in it. In this context, students' entry into algebraic culture is analogical with "cultural shock." This analogy is particularly true considering that (in)-consistencies between arithmetic and algebra is the main considerations for the cognitive obstacles (English & Halford, 1995; Herscovics, 1989; Herscovics & Linchevski, 1994; Kieran, 1989, 1992). The notion of equality and equation sign, different use of letters in arithmetic and algebra as well as the procedural nature of arithmetic versus the relational nature of algebra are some of the differences between the cultures of arithmetic and algebra. Among others, student perceptions about the

[1] Kieran (1992) outlines the historical development of algebra in three stages. The first stage is *rhetorical algebra* in which no symbols were used at all. Equations and problems were posed and solved in prose form. The second stage is *syncopated algebra* in which some abbreviations for the frequently recurring quantities and operators were used. The third stage is *symbolic algebra,* which is the algebra that we use today.

use of letters in algebra are one of the key factors in understanding the elements of the algebraic culture. Different uses of letters are also among the sources of difficulty in learning algebra (Kuchemann, 1981; Wagner & Parker, 1993). In general, the view that algebra is "generalized arithmetic" or "completion of arithmetic" (Usiskin, 1988) reminds us of the importance of justification and generalization in algebra. However, over generalization, or generalization of rules in wrong context is one of the frequent and persisting difficulties in algebra.

With the development of the cognitive psychology, the leading domain to perceive and explain the student difficulties has become "cognitive obstacles." One theoretical framework to understand the cognitive obstacles in the learning process is drawn by Herscovics (1989) from Piaget's theory of equilibration.

> …This process of equilibration involves not only *assimilation*-the integration of the things to be known into some existing cognitive structure-but also *accommodation*-changes in the learner's cognitive structure necessitated by the acquisition new knowledge. However, the learner's existing cognitive structures are difficult to change significantly, their very existence becoming cognitive obstacles in the construction of new structure. (p. 62)

From this perspective, cognitive obstacles are the natural consequences of the accommodation process of the elements (i.e. beliefs, way of communicating, etc.) of the new culture.

Many students have difficulty formulating linear algebraic equations from information presented in words (Clement, 1982; Clement, Lochhead & Monk, 1981; MacGregor & Stacey, 1993; MacGregor & Stacey 1997a; MacGregor & Stacey, 1997b;

Rosnick, 1981). *Syntactic translation*, in which translation of a statement of English into an equation occurs by replacing key words by mathematical symbols sequentially from left to right, is accepted as a procedure frequently used by students for formulating equations from natural language expressions, and is thought to be an important cause of errors, particularly the *reversal error* (Clement et al., 1981). Although syntactic translation was the first source blamed for reversed equations, Clement et al. (1981) observed that another approach to writing equations was used frequently. In this approach, which they called *static comparison* and Hesrcovics (1989) called it *semantic translation*, the equation is used to represent an association of related groups, rather than equal numbers. MacGregor and Stacey (1993) reported that the majority of secondary school students did not use a syntactic translation procedure for writing simple linear algebraic equations; instead they tried to express the meaning of the statement and wrote incorrect equations. MacGregor and Stacey suggested that the majority of the incorrect equations, particularly the reversal equations, are consequences of cognitive models attempting to represent compared unequal quantities. In a different cultural context, Mestre (1989) reported that many Hispanic students have misconceptions in solving the Student-Professor problem (Clement, 1982) such as "6S = 6P and 6S + P = T," which originate in language differences. He concludes that differences in language cause Hispanics to commit the same types of errors as Anglos, but with a higher frequency.

Mis-generalization and justification are among major sources of students' difficulties in algebra as Steve Maurer's (1987, no reference) observed, "… the research brings Good News and Bad News. The Good News is that, basically, students are acting like creative young scientists, interpreting their lessons through their own generalizations. The Bad News is that their methods of generalizing are often faulty." For the question "Is the statement (2x + 1)/(2*x* +1 + 7) = 1/8 definitely true? /possibly true? /never true?" Lee & Wheeler (as cited in Kieran, 1990) reported that most of the 354 students' conclusion were "definitely true" for two distinct reasons: either the 2*x*'s were cancelled, or cross-multiplying led to an equation which was solved or at least appeared to be solvable. Barnord (1989) found that 40.7% of the subjects selected $1/y + 1/x$ when they were wanted to simplify $x/x + y + y/x - y$. Similar results were reported for students' simplification of $(a^2 + a/a)$ as a^2. Margilues (1993) and Parish & Ludwig (1994) reported that most students commute errors of the types $3/a + 3/b = 3/(a+ b)$, $(a + b)/(c + d) = a/c + b/d$, $(a + b)/b = a$, $(x + 2)^2 = x^2 + 4$, $(a - b)^2 = a^2 - b^2$.

On Teaching (of Algebra) and Teachers

It is well recognized that to educate a generation for tomorrows' world where demands will be placed on problem solving and communication skills, we need to shift instruction beyond routine symbolic manipulation and procedures toward an in depth, "relational understanding" (Skemp, 1987) of algebra (Booth, 1989). Thus, students should be provided with meaningful tasks to develop critical thinking skills and enhance appreciation for the usefulness of algebra. As Davis (1989) cautioned these mathematical tasks should be of a genuine nature, not the artificial contrived word problems found in many textbooks. Thorpe (1989) draws a framework that a topic should not be included in any algebra curriculum unless it has at least one of *intrinsic value*, *pedagogical value*, or *intrinsic excitement or beauty*.

Bertrand Russell (no reference) once said that "When it comes to algebra we have to operate with x and y. There is a natural desire to know what x and y really are. That, at least, was my feeling. I always thought the teacher knew what they were but wouldn't tell me." If we need to draw some quick guidelines from research about how to teach algebra, the first one should be that the focus of instruction should be on the meaningful development of important algebraic/mathematical ideas (Hiebert & Carpenter, 1992; Koehler & Grouws, 1992; Skemp, 1987). Instruction should emphasize the mathematical and practical meanings of ideas, including how the idea, concept or skill is connected in multiple ways to other mathematical ideas as well as to real life wherever possible. Within that context, teaching should be toward the big ideas of algebra (Edwards, 2000; Woodbury, 2000). Thus, a classroom context should be provided for students to construct meaning and make connections within algebra/mathematics and across other disciplines. Research also tells us that students can learn new skills and concepts without sacrifice of traditional pencil and paper skills for solving problems (Heid, 1996; O' Callaghan, 1998). Furthermore, teaching should incorporate students' intuitive solution methods with a combination of opportunities for student interaction and discussion (Boaler, 1998; Fennema, et al. 1993; Filloy & Rojano, 1989; Swafford, and Langrall, 2000; Thompson, 1988).

Falkner, Levi, & Carpenter (1999) mention that when they presented the problem "8 + 4 = c + 5" in a project to a six-grade teacher to give her students, her approach was "Sure, I will help you out and give this problem to my students, but I have no idea why this will be of interest to you." (p. 232). Finding out that all twenty-four of her students answered that 12 was the answer, she disseminated the problem to all 145 sixth-grade students and found out that those students thought that either 12 or 17 was the answer (p. 232). I think this particular example is a good indication of teachers' limited knowledge about research in student thinking in algebra, unless they had a specific in-service or graduate level course that presented the research. In other words, my expectation is that mathematics teachers do not have much familiarity with how algebra is taught and learned. I rather expect, however, their familiarity with how algebra should be taught is determined more by the national or statewide standards for school mathematics such as NCTM (1989, 1991, 2000) and Quality Core Curriculum (Georgia Department of Education, 1997).

Although a rich and extensive body of research focuses on students' understanding and learning in algebra, few studies have investigated teachers’ conceptions or professional knowledge in algebra, and those studies typically have investigated prospective teachers’ (Even, 1993; Wilson, 1994) and practicing teachers’ (Llinares, 2000) concept of function. In terms of teacher knowledge or awareness of student thinking there are few studies, which I think are quite significant. They reflect practicing algebra teachers’ professional knowledge and beliefs about teaching algebra.

Studies suggest that teachers do not study explicitly or do not give attention to students' conceptions and ways of thinking in mathematics and so are not knowledgeable about these areas. Research on teacher reactions to student thinking and conceptions has shown that the majority of teachers judged the students' answers only in terms of being right or wrong, and provided them with their own explanations for the right answer. Many of them made no attempt at understanding the sources of students' responses including right/wrong ideas (Even & Martovits, 1995; Even & Tirosh, 1995). Even & Martovits (1995) found that teachers did not pay enough attention to the student reasoning and knowledge construction but rather they tried to explain student reasoning and difficulties from their own perspective. Furthermore, Even & Tirosh (1995) reported that "…the majority of the teachers made no attempt at understanding the sources of student responses. When asked directly, they found it difficult to explain why students reacted the way they did. Sensitivity to student thinking becomes even more difficult under the pressure of real teaching instead of an interview setting." (p. 17-18).

Tirosh, Even and Robinson (1998) investigated 2 novices and 2 expert teachers' awareness of students' tendency to conjoin algebraic expressions (i.e., 3x+2 = 5x or =5) and the attributed sources for it when teachers were made aware of the situation. For all four teachers, the main teaching approach was some version of "collecting like terms." Although the two novice teachers used that method as soon as they introduced the topic of simplifying algebraic expressions, one of the experienced teachers, Gilah, on the other hand, introduced like and unlike terms and gave students enough time to practice and master "collecting like terms" before she introduced the topic of simplification. Some scholars see this approach (i.e., skills presented to students as rituals to be practiced) as problematic, considering the long-term effect of curtailing the conceptual learning of mathematics although it gives immediate results. This dilemma is closely related to the teachers' judgments of the importance of skills in mathematics.

Another approach, used by one novice teacher, was *the fruit salad approach* (i.e., using fruit names to interpret an algebraic expression such as interpreting 4*a* as four apples). This approach is also problematic and may lead to some other wrong generalizations although it may work in some cases. Thus, it may cause harm rather than benefit. A third approach identified was going backward, used by Betia. Furthermore, order of operations in which unknown nature of *x*, thus the impossibility to add something to it (e.g., in *x* + 3, we can not add 3 to *x* since we do not know what x is), was emphasized. This approach was also presented as problematic since it may lead students to view 3 + 4*x*, for example, not as a final answer but rather as a partial answer that one can not 'finish.' The last approach, used by

Batia was substitution, which was mentioned in the textbook she used. In that approach students substitute numbers in two expressions and check if the results are same for both. Although this approach, like the order of operations approach, does not provide a view of algebraic expressions as objects rather than process as final answers, it helps students to see whether the expressions are equivalent or not.

In another study, Nathan and Koedinger (2000) investigated the accuracy of teacher beliefs about relative difficulty of solving various algebra problems and how teacher judgments are influenced by their general beliefs about mathematics teaching and student learning. The research participants were 107 Grade 2 through Grade 12 teachers from the same district enrolled in an obligatory school district-sponsored workshop on primary and secondary mathematics teaching. Nathan & Koedinger previously studied high school students' problem solving performance in various types of algebra problems and they found that high school teachers inaccurately predict students' problem solving performance and thus misjudge their symbolic- and verbal-reasoning abilities. This led them to a conclusion that those teachers were dominated by *symbol-precedence view* of student mathematical development, "wherein arithmetic reasoning strictly precedes algebraic reasoning and symbolic problem solving develops prior to verbal reasoning" (p. 209). Two instruments were used to collect data: a survey (the problem difficulty ranking task) and a questionnaire (the belief instrument).

In the survey, teachers were asked to rank six algebra problems, which varied with respect to two factors, from easiest to most difficult. The first factor was the position of the unknown value: either the result of a problem (arithmetic type) or the beginning of the problem (algebra type). The second factor included problems in one of three presentation formats: verbal with a context (a story problem), verbal with no context (a word equation problem), or symbolic (a symbol equation). For all grade levels, teachers ranked the arithmetic problem as more difficult than matched algebra problems. Moreover, on average, teachers ranked verbal problems as more difficult than symbolic problems and word-equation problems as harder to solve than story problems (p. 223). On the other hand, high school teachers were least accurate in their prediction of student performance in problem solving whereas middle school teachers were the most accurate in their predictions based on student data from authors' previous studies with students.

In the belief instrument, the researchers asked teachers to express their agreement/views addressing current reform movements that were adopted from past literature. Nathan and Koedinger found that in general high school teachers were least likely to agree with reform views expressed in the survey in contrast to elementary school teachers who expressed the strongest agreement and middle school teachers falling midway on each construct. Although the majority of teachers participating in the study expressed reform-based views of mathematics learning and instruction, they did not seem affected by these particular beliefs when they are asked to judge how students would perform on a set of algebra and arithmetic problems. The symbol-precedence view was concluded to be the dominant belief affecting teachers' views of problem difficulty. Grade level was concluded to be a more important factor than symbol-precedence in determining problem difficulty. Since formal representations and solution methods are increasingly emphasized from elementary to middle to high school, and since high school teachers tend to have greater expertise in their content areas, they were least aware of difficulties of their novice students and they were usually the ones to least agree with reform views.

METHODOLOGY

Design of the Study

One needs to get close to people (e.g., talking to people, observing them in their day-to-day lives) to understand the way that they think, what they know about their world, how these are constructed and come to play in different contexts (Bogdan & Biklen, 1998). I plan to follow a qualitative research design to generate data relevant to teacher knowledge and beliefs about student thinking and how they use this in goals, decision-making, or basically in the context of teaching and learning algebra.

Qualitative research, stressing the socially constructed reality, studies phenomena in their actual settings to make sense of, or interpret the meanings people bring to them (Bogdan & Biklen, 1998; Denzin & Lincoln, 2000). Qualitative research encourages emergent (opposed to predetermined) design, and researchers focus on process rather than simply on products or outcomes. Researchers "do not search out data or evidence to prove or disprove hypotheses they hold before entering the study; rather, the abstractions are built as the particulars that they have been gathered are grouped together" (Bogdan & Biklen, 1998, p. 6) The researcher acts as the “human instrument” (Patton, 1990) of data collection. In other words, “qualitative researchers set up strategies and procedures to enable them to consider experiences from the informants' perspectives” (Bogdan & Biklen, 1998, p. 7).

Theoretical Framework

Considering the decisions made during teaching, some of the major roles of mathematics teachers are defined *by NCTM's The Professional Standards for Teaching Mathematics* as:

- Setting goals and selecting or creating mathematical tasks to help students achieve these goals;
- Stimulating and managing classroom discourse so that both the students and the teacher are clearer about what is being learned;
- Creating a classroom environment to support teaching and learning mathematics;
- Analyzing student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions. (NCTM, 1991, p. ?)

Fulfilling these roles is assumed to be the major part of becoming a good mathematics teacher so that students have the opportunity to learn mathematics that is meaningful and structural, as envisioned by National Council of Teachers of Mathematics (NCTM, (1989, 2000). The first natural question to ask would be about what teachers do and why they do it before answering the question what they need to know to know how to do the job “well.”

For the purposes of this study, I have constructed and adopted a theoretical model based on the perspectives suggested in *Teaching in Context* (Schoenfeld, 1998), *Cognitively Guided Instruction* (Carpenter & Fennema, 1991; Fennema & Carpenter, 1996), and *Pedagogical Content Knowledge* (Shulman, 1986). I have represented the model in Figure 1. I must note that the model is similar to the one in *Teaching in Context* (Schoenfeld, 1998) with a focus on student thinking.

Based on the cumulative knowledge that the research provided, Schoenfeld (1998) proposes that teacher knowledge, beliefs, and goals are three critical identifiers that affect what teachers do and why they do it. Although there are many components forming or consisting of teacher knowledge, beliefs, and goals, I plan to focus on those related to student thinking. Thus, in figure 1, multiple layers with “student thinking” up front represent this point. Equally important to beliefs, knowledge, and goals, Schoenfeld, also points out the importance of the context that activates the valued knowledge, belief or goal. Naturally, the current context is affected by the past contexts and experiences.

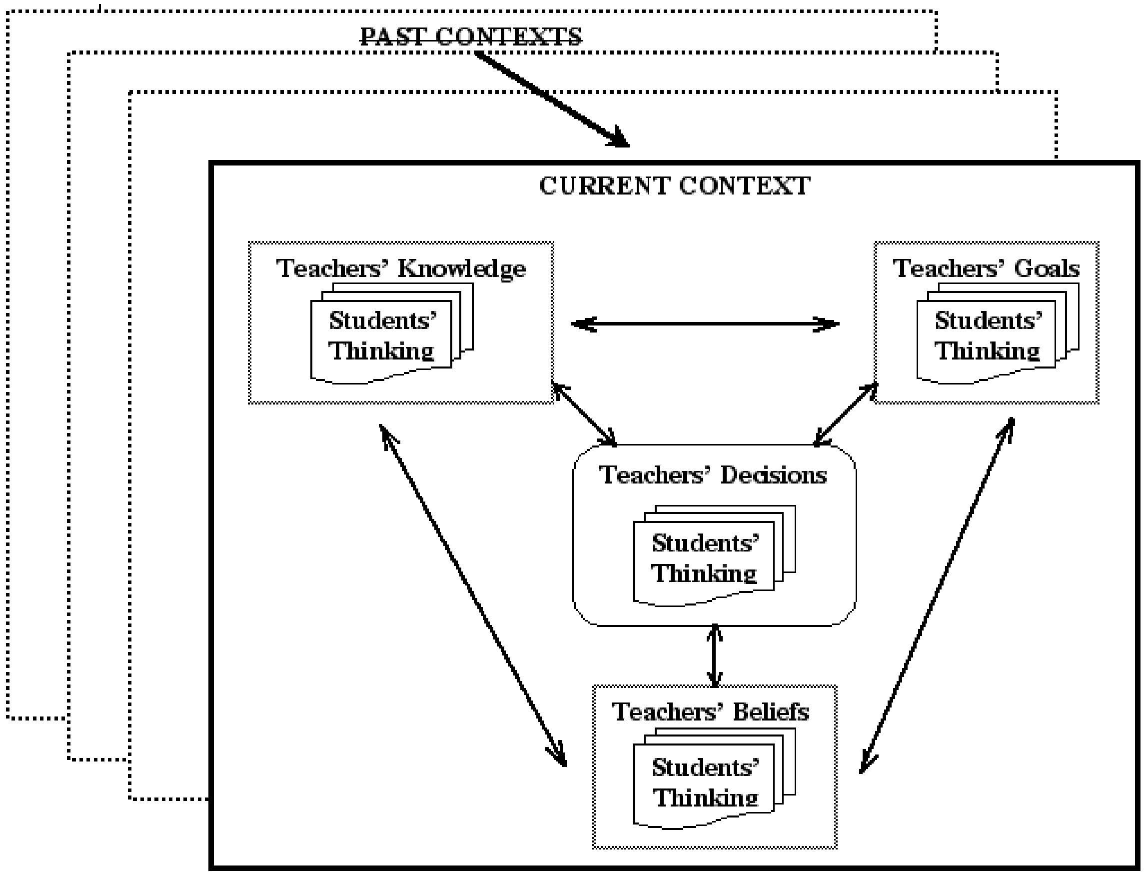

Figure 1. General model for research proposed in this dissertation research prospectus.

Why Focus on Student Thinking?

Several studies attempted to enhance teacher knowledge of student thinking so that they can improve the quality of their mathematics instruction. Among these, *Cognitively Guided Instruction* (CGI) seems to be promising with lots of success stories (Fennema & Carpenter, 1996; Lubinski & Fox, 1998; Swafford and Jones, 1997; Vacc & Bright, 1999). Focusing on teachers' pedagogical content knowledge, CGI has the goal "to help teachers develop an understanding of their own students' mathematical thinking and its development and how their students' thinking could form the basis for the development of more advanced mathematical ideas" (Fennema and Carpenter, 1996, p. 404).

Carpenter & Fennema (1991) emphasize that the approach CGI brings is not an attempt to provide a prescription or a series of procedures for instruction, but it is an attempt to help teachers to improve their own instructional decisions using the knowledge from cognitive science. Thus, the premise behind CGI is the perception that “the teaching-learning process is too complex to specify in advance, and as a consequence, teaching essentially is problem solving. Instruction must necessarily be mediated by teachers’ decisions, and we can ultimately bring about the most significant changes in instruction by helping teachers to make more informed decisions rather than by attempting to program them to perform in a particular way.” (pp. 10-11). Student knowledge and instructional goals should be the bases for teachers’ instructional decisions. Teachers’ knowledge of content (i.e., its difficulty, depth, multiple ways to represent, etc.) is important in this decision-making process. On the other hand, in order to assess their students’ thinking, teachers should have the knowledge of the general stages that the students pass through during the event of learning the concepts and procedures in a specific domain or topic. The most fundamental principle of all is that “instruction should be appropriate for each student.” (p. 11) In other words, the task, concepts, procedures should be all meaningful to students. Furthermore, instruction should be designed in a way that it emphasizes the relationships among concepts, procedures, and problem solving and allow students to construct their own knowledge with understanding. Therefore, promoting teachers' content knowledge and pedagogical content knowledge in terms of the "knowing that" and "knowing why" (Even & Tirosh, 1995; Shulman, 1986) aspects of student thinking seems to be promising for more informed instructional decisions and thus quality mathematics education.

I must note that so far, the research on CGI was done in elementary school level mathematics: Fennema & Carpenter (1996) on change in the beliefs and instructions of word problems of 21 primary grade teachers, Lubinski & Fox (1998) on one pre-service teachers' knowledge of division of fractions, Vacc & Bright (1999) on 34 elementary pre-service teachers' beliefs of teaching and learning mathematics, Swafford & Jones (1997) on how enhancing teachers' knowledge of geometry and their knowledge of research on students' cognition in geometry effect their instruction. They all have provided evidence that consideration of students' thinking was related to the change of instruction. Considering that there is a considerable amount of studies on students' cognition in algebra, I assume that the results obtained from elementary teachers teaching arithmetic can be replicated by secondary teachers teaching algebra.

Teacher Knowledge: Student Thinking and Pedagogical Content Knowledge

Shulman (1986) distinguishes three types of content knowledge domains relevant to teaching: *subject matter content knowledge*, *pedagogical content knowledge*, and *curricular knowledge.* Within each, three types of knowledge categories are defined: *propositional knowledge*, *case knowledge*, and *strategic knowledge.* Although I will focus on pedagogical content knowledge for my purposes, I shall briefly describe the knowledge domains, and the forms that they take in each.

Subject matter content knowledge comprises "the amount and organization of knowledge" (p. 9) by the teacher. It should be understood that this merely includes the basic facts, and concepts in the subject matter, but beyond this it requires substantive and syntactic structures understanding. Curricular knowledge is the understanding of the methods and materials for teaching the subject. Propositional knowledge covers the "facts" deduced from disciplined empirical or philosophical inquiry, practical experience, and moral or ethical reasons. Case knowledge is knowledge of “specific, well-documented, and richly described events” (p. 11). How to deal with a specific type of student or an emotion in the classroom, or how an instructional event occurred are forms of case knowledge. Strategic knowledge, the "highest" form of knowledge, covers events that conflict with known facts or cases. Critiquing the order or presentation of topics in a textbook, resolving a conflict are forms of strategic knowledge.

In his discussion of pedagogical content knowledge, Shulman describes the pedagogical content knowledge as “the ways of representing a formulating the subject that make it comprehensible to others.” (p. 9). This includes “the most regularly taught topics in one’s subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations. ” (p. 9). Furthermore, he points out that teachers also need to know what makes a topic easy or difficult to learn including student difficulties, and conceptions (i.e., preconceptions, conceptions, misconceptions) and strategies to deal with them. According to Shulman, pedagogical content knowledge is the domain where research on learning and teaching come most closely. In his words, as an important component of pedagogical understanding, research based knowledge of students’ misconceptions, their influence on learning, and instructional strategies to overcome and transform initial conceptions “should be included at the hearth of our definition of needed pedagogical knowledge.” (p. 10).

Like Shulman, Oliver (1989) points out that errors and misconceptions are important parts of learning, since they form part of an individual's conceptual structure that influence, mostly in a negative way, the learning of new concepts. Similarly Nesher (1987) describes the students' errors and misconceptions as the students' "expertise, his contribution to the process of learning", while she discusses the role of students in learning situation according to their contribution as expertise.

Teacher Beliefs (on Student Thinking)

Teacher and student behaviors and actions are not random but reflections of certain beliefs and values (Marshall & Rossman, 1999). Schoenfeld (1998) defines beliefs as “mental constructs that represent the codifications of people's experiences and understandings.” The line between beliefs and knowledge is often fuzzy and often teachers treat their beliefs as their knowledge (Thompson, 1992). However, beliefs might have different degrees of conviction and so up to change and may not need to be verified as it is the case for knowledge (Thompson, 1992). In general, studies show that beliefs have a strong impact on actions of teachers (Cooney, 1985; Fennema & Franke, 1992; Schoenfeld, 1998; Thompson, 1992). It has also been shown that a contradiction exists between what teachers state as their beliefs and what they do in action (Cooney, 1985). As an explanation for this situation, Schoenfeld (1998) suggests the effect of the current context that activates and prioritizes the beliefs, knowledge, and goals. Although it may not be possible to know what teachers truly believe, Schoenfeld (1998) suggests looking at the following beliefs affecting teachers’ actions in any study of teaching:

- beliefs about the nature of subject matter (in general and with regard to the specific topics being taught);
- beliefs about the nature of the learning process (both cognitive and affective);

- beliefs about the nature of the teaching process and the roles of various kinds of instruction;
- beliefs about particular students and classes of students. (p. 23)"

In this study, I plan to focus on teachers' beliefs about student thinking in particular.

Sample/Participant Selection

Purposeful sampling, which seeks information-rich cases that can be studied in depth, is the dominant strategy in qualitative research. Patton (1990) identifies and describes 16 types of purposeful sampling. Among all, I plan to use the *maximum variation sampling*, which "aims to capture and describe the central themes or principal outcomes that cut across a great deal of participant or program variation" (Patton, 1990, p. 172). *Maximum variation sampling* potentially yields detailed descriptions of each case, in addition to identifying shared patterns that cut across cases.

In order to focus on the context specific knowledge, beliefs, goals, and instructional decisions of individual teachers, I plan to choose four Algebra-1 teachers representing varying characteristics such as program/curriculum used, sex (M or F), experience (years in professions), and education (BS, MSc, or Ph.D.). The pool of mathematics teachers for sampling process will be the ones with whom I have contact, mathematics teachers who are participating or whom have participated in workshops coordinated by the Department of Mathematics Education at UGA.

Data Collection

The related data will be collected by (1) contextual and biographical interviews; (2) video recorded classroom observations; (3) interviews on teacher planning and assessing of algebra lessons; (4) interviews to analyze hypothetical situations that described students' answers to questions about some Algebra 1 topics; (5) interviews following the critical incidents observed during the classes. A description and rationale for each of the methods is given in this section.

(1) Contextual and biographical interviews: Semi-structured interviews will be used to gather data about participant teacher's background (experience, education, etc.), knowledge and beliefs about algebra as a branch of mathematics (i.e., content perspective), about how algebra should be taught, about how algebra curriculum should be organized, and lessons prepared and delivered as well as their beliefs about nature and remedy of student difficulties. In three main interviews, at the beginning, middle and the end of the study, teachers will be asked for their beliefs, ideas and experiences with the nature of the student difficulties in algebra. By interviewing at the beginning, middle and the end of the study, it is also intended to study the change in teachers' beliefs and action toward student thinking and so difficulties as their course moves along. I plan the interview questions similar to the following ones:

"Tell me about yourself: Your education as a math teacher; your background on algebra."

"Tell me about yourself as a math teacher in general. Who are you in the classroom?"

"Tell me what you think about algebra (content)?"

"What do you think about the continuity and discontinuity among algebra, arithmetic and geometry from content perspective?"

"Tell me your ideas about how to teach algebra. Why do you think so?"

"How could you define your algebra classroom (culture)?"

"What can you say about the interaction and communication between you and the students?"

"What do you think about students' difficulties in Algebra-1?"

"What can/can't you do as a teacher to help students to solve their difficulties?"

(2) Video recorded classroom observations: Classroom observations, for me, is a place to validate the information provided by interviews and archival analysis as well as to seek for the nature of interaction within the context of classroom setting. The main focus of the classroom observations will be the teachers' interpretation of topic, questions to students', answers to students' questions, and dealing with students' difficulties. Furthermore, the classroom observations will help to understand the teachers' instructional practices. More specifically, I mainly plan to observe the context of teachers' delivery of subject matter, which includes teacher's perception of the subject matter, questioning the students, nature of feedback provided to students, etc. My positioning will be observer rather than participant observer. I do not intend to participate in anything but watch and make sense of verbal and non-verbal behavior (Marshall & Rossman, 1999) by being there (Wolcott, 1999). In other words, I will be a participant (i.e., being there) as observer (Wolcott, 1999). I will observe and take field notes during the lessons and if it will be allowed, all of the class periods will be video-recorded and then related extracts will be transcribed for the data analysis.

(3) Interviews on teachers' planning and assessment: Teacher lesson plans on some selected topics will be examined to search for the questions: "How do the teachers plan their lessons taking student difficulties (errors and misconceptions) into consideration?" or equivalently "What is the place of student thinking, difficulties, errors and misconceptions in lesson plans?" Moreover, the lesson plans are thought to be a good indication of teacher interpretation of content. In order to understand which criteria teachers use to prepare their lesson plans, I plan to conduct interviews with teachers and ask about criteria and circumstances they consider when planning their lessons.

Teachers' measurement and evaluation strategies are considered to provide additional data to understand the role of student thinking in teachers' instructional practices. Tests, homework, and journals can be used to inform about the difficulties, errors and possible misconceptions that the students hold. From that perspective, teachers' purpose in asking a particular item or giving a particular project reflects what priorities he/she has in teaching and learning of algebra. On the other hand, teachers' questions asked in classroom will also be considered in the frame of measurement and evaluation. So, post interviews will address teachers' measurement and evaluation policies in and outside of classroom. I plan the interview questions similar to the following ones:

"Tell me how you prepare for a lesson. What are your goals, objectives, representations used, type of tasks, and concepts?

"What are the main factors affecting your preparation? How?"

"Tell me how you apply your lesson plan in classroom. Why?"

"Tell me about your assessment strategies. Tests, projects, journals? How often? Why do you prefer these?"

"Tell me about your criteria to decide which questions to ask in classroom."

"Tell me about your criteria to decide which items to ask on a test or quiz."

(4) Interviews to analyze hypothetical situations: Hypothetical situation will be used as a tool to initiate teacher talk about his/her students' learning of selected algebra topics. The cases will be constructed based on research literature on students' cognitions in algebra. I plan to structure the cases consisting of "(i) the description of a pupil's hypothetical response to a problem and (ii) some questions for the teacher (diagnosis and action). Each cases described a situation in which the teacher had to react to pupil's observation. The teacher had

to explain what the pupil might be thinking. Then the teacher was asked to describe how she might answer (Llinares, 2000)”

(5) Interviews about the critical incidents occurring during the observed classes: After each observed class period, I plan to conduct further interviews to reveal or allow each teacher to comment about incidents that occurred during the class, that I or the teacher found interesting, as well as allow them to describe their experience with students during the lesson. I particularly expect to point out or expect them to describe their experiences with student thinking, how and why they responded in particular ways. Furthermore, with these interviews I expect to gain insights into what and to what extend the teachers know about student thinking, errors and misconceptions in particular. All of the interviews will be video or audio-recorded (to the extent that the participants allow) and then transcribed for the data analysis. The timing and number of them will depend on the on-going analysis of the data obtained from the classroom observations.

Data Analysis

Data analysis in qualitative inquiry is both an iterative and an on-going process. With rigorous standards, data and interpretations are continuously checked with respondents who have acted as sources, differences of opinions are negotiated and outcomes are agreed upon, understood and reflected. Glesne (1999) defines qualitative data analysis as "working with the data, you describe, create explanations, pose hypotheses, develop theories, and link your story to other stories" (p. 130) Although there are various ways of analyzing data in a qualitative research, it becomes a common sense in qualitative inquiry that data analysis is a continuous task right from the beginning of research. According to LeCompte (2000), the first step in data analysis should be identification of bias since it affects what the researcher sees and interprets. I will take this point into consideration in my data collection and analysis. On the other hand, handling and keeping track of data as study moves along is the initial step in data analysis. This issue is referred as “managing data" by Dey (1993, p. 75), "tidying up" by LeCompte (2000, p. 148), and "early data analysis" by Glesne (1999, p. 131). I am considering analyzing my data analysis strategies under four categories:

1. Data Management: I will accumulate jottings, expanded fieldnotes, interview notes, interview -transcripts, documents and artifacts to categorize/organize the data into file folders on the computer and/or as tangible texts.

2. Description: Using data from multiple sources I will articulate a "thick" and rich description of the persons and context under investigation.

3. Analysis: I will manipulate the data by creating and applying abstract categories and then use those categories to compare, contrast, sort, and refine distinguishable thoughts, behaviors, and events.

4. Interpretation: Informed by descriptions and analysis it is the stage where I will make inferences by connecting the data with the theoretical structure that frames the study.

In order to make connections among stories I will have from the study, analysis will begin with identification of the themes emerging from the raw data, a process sometimes referred to as open coding (Strauss and Corbin, 1990). During analytic coding, I will identify and tentatively name the conceptual categories into which the phenomena observed would be grouped. The goal will be to create descriptive, multi-dimensional categories, which form a preliminary framework for analysis. Words, phrases or events that appear to be similar will be grouped into the same category. These categories will be gradually modified or replaced during the subsequent stages of analysis that follow.

As the raw data are broken down into manageable chunks, I will also devise an "audit trail"; that is, a scheme for identifying these data chunks according to their speaker and the context. The particular identifiers developed may or may not be used in the research report, but speakers are typically referred to in a manner that provides a sense of context. I will use "the voice" in the text; that is, participant quotes that illustrate the themes being described. The next stage of analysis will involve re-examination of the categories identified to determine how they are linked. The discrete categories identified in open coding are compared and combined in new ways as I begin to assemble the "big picture." During this time, I may determine that the initial categories identified must be revised, leading to re-examination of the raw data. Additional data collection may occur at any point if I uncover gaps in the data. The next step will be to translate the conceptual model into the story line that will be read by others. Ideally, the report will be a rich, tightly woven account that "closely approximates the reality it represents" (Strauss and Corbin, 1990, p. 57). To facilitate the above processes, I will try to use a computer-assisted data analysis program like Nud/ist or Ethnograph.

On the other hand, data analysis will also be an inductive process in which interview transcriptions, fieldnotes and documents will be deconstructed, coded, and synthesized using conceptual themes. In analytic induction, a procedure for verifying theories and propositions/hypotheses based on qualitative data, to develop a working hypothesis, a hypothesis will be tested, revised, and re-tested case by case across a broad range of cases until the hypothesis adequately explains in all cases the phenomenon being studied.

Representation

At the first hand, the audience of this study will be the community of academics, my committee members, in mathematics education. Thus, the writing up of the study will be considered in formal or traditional dissertation format. I will try to enrich this formal structure by writing up realistic and confessional tales (Glesne, 1999). On the other hand, I do not want to leave the results of this study as a dissertation. Rather, I will write up several research articles in the form of “Pradley's Narrowing and Expanding the Focus: The author moves from descriptive detail to theoretical abstraction or vice versa (Glesne & Peshkin, 1992, p. 164).” My role in the writings will be somewhere between *translator/Interpreter* and *transformer* (Glesne, 1999, Glesne & Peshkin, 1992). I do not only want to understand the others' world and then to translate the text of lived actions into a meaningful account, but also we want our readers, as they read, to identify with the problems, worries, joys, and dreams that are the collective human lot. I also want to develop a multimedia material packet for pre- and in-service teacher education. The audio and video materials will be developed from selected episodes of classroom events and interviews. However, not to cause any ethical problem, the episodes will be filmed or transcribed with different people like a movie. On the other hand, some text material will be developed for the interpretation and theorizing of video and audio materials as well as on how to implement these materials in a pre- or in-service teacher education. The researcher’s role in developing (or writing up) of these materials will be *artist* (Glesne & Peshkin, 1992).

Validity and Reliability

Probably many qualitative researchers have been criticized by critics who have claimed that their research it is too subjective, or the number of cases is too small, or that mere talking is never a scientific method, and so on (Merriam, 1995). Today, it is widely accepted by academia that qualitative research is no less rigorous than more traditional forms of inquiry in terms of validity and reliability. In general, eight verification procedures are used to build the trustworthiness in qualitative research: prolonged engagement and persistent observation, triangulation, peer review and debriefing, negative case analysis, clarification of researcher bias, member checking, rich/thick description and external audit (Glesne, 1999; Lincoln & Guba, 1985; Merriam, 1995).

In this study, I plan to employ a variety of techniques to ensure credibility of data and data analysis. Through prolonged engagement with the participants, both with interviews and in their classrooms, an attempt will be made to establish and maintain an effective vehicle for obtaining and processing reliable information. Additionally, I will maintain credibility through on-going dialogue with participants, numerous observations, peer debriefing and by the use of member checks. Data and their interpretations will be constantly scrutinized by me, the participants, my major professor, and the colleagues in my writing group who won't be directly involved in the study. On the other hand, triangulation, to seek multiple and comparative opinions about the same topic or issue, will be another means of strengthening data collection and analysis through confirmation and completeness. Furthermore, triangulation will provide the convergence, inconsistencies and contradictions in the data and/or phenomenon under study (Mathison, 1988). I plan to employ the following types of triangulation: data triangulation (including time, space and person), investigator triangulation, and methodological triangulation (Mathison 1988).

Consistency, which may also be known as reliability, is dependent upon stability, consistency and predictability (Lincoln & Guba, 1985 p. 296). Considering the unstability of human behavior and social factors, qualitative researchers seek a means for taking into account both factors of instability and factors of phenomenal or design induced change. Dependability in interpretive research is often accomplished using an audit trail (Lincoln, and Guba, 1985 p. 382-385) in which the researcher maintains a log containing personal notes, which allow for reflection upon what happens in relationship to personal values or perceptions. In addition to triangulation and peer examination, the audit trail used in this study will include raw data, data reduction and analysis procedures, data reconstruction and synthesis and processing notes.

Working hypotheses, *concrete universals* and *reader/user generalizabilit*y are considered as bases for interpretation of generalizability of findings of qualitative research (Merriam, 1995). In that context, interpretations are neither value nor context free. Thus, as long as similar context is provided, there is no reason not to generalize (or transfer) research findings in other types of situations. Moreover, "the general lies in the particular" (Merriam, 1995, p. 58). Most of the time, we make general conclusions from similar particulars. On the other hand, generalizability is useful to the readers of the research findings. In other words, it is up to the reader to decide the extent of generalizability of the research findings, not up to the researcher (Merriam, 1995). For this, the readers will be provided *thick descriptions* (Geertz, 1973) to enter the research context. On the other hand, in this study, I plan to maintain confirmability by providing raw data that can be traced to original sources and by describing how the data is to be interpreted and placed into categories or conclusions (Lincoln and Guba, 1985, p. 384-385).

Proposed Timeline for Study

Phase I: Pre-field Work

Dates: June 2002 - August 2002

- IRB approval
- Field and participant selection
- Design of the interview guides, hypothetical situations, and other materials

Phase II: Implementation of the Study - Data Collection

Dates: August 2002 - November 2002

- Contextual and biographical interviews (mid August, mid October, and end of November)
- Video recorded classroom observations (daily or weekly)
- Interviews on teachers' planning and assessment (daily or weekly)
- Interviews to analyze hypothetical situations (weekly)
- Interviews on the critical incidents observed during the observed classes (daily or weekly)

ß Transcription of audio and video tapes (weekly)

ß Member checking and co-construction of data (weekly)

Phase III: Analysis & Interpretation

Dates: August 2002- February 2003

Phase IV: Writing it Up - Representation

Dates: March 2002 - July 2003

REFERENCES

Barnard J. J. (1989). *Poor concept formation in mathematics: A diagnostic perspective.* (Report No. RGN/HSRC-P-105). Pretoria, South Africa: Human Sciences Research Council. (ERIC Document Reproduction Service. No. ED 310 926)

Boaler, J. (1998). Open and closed mathematics: student experiences and understandings. *Journal for research in mathematics education, 29*, 41–62.

Bogdan, R. C. & Biklen, S. K. (1998). Foundations of Qualitative Research in Education. In R. C. Biklen & S. K. Bogdan (Eds.), *Qualitative research in education: An introduction to theory and methods* (pp. 1-48). Boston: Allyn Bacon.

Booth, L.R. (1989). A question of structure. In S. Wagner and C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra* (pp. 57-59). Reston, VA: National Council of Teachers of Mathematics.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12*. (pp. 20-32). Reston, VA: National Council of Teachers of Mathematics.

Booth, L. R. (1984). *Algebra: Children's strategies and errors.* Windsor, UK: NFER Nelson.

Carpenter, T. P., & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 1–16). Albany, NY: SUNY Press.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*, 16-30.

Clement, J., Lochhead, J., & Monk, G. S. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly, 88*, 287-290.

Cooney, T. J. (1985). A Beginning Teachers' View of Problem Solving. *Journal For Research in Mathematics Education, 16*, 324-336.

Davis, R. B. (1989). Three ways of improving cognitive studies in algebra or a reaction to: “Cognitive studies of algebra problem solving and learning.” In S. Wagner and C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra* (pp. 115-119). Reston, VA: National Council of Teachers of Mathematics.

Denzin, N. K. & Lincoln, Y. S. (2000). Introduction: The discipline and practice of qualitative research. In Norman K. Denzin & Yvonna S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 1-28). Thousand Oaks, CA: Sage Publications.

Dey, I. (1993). *Qualitative data analysis*. NY: Routledge.

Edwards, T. G. (2000). Some big ideas of algebra in the middle grades. *Mathematics Teaching in the Middle School, 6*(1), 26-32.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes.* Mahwah, NJ: Lawrence Erlbaum.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), 94-116.

Even, R., & Martovits, Z. (1995). Some aspects of teachers' and students' views on student reasoning and knowledge construction *International Journal of Mathematics Education in Science and Technology*. *26*(4) 531-544

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and the knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 29*, 1-20.

Falkner, K., Levi, L., and Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*(4), 232-236.

Fennema, E., & Carpenter, T. P. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-435.

Fennema, E., Franke, M.L. (1992). Teachers' knowledge and its impact. In D. G. Grouws (Ed.) *Handbook of Research in Mathematics Teaching and Learning*, 147-164. New York: Macmillan.

Fennema, E., Franke, M. L., Carpenter, T. P., & Carey, D. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal, 30*(3), 555–583.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19-25.

Geertz, C. (1973). Thick description: Toward an interpretive theory of culture. In *The interpretation of cultures: Selected essays* (pp. 3-30). New York: Basic Book.

Georgia Department of Education. (1997). *Quality Core Curriculum*. [On-line]. Available: http://www.doe.k12.ga.us

Glesne, C. (1999). *Becoming qualitative researchers: An introduction.* New York: Longman.

Glesne, C. & Peshkin, A. (1992). *Becoming qualitative researchers: An introduction*. White Plains, NY: Longman.

Heid, M. K. (1996). A technology-intensive approach to the emergence of algebraic thinking. In Kieran, N. Bednarz, & L. Lee (Eds.), *Approaches to algebra: Perspectives for Research and teaching* (pp. 239-255). Dordrecht: Kluwer.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In Sigrid Wagner & Carolyn Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60-86). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., & Carpenter, T. (1992).Learning and teaching with under-standing. In: D. A. Grouws, (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kieran, C. (1990). *Cognitive processes involved in learning school algebra*. In P. Nesher, & J. Kilpatrick (Eds.), Mathematics and Cognition. (pp. 96-112). New York: Cambridge University Press.

Kieran, C. (1989) The Early Learning of Algebra: A Structural Perspective. In S. Wagner and C. Kieran (eds.) *Research Issues in the Learning and Teaching of Algebra* (pp. 33-56). Reston, VA: National Council of Teachers of Mathematics.

Koehler, M., & Grouws, D.A. (1992). Mathematics teaching practices and their effects. In D. A. Grouws, (Ed.), *Handbook of research on mathematics teaching and learning* (p.

115–26). New York, NY: Macmillan.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics*: 11-16 (pp. 102-119). London: John Murray.

LeCompte, M. D. (2000). Analyzing qualitative data. *Theory into Practice*, *39*(3), 146-154.

Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bernardz et al. (eds.), *Approaches to Algebra*. (pp. 87-106). Kluwer Academic Pub., Netherlands.

Lincoln, Y. S., & Guba, E. (1985). *Naturalistic Inquiry*. Newbury Park, CA: Sage.

Llinares, S. (2000). Secondary school mathematics teachers' professional knowledge: a case from the teaching of the concept of function. *Teacher and Teaching: Theory and Practice, 6*(1), 41-62.

Lubinski, C. A., & Fox, T. (1998). Learning to make sense of division of fractions: One k-8 preservice teacher's perspective. *School Science & Mathematics*, 98(5), 247-260.

MacGregor, M. & Stacey, K. (1997a). Students` understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33*, 1-19.

MacGregor, M. & Stacey, K. (1997b). Ideas about symbolism that students bring to algebra. *Mathematics Teacher*, *90*(2), 110-113.

MacGregor, M. & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education*, *24* (3), 217-232.

Mestre, J. (1989). *Hispanic and Anglo students' misconceptions in mathematics*. Charleston,WV: ERIC Clearinghouse on Rural Education and Small Schools. (ERIC Document Reproduction Service. No. ED 313 192)

Margilues, S. (1993). Algebraic A-S. *Mathematics Teacher, 86*, 40-41.

Marshall, C.B. & Rossman, G.B. (1999). *Designing Qualitative Research*. Thousand Oaks, CA: Sage.

Mathison, S. (1988). Why triangulate? *Educational Researcher, 17*(2), 13-17.

Merriam, S. (1995). What can you tell from an N of 1?: Issues of validity and reliability in qualitative research. *PAACE Journal of Lifelong Learning, 4*, 51-60.

Nathan, M. J. & Koedinger, K. R. (2000). An investigation of teachers' beliefs of students' algebra development. *Cognition and instruction, 18,* 209-237.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nesher, P. (1987). Towards an instructional theory: the role of students' misconceptions. *For the Learning of Mathematics*. *7*(3) .33-39

O' Callaghan, B. R. (1998). Computer-intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education, 29 (1),* 21 40.

Oliver, A. (1989, July). *Handling pupils' misconceptions.* Paper presented as the presidential address at the thirteenth national convention on mathematics, physical science and biology education, Pretorio, South Africa.

Parish, C. R. and Ludwig, H. J. (1994). Language, intellectual structures and common mathematical errors: a call for research. *School Science and Mathematics*, 94(5), 235-238.

Patton, M. Q. (1990). *Qualitative Evaluation and Research Methods*. Newbury Park, CA: Sage.

Rojano, T. (1996). The role of problems and problem solving in the development of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 137–145). Dordrecht, Netherlands: Kluwer Academic Publishers.

Rosnick, P. (1981). Some misconceptions concerning the concept of variable. *Mathematics Teacher, 74*, 418-420.

Schoenfeld, A. (1998). Toward a Theory of Teaching-in-Contexts. [On-line]. Available: http://www-gse.berkeley.edu/Faculty/aschoenfeld/TeachInContext/teaching-in-context.html

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14*, 15-39.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Skemp, R. R. (1987). *The psychology of learning mathematics.* Hillsdale, NJ: Erlbaum.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Thousand Oaks, CA: Sage.

Swafford, J.O., & Jones, G.A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education, 28*(4), 467-484.

Swafford, J. O., and Langrall, C. W. (2000). Grade six students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education 31*, 89-112.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In *Handbook of Research in Mathematics Teaching and Learning* (pp. 127-146). New York: Macmillan.

Thompson, F. M. (1988). Algebraic instruction for the younger child. In A. F. Coxford (Ed.), *The ideas of algebra, K-12.* (pp. 69-77). Reston, VA: National Council of Teachers of Mathematics.

Thorpe, J. A. (1989). Algebra: What should we teach and how should we teach it? In S. Wagner and C. Kieran (Eds.) *Research Issues in the Learning and Teaching of Algebra* (pp. 11–24). Reston, VA: National Council of Teachers of Mathematics.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*, 51-64.

Ususkin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford, (Ed.). *The ideas of algebra, K-12* (1988 Yearbook, pp. 8-19). Reston, VA; National Council of Teachers of Mathematics.

Wagner, S. & Parker, S. (1993). Advancing algebra. In P. Wilson (ed.), *Research ideas for the classroom: High school mathematics* (pp. 119-139). Reston, VA: National Council of Teachers of Mathematics.

Wilson, M. R. (1994). One preservice secondary teacher's understanding of function: The impact of a course integrating mathematical content and pedagogy. *Journal for Research in Mathematics Education, 25*(4), 346-370.

Wolcott, H. (1999). *Ethnography: A way of seeing.* Walnut Creek, CA: Altamira Press.

Woodbury, S. (2000). Teaching to the big ideas of algebra. *Mathematics Teaching in the Middle School, 6*(4), 226-232.

Vacc, N.N., & Bright, G. W. (1999). Elementary preservice teachers' changing beliefs and instructional use of children's mathematical thinking. *Journal for Research in Mathematics Education, 30*(1), p89-101.