

Chapter 1: Background for the Mathematical Understanding Framework

The evolution of the Mathematical Understanding for Secondary Teaching framework began with a desire to characterize mathematical knowledge for teaching at the secondary level. Our initial characterization was much influenced by the work of Deborah Ball and her colleagues at the University of Michigan (e.g., Ball, 2003; Ball & Sleep, 2007a, 2007b; Ball, Thames, & Phelps, 2008). In particular, Ball et al. partitioned mathematical knowledge for teaching into components that distinguish between subject matter knowledge and pedagogical content knowledge (Shulman, 1986). As we worked on developing our own framework, we considered attempts to develop similar frameworks (e.g., Adler & Davis, 2006; Cuoco, 1996, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004; Tatto et al., 2008). Our intention has been to add to the work in this area, which continues to expand. We believe that our approach brings something new to the conversation about teachers' mathematical knowledge.

A New Framework: Mathematical Understanding for Secondary Teaching

Mathematical understanding for secondary teaching (MUST) is related but not identical to mathematical knowledge for teaching (MKT). In examining the work that others have done in developing frameworks for MKT, we became increasingly convinced that whatever framework we developed should reflect a more dynamic view of mathematical knowledge. Therefore we have chosen to characterize mathematical *understanding* rather than mathematical *knowledge*. In our approach, understanding underlies the observable application of a teacher's knowledge and therefore reveals knowledge held by the teacher. Furthermore, MUST is related to, but different from, pedagogical content knowledge (Shulman, 1986). We focus on mathematics and do not attempt to describe pedagogical knowledge or proficiency.

Our framework has been developed out of classroom practice, much like the work of Ball and her colleagues (e.g., Ball & Bass, 2003). A unique characteristic of our framework is the variety of classroom contexts from which we have drawn examples. We have observed the work of practicing teachers, preservice teachers, and mathematics educators and have used episodes from classrooms to examine and characterize MUST, as described in the following discussion of our development of *situations*.

Situations

Starting from the bottom up, we developed a collection of sample *situations* as a way of capturing classroom practice. Each situation portrays an incident in teaching secondary mathematics in which some mathematical point is at issue. (For details of our approach, see Kilpatrick, Blume, & Allen, 2006.) Looking across situations, we attempted to characterize the knowledge and understanding of mathematics that are beneficial for secondary school teachers to have but that other users of mathematics may not necessarily need.

Each situation begins with a *prompt*—an episode that has occurred in a mathematics classroom and raises issues that illuminate the mathematics understanding that would be beneficial for secondary teachers. The prompt may be a question raised by a student, an interesting response by a student to a teacher’s question, a student error, or some other stimulating event. We then outline, in descriptions called *mathematical foci*, mathematics that is relevant to the prompt. The set of foci is not meant to be an exhaustive accounting of the mathematics a teacher might draw upon, but we believe the foci include key points to be considered. These foci, each of which describes a different mathematical idea, constitute the bulk of each situation. There is no offer of pedagogical advice or comment about what mathematics the teacher should actually discuss in a class in which such an episode may occur. Rather, we describe the mathematics itself and leave it to the teacher or mathematics educator to decide what to use and how to do so. Along with the foci, each situation includes an opening paragraph, called a *commentary*, to set the stage for the mathematical foci. The commentary gives an overview of what is contained in the foci and serves as an advance organizer for the reader. Some situations also include a *post-commentary* to include extensions of the mathematics addressed in the situation.

Throughout the process of writing and revising the situations, we used aspects of what we would come to include in our MUST framework. For example, various *representations* helped us to think about the mathematics in the prompt. Perhaps there was a geometric model that was helpful or a graph or numerical representation to provide insight or clarification. At times a particular analogy was pertinent to the prompt. We were not interested in making every situation follow a particular format in which the same representations (such as analytical, graphical, verbal) were used again and again. We wanted to emphasize representations that we perceived as particularly helpful or relevant in relation to the prompt.

Another example of our use of aspects of mathematical understanding in writing and revising situations was the use of *connections* to other mathematical ideas, or *extensions* to concepts beyond those currently at hand. For example, if a prompt addressed sums of integers, we described (though not in great detail) sums of squares. This is an example of a topic to be discussed in a post-commentary at the end of the situation. Another way to extend a mathematical focus is to adjust the assumptions. For example, in a geometry problem, one could consider the implications of relaxing the constraint of working only in Euclidean space.

Our use of these aspects of what would eventually constitute the MUST framework drew our attention to what we believed were pertinent elements of mathematical understanding for teaching. This process helped us construct, clarify, and comprehend the framework and also provided us with examples to illustrate the elements of the framework.

Evolution of the Framework

By examining mathematical foci for about 50 situations, we developed a framework characterizing MUST for secondary school mathematics. In the situations,

we could see the need, for example, for a teacher to be skilled in such tasks as using multiple representations of a mathematical concept, making connections between concepts, proving mathematical conjectures, determining the mathematics in a student comment or error, understanding the mathematics that comes before and after the task at hand, or discerning when students' questions raise mathematical issues that should be explored given the time available. In seeking to develop and improve the framework, we have responded to comments and suggestions given by experts (mathematicians, teacher educators, and teacher leaders [e.g. department heads]) in the field of mathematics education. We gathered this input at two Situations Development Conferences at the Pennsylvania State University, the first in May 2007 and the second in March 2009.

The purpose of the first conference was to present our work on ten of the situations to a group of mathematicians and mathematics educators. At that point we had not developed a framework; rather, we were at the stage of writing and revising situations with the goal of being able to characterize mathematical *knowledge* (then *proficiency* and finally *understanding*) for teaching at the secondary level and constructing a framework for doing so. We received input from the experts about the situations themselves, and that input challenged us to continue to refine our work and to consider some additional mathematical ideas that we had not included in the foci of the ten situations we shared. We also sought advice from the participants about what they considered to be key aspects of mathematical knowledge for teaching at the secondary level. A few of the ideas arising from that discussion were analysis of student thinking and student work, mathematical reasoning, mathematical connections, and mathematical habits of mind. We went back to work on the framework, trying to incorporate advice we had received at the conference, so as to continue the process of characterizing MKT (later MUST). We began to build lists of items (content and processes) to be included in the framework (e.g., entities such as mathematical connections and representations, and actions like choosing appropriate mathematical examples).

In March 2009, we presented a version of our framework to a group of mathematicians, mathematics educators, and teacher leaders for the purpose of seeking feedback and advice, as well as to discuss ideas about how the framework and situations could be used and disseminated. We received positive responses from participants regarding how they envisioned using the situations in their work with prospective or practicing teachers. The feedback we received on the framework document included comments about both the content and the format of the document. In small- and large-group sessions, we had discussions about ideas for improving the framework—what to change or clarify, what to leave out, and what to add. Following the conference we began to work on incorporating these recommendations into our framework document.

At different times over the course of our work, we have focused on the situations, the framework, or both. Working on these two parts of our project in parallel has been helpful in keeping them both in view, particularly as our development of the situations has informed our construction of the framework. We believe that the framework now

can be used to better interpret the situations, to write new situations, and to further our understanding of mathematical understanding for teaching.

Conclusion

The work of describing MUST continues, but we believe this framework is already an important contribution to the mathematics education community. Teachers require mathematical understanding that is different from that needed in other professions. A teacher's work requires general mathematical knowledge as well as expertise in the kinds of tasks described in this framework: accessing the mathematical thinking of learners, developing multiple representations of a mathematical concept, knowing how to use the curriculum in a way that will help further the mathematical understanding of students, and so on.

As we have said, a unique feature of our work is that we have chosen to develop a framework for mathematical *understanding* for secondary teaching (MUST), highlighting the dynamic nature of teacher knowledge. We believe that this focus is a valuable contribution to the field. That MUST is dynamic is one reason we have not arrived at a final formulation of MUST, and see this project as a work in progress. Mathematical understanding for teaching should grow and deepen over the course of a teacher's career, and we expect our grasp of that understanding to grow and deepen as well.

A second unique feature of the framework is that we focus solely on mathematical understanding at the secondary school level. We believe that this focus is essential to the profession's conversation about teacher knowledge. Secondary mathematics differs from elementary school mathematics in its breadth, rigor, abstraction, explicitness of mathematical structure, and level of reasoning required. Therefore, teaching at the secondary level requires a special kind of mathematical understanding.

Finally, we believe we bring a unique perspective in that our framework has arisen from the practice of classroom teachers in a wide variety of settings including courses for prospective teachers, high school classes taught by practicing teachers, and classes taught by student teachers.

Just as we have sought the input of many mathematicians, mathematics teachers, and teacher educators during construction of this framework, we welcome comments from those in the field on our final product. Furthermore, we would like to gain further insight from others into MUST, perhaps by building on the ideas presented here.

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