Chapter 2: Mathematical Understanding for Secondary Teaching: A Framework

Secondary school mathematics comprises far more than facts, routines, and strategies. It includes a vast array of interrelated mathematical concepts, ways to represent and communicate those concepts, and tools for solving all kinds of mathematical problems. It requires reasoning and creativity, providing learners with mathematical competence while also laying a foundation for further studies in mathematics and other disciplines.

To facilitate the learning of secondary school mathematics, teachers need a particular kind of understanding. Mathematical understanding for teaching at the secondary level is the mathematical expertise and skill a teacher has and uses for the purpose of promoting students' understanding of, expertise with, and appreciation for mathematics. It requires that teachers not only know more mathematics than they teach but also know it more deeply.

Mathematical understanding for secondary teaching (MUST) is unique to the work of teaching. It is different from the mathematical understanding needed for engineering, accounting, or the medical professions. It is even different from the mathematical understanding a mathematician needs. For example, a mathematician may prove a theorem, and an architect may perform geometric calculations. For these users of mathematics, it is sufficient that they have the skills and abilities for the task at hand. But a teacher's work includes these tasks as well as interpreting students' mathematics, developing multiple representations of a mathematical concept, knowing where students are on the path of mathematical understanding, and so on.

Mathematical understanding for secondary teaching is *dynamic*. We make a distinction between knowledge and understanding. Knowledge may be seen as static and something that cannot be directly observed, whereas understanding can be viewed as the dynamic use of the knowledge one has. Understanding can be observed in a teacher's actions and the decisions he or she makes. Also, because of its dynamic nature, MUST grows and deepens in the course of a teacher's career.

The focus of our framework is on *secondary school* mathematics. That is, we seek to characterize the mathematical understanding that is useful to secondary teachers as distinct from the understanding needed by elementary school mathematics teachers. We believe that MUST is different from mathematical understanding for elementary school teaching in at least four ways: (1) There is a wider range of mathematics content (i.e., more topics are studied); (2) there is a greater emphasis on formality, axiomatic systems, and rigor in regard to mathematical proof; (3) there is more explicit attention to mathematical structure and abstraction (e.g., identities, inverses, domain, and undefined elements); and (4) the cognitive development of secondary students is such that they can reason differently from elementary school children about such matters as proportionality, probability, and mathematical induction.

Our framework has been developed out of *classroom practice,* and we have drawn examples from a wide variety of classroom contexts. We have examined episodes occurring in the work of prospective and practicing secondary mathematics teachers and mathematics educators at the college level. From this collection, we have determined elements of mathematical understanding that would be beneficial to secondary teachers. We describe a wide sample, as opposed to a comprehensive catalog, of mathematical understanding for teaching that comes from our analyses of these classroom episodes.

Mathematical understanding for teaching is not the same as understanding in pedagogy. Being equipped with the understanding described in our MUST framework is not simply a matter of "knowing the mathematics" plus "knowing how to teach." The task of teaching mathematics cannot be partitioned into such simple categories.

A Framework for MUST

Mathematical understanding for secondary teaching (MUST) can be viewed as having three overlapping components: mathematical proficiency, mathematical activity, and mathematical work of teaching (Figure 1). Each component emphasizes a different aspect of MUST. MUST is a developing quality and not an endpoint.

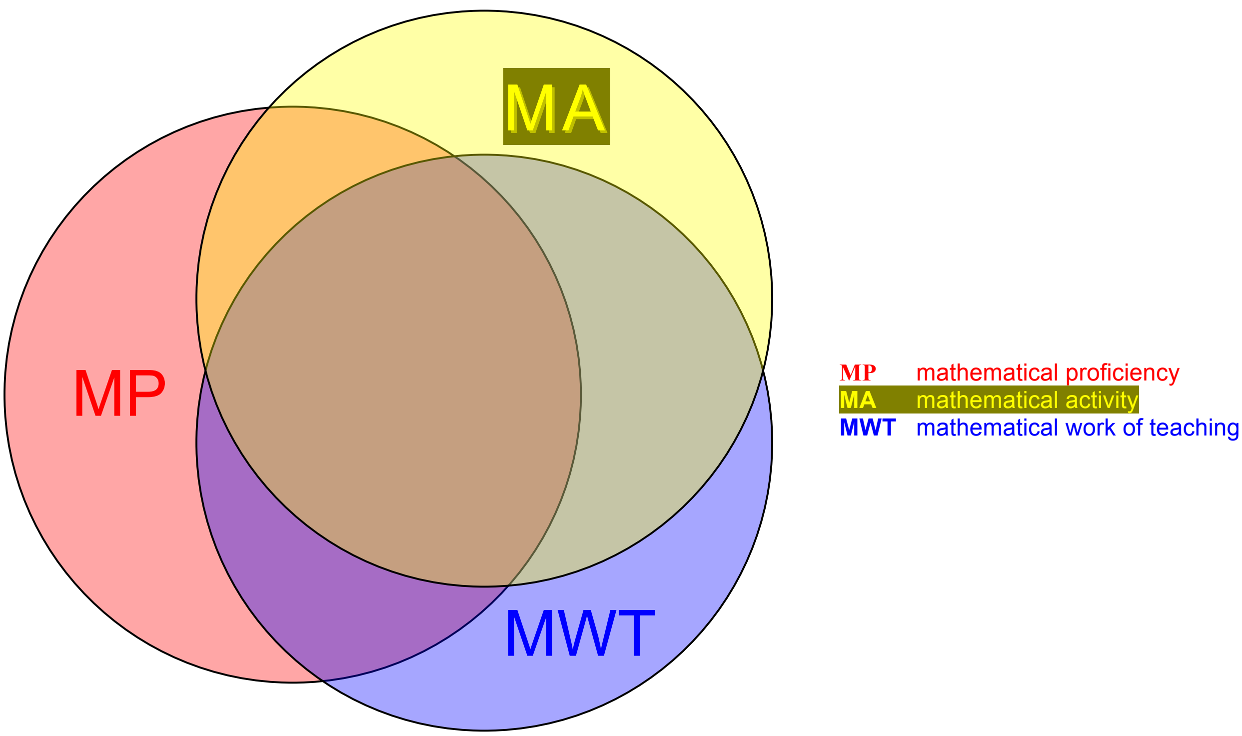

Figure 1. Three components of mathematical understanding for secondary teaching.

Mathematical proficiency includes aspects of mathematical knowledge and ability, such as conceptual understanding and procedural fluency, that teachers need themselves and that they seek to foster in their students. The mathematical proficiency teachers need goes well beyond what one might find in secondary students or even the average educated adult. Students' development of mathematical proficiency usually depends heavily on how well developed their teacher's proficiency is. Secondary teachers of mathematics need proficiency with the mathematics their students should have learned in elementary school, and they need proficiency with the mathematics their students may encounter when taking mathematics and related subjects in college.

Engaging in *mathematical activity* can be thought of as "doing mathematics." It is like, and overlaps, the mathematical proficiency component, but the emphasis is on those mathematical activities that teachers employ and that they want their students to learn. Other users of mathematics may engage in such activities from time to time, but teachers need a more conscious, elaborated command of their nature and particulars. Examples include representing mathematical objects and operations, connecting mathematical concepts, modeling mathematical phenomena, and justifying mathematical arguments. This facet of mathematical understanding for teaching is on display as teachers engage students in the day-to-day study of mathematics. Teachers need a deep knowledge, for example, of what characterizes the structure of mathematics (as opposed to conventions that have been adopted over the centuries) and how to generalize mathematical findings. The more a teacher's expertise in mathematical activity has developed, the better equipped he or she will be to facilitate the learning and doing of mathematics.

Engaging in the *mathematical work of teaching* diverges sharply from the mathematical understanding needed in other professions requiring mathematics. One of its aspects is an understanding of the mathematical thinking of students, which may include, for example, recognizing the mathematical nature of their errors and misconceptions. Teachers need to be able to decide whether a proof might be circular or incomplete, how well a proposed solution satisfies the conditions of a problem, and whether an alternative definition is equivalent to one already proposed. Another aspect of the mathematical work of teaching is knowledge of and expertise in the mathematics that comes before and after what is being studied currently. A teacher benefits from knowing what students have learned in previous years so that he or she can help them build upon that prior knowledge. The teacher also needs to provide a foundation for the mathematics they will be learning later, which requires knowing and understanding the mathematics in the rest of the curriculum.

The three components of MUST—mathematical proficiency, mathematical activity, and mathematical work of teaching—together form a full picture of the mathematics required of a teacher of secondary mathematics. It is not enough to know the mathematics that students are learning. Teachers must also possess a depth and extent of mathematical understanding that will equip them to foster their students' mathematical proficiency. Mathematical understanding informs the other two perspectives on MUST: Mathematical activity and the mathematical work of teaching emerge from, and depend upon, the teacher's mathematical understanding.

An Example of MUST Use

In responding to the following situation, no matter how it is handled pedagogically, the teacher needs to make use of all facets of his or her MUST:

> In an Algebra II class, students had just finished reviewing the rules for exponents. The teacher wrote $x^m \bullet x^n = x^5$ on the board and asked the students to make a list of values for *m* and *n* that made the statement true. After a few minutes, one student asked, "Can we write them all down? I keep thinking of more."

To decide whether the student's question is worth pursuing, frame additional questions appropriately, and know how to proceed from there, the teacher needs conceptual understanding and productive disposition (two aspects of mathematical proficiency). The concept of an exponent is more complicated than might be initially apparent. Does the rule $x^m \bullet x^n = x^{m+n}$ always apply? Must the domain of x be restricted? Must the domain of m and n be restricted? These are questions the teacher needs sufficient mathematical proficiency to address. With respect to mathematical activity, the teacher's skill in representing exponents, knowing constraints that may be helpful in dealing with them, and making connections between exponents and other mathematical phenomena are all crucial to successfully teaching the concept. What are the advantages of a graphical representation of an exponential function as opposed to a symbolic representation? How is the operation of exponentiation connected to the operation of multiplication? Does an exponent always indicate repeated multiplication? With respect to the mathematical work of teaching, it is critical that the teacher knows and understands the mathematics that typically comes before and after the point in the curriculum where a problem like the one involving the rule $x^m \bullet x^n$ is addressed. For example, if this problem is being discussed in a beginning algebra course, it is important to realize that students have probably had limited exposure to exponents and may think about them only in terms of the repeated multiplication of natural numbers. And to lay a good foundation for later studies of exponential functions, the teacher needs to know that there may be discontinuity in the graph of x^n depending on the domain of both the base and the exponent.

Elaboration of the MUST Components

The philosopher Gilbert Ryle (1949) claimed that there are two types of knowledge: The first is expressed as "knowing that," sometimes called *propositional* or *factual* knowledge, and the second as "knowing how," sometimes called *practical* knowledge. Because we wanted to capture this distinction and at the same time to enlarge the construct of *mathematical knowledge for teaching* to include such mathematical aspects as reasoning, problem solving, and disposition, we have adopted the term *understanding* throughout this document instead of using the term *knowledge*. An outline of our framework for the three components of MUST is shown in Figure 2. In this section, we amplify each component in turn.

1. **Mathematical proficiency**
    - Conceptual understanding
    - Procedural fluency
    - Strategic competence
    - Adaptive reasoning
    - Productive disposition
    - Historical and cultural knowledge

2. **Mathematical activity**
    - Mathematical noticing
        - Observing structure of mathematical systems
        - Discerning symbolic form
        - Detecting form of an argument
        - Connecting within and outside mathematics
    - Mathematical reasoning
        - Justifying/proving
        - Reasoning when conjecturing and generalizing
        - Constraining and extending
    - Mathematical creating
        - Representing
        - Defining
        - Modifying/transforming/manipulating
    - Integrating strands of mathematical activity

3. **Mathematical work of teaching**
    - Probe mathematical ideas
    - Access and understand the mathematical thinking of learners
    - Know and use the curriculum
    - Assess the mathematical knowledge of learners
    - Reflect on the mathematics of practice

Figure 2. Framework for mathematical understanding for secondary teaching (MUST).

Mathematical Proficiency

The principal goal of secondary school mathematics is to develop all facets of the learners' mathematical proficiency, and the teacher of secondary mathematics needs to be able to help students with that development. Such expertise on the teacher's part requires that the teacher not only understand the substance of secondary school mathematics deeply and thoroughly but also know how to guide students toward greater proficiency in mathematics. We have divided the teacher's mathematical proficiency into six strands, shown in Figure 2, to capture the multifaceted nature of that proficiency. We use *proficiency* in much the same way as it is used in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) except that we are applying it to teachers rather than students, and we have added a sixth strand to represent the historical and cultural knowledge of mathematics that teachers need.

There is a range of proficiency in each strand, and a teacher may become increasingly proficient in the course of his or her career. At the same time, certain categories may involve greater depth of mathematical knowledge than others. For example, *conceptual understanding* involves a different kind of knowledge than *procedural fluency,* though both are important. Only rote knowledge is required in order to demonstrate procedural fluency in mathematics. Conceptual understanding, however, involves (among other things) knowing *why* the procedures work.

Conceptual Understanding

Conceptual understanding is sometimes described as the "knowing *why*" of mathematical proficiency. A person may demonstrate conceptual understanding by such actions as deriving needed formulas without simply retrieving them from memory, evaluating an answer for reasonableness and correctness, understanding connections in mathematics, or formulating a proof.

Some examples of conceptual understanding are the following:
1. Knowing and understanding where the quadratic formula comes from (including being able to derive it).
2. Seeing the connections between right triangle trigonometry and the graphs of trig functions.
3. Understanding how the introduction of an outlying data point can affect mean and median differently.

Procedural Fluency

A person with procedural fluency knows some conditions for when and how a procedure may be applied and can apply it competently. Procedural fluency alone, however, would not allow one to independently derive new uses for a previously learned procedure, such as completing the square to solve $ax^6 + bx^3 = c$. Procedural fluency can be thought of as part of the "knowing *how*" of mathematical proficiency. Such fluency is useful because the ability to quickly recall and accurately execute procedures significantly aids in the solution of mathematical problems.

The following are examples of procedural fluency:
1. Recalling and using the algorithm for long division of polynomials.
2. Sketching the graph of a linear function.
3. Finding the area of a polygon using a formula.

Strategic Competence

Strategic competence requires procedural fluency as well as a certain level of conceptual understanding. Demonstrating strategic competence requires the ability to generate, evaluate, and implement problem-solving strategies. That is, a person must first be able to generate possible problem-solving strategies (such as utilizing a known formula, deriving a new formula, solving a simpler problem, trying extreme cases, or graphing), and then must evaluate the relative effectiveness of those strategies. The person must then accurately implement the chosen strategy. Strategic competence could be described as "knowing how," but it is different from procedural fluency in that it requires creativity and flexibility because problem-solving strategies cannot be reduced to mere procedures.

Examples of strategic competence are the following:

1. Recognizing problems in which the quadratic formula is useful (which goes beyond simply recognizing a quadratic equation or function).
2. Figuring out how to partition a variety of polygons into "helpful" pieces so as to find their areas.
3. Investigating a special case as a way to approach a problem whose solution for the general case is not immediately apparent.

Adaptive Reasoning

A teacher or student with adaptive reasoning is able to recognize current assumptions and adjust to changes in assumptions and conventions. Adjusting to these changes involves comparing assumptions and working in a variety of mathematical systems. For example, since they are based on different assumptions, Euclidean and spherical geometries are structurally different. A person with adaptive reasoning, when introduced to spherical geometry, would consider the possibility that the interior angles of a triangle do not sum to 180°. Furthermore, he or she would be able to construct an example of a triangle, within the assumptions of spherical geometry, whose interior angles sum to more than 180°.

Examples of adaptive reasoning are as follows:

1. Recognizing that division by an unknown is problematic.
2. Working with both common definitions for a trapezoid.
3. Operating in more than one coordinate system.

Productive Disposition

Those with a productive disposition believe they can benefit from engaging in mathematical activity and are confident that they can succeed in mathematical endeavors. They are curious and enthusiastic about mathematics and are therefore motivated to see a problem through to its conclusion, even if that involves thinking about the problem for an extended time so as to make progress. People with a productive disposition are able to notice mathematics in the world around them and apply mathematical principles to situations outside the mathematics classroom. They possess Cuoco's (1996) "habits of mind."

Examples of productive disposition are as follows:

1. Noticing symmetry in the natural world.
2. Persevering through multiple attempts to solve a problem.
3. Taking time to write and solve a system of equations for a real-world application such as comparing phone service plans.

Historical and Cultural Knowledge

Having knowledge about the history of mathematics is beneficial for many reasons. One prominent benefit is that a person with such knowledge will likely have a deeper understanding of the origin and significance of various mathematical conventions, which in turn may increase his or her conceptual understanding of mathematical ideas. For example, knowing that the integral symbol ∫ is an elongated *s*, from the Latin *summa* (meaning *sum* or *total*) may provide a person with insight about what the integral function is. Some other benefits of historical knowledge include an awareness of which mathematical ideas have proven the most useful in the past, an increased ability to predict which mathematical ideas will likely be of use to students in the future, and an appreciation for current developments in mathematics.

Cross-cultural knowledge (i.e., awareness of how people in various cultures or even in various disciplines conceptualize and express mathematical ideas) may have a direct impact on a person's mathematical understanding. For example, a teacher or student may be used to defining a rectangle in terms of its sides and angles, but people in some non-Western cultures define a rectangle in terms of its diagonals. Being able to conceptualize both definitions can strengthen one's mathematical proficiency.

The following are additional examples of historical and cultural knowledge:

1. Being familiar with the historic progression from Euclidean geometry to multiple geometric systems.
2. Being able to compare mathematicians' convention of measuring angles counterclockwise from horizontal with the convention (used by pilots, ship captains, etc.) of indicating directions in terms of degrees clockwise from North.
3. Understanding similarities and differences in algorithms typically taught in North America and those taught elsewhere.
4. Knowing that long-standing "open problems" in mathematics continue to be solved and new problems posed.

Mathematical Activity

Through a mathematical activity perspective, we acknowledge that mathematical knowledge has a dynamic aspect by describing mathematical actions. The categories in the dimension of mathematical activity organize the verbs of doing mathematics—the actions one takes with mathematical objects. The three strands—mathematical noticing, mathematical reasoning, and mathematical creating—intertwine in mathematical activity.

Fundamental to each strand is constraining, extending, or otherwise altering conditions and forms. Constraints can be removed, altered, or replaced to explore the resulting new mathematics. Mathematical relationships and properties can be tested for extended sets of numbers. As Cuoco, Goldenberg, and Mark (1996) argue, "Mathematicians talk small and think big" (p. 384). Teachers need to move flexibly between related small and big ideas. They use constraining, extending, and altering as ways to refine ideas to create valid statements from intuitive notions and observations.

Mathematical Noticing

The first category of mathematical activity, mathematical noticing,[1] involves recognizing similarities and differences in structure, form, and argumentation both in mathematical settings and in real-world settings. Mathematical noticing requires identifying mathematical characteristics that are particularly salient for the purpose at hand and focusing on those characteristics in the presence of other available candidates for foci.

Structure of mathematical systems. Noticing structure is foundational to making mathematical conclusions. An example of the structure on which one focuses can be the definitions and axioms that govern a mathematical system. Noticing and using the structure of mathematical systems underpins other mathematical activities such as deriving properties of a system. Whereas many users of mathematics rely on these system, form, and argumentation structures, teachers need to notice similarities and differences among the structures in varied mathematical settings.

As students proceed through secondary school mathematics, the rate of introduction of new mathematical systems increases. Although the new systems use similar operations on similar objects, teachers need to be constantly vigilant regarding the constraints under which each system operates. Teachers need to notice invariant as well as changing aspects of mathematical structure as the curriculum moves from the study of rational numbers to the study of real and complex numbers, variables, polynomials, matrices, and functions.

Examples of noticing mathematical structure are as follows:

1. Noticing the effects on a geometry when the parallel postulate is not assumed.
2. Being aware that familiar operations do not have the same meaning when applied to different mathematical objects and structures, and hence knowing not to generalize properties of multiplication over the set of real numbers to multiplication over the set of matrices.
3. Recognizing the entities of inverses and compositions across a broad range of mathematical settings.
4. Noticing connections between (and features of) different methods for solving problems (e.g., noticing the structural similarities between the Euclidean algorithm and the long division algorithm).
5. Noticing differences between the same objects in different systems (e.g., noticing the difference in solutions when solving an equation in the real number system and in the complex number system).
6. Noticing differences in algebraic structure (e.g., noticing properties of a system such as the field properties, properties of equivalence relations, and properties of

[1] Although our notion of mathematical noticing may have some features in common with Goodwin's (1994) *professional noticing,* the terms are not equivalent, and one is not derived from the other.

equality) and applying the knowledge of this structure to algebraic transformations.[2]

Symbolic form. Recognizing algebraic symbolic forms allows teachers to identify and use potential symbolic rules with those forms.

Examples of noticing symbolic forms are as follows:

1. Being aware that that the truth of $f(a) + f(b) = f(a + b)$ depends on the nature of the function f, and that students tend to apply this "student's distributive property" indiscriminately.
2. Noticing differences and similarities in notation and distinguish among the meanings of notations that are similar in appearance (e.g., noticing differences in the uses of familiar notation such as the superscript -1—as in x^{-1} and f^{-1}—depending on context, and being able to identify and explain the conditions under which specific meanings for the notation are appropriate).

Form of an argument. Secondary teachers have a particular need to notice the form of mathematical arguments, whether advanced in a textbook or by a student. Noticing the form of a mathematical argument allows teachers to identify missing elements or redundant portions of the argument.

Connect within and outside mathematics. Connecting within mathematics requires teachers to extract the characteristics and structure of the mathematics they are teaching and notice those characteristics and structure in other areas of mathematics. Teachers who notice connections between mathematical representations of the same entity and between mathematical entities and their properties can provide rich and challenging environments for their students. Such teachers are able to move smoothly from question to question, both fielding student questions and posing challenges that require students to connect mathematical ideas.

Examples of noticing connections within mathematics are as follows:

1. Noticing different manifestations and representations of the same mathematical system (e.g., noticing that paper-folding, symmetries of a triangle, paper-and-pencil games, and Escher-type drawings are all venues for studying transformations such as reflections, rotations, translations, and glide reflections, and that transformations can be represented and manipulated through matrix operations as well as through mappings on a plane).
2. Recognizing relationships between alternative algorithms, student-generated algorithms, and standard algorithms (e.g., noticing that Peasant multiplication and standard multiplication algorithms can be derived using the field properties).
3. Noticing the affordances of the different representations and that different representations highlight different strengths and weaknesses of what is being represented.

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[2] Algebraic transformations such as the production of equivalent expressions and equivalent equations are at the core of many school algebra courses.

Connecting to areas outside of mathematics requires teachers to have a disposition to notice mathematics outside of their classroom and to seek mathematical explanations for real-world quantitative relationships. The point is not that there is not some ordained list of applications that a teacher needs to know, but rather that there are intriguing topics that teachers can explore with their students by applying secondary school mathematics and that teachers should be willing and able to seek out the resources to investigate these topics. Connecting within and outside mathematics means looking for and noticing applications of mathematics as well as circumstances from which to extract mathematics, while at the same time recognizing the constraints that the context places on a mathematical result. Every teacher may not need to know something about a particular connection, but all teachers need to know the properties of the mathematical entities about which they are teaching well enough to recognize an application when they see it. This recognition involves seeing the properties of the mathematical entities well enough to match them to the situation (Zbiek & Conner, 2006).

Examples of noticing connections to the world outside of mathematics are as follows:

1. Noticing the mathematics that underpins today's electronic technology (e.g., noticing that video games employ matrix operations to animate images on the screen through geometric transformations).
2. Noticing ways that mathematics underpins different industries (e.g., noticing that designers of automobiles use Bezier curves to render pictures of new designs for cars).

Mathematical Reasoning

The second category of mathematical activity is mathematical reasoning. Mathematical reasoning includes justifying and proving as well as reasoning in the context of conjecturing and generalizing. Mathematical reasoning results in the production of a mathematical argument or a rationale that supports the plausibility of a conjecture or generalization.

Justifying/proving. Teaching mathematics well requires justifying mathematical claims through logically deduced connections among mathematical ideas. Formal justification, or proof, requires basing arguments on a logical sequence of statements supported by definitions, axioms, and known theorems, whereas informal arguments involve reasoning from empirically derived—but often limited—data, reasoning by analogy, establishing plausibility based on similar instances, and the like. When creating formal or informal arguments, teachers need to be on the alert for special cases they need in order to recognize or generate an exhaustive list of cases, and they need to recognize the limitations of reasoning from diagrams.

Teachers of secondary mathematics need a different sort of justification ability from that of other users of mathematics because they are required to formulate and structure arguments across a range of appropriate levels. Teachers need to be comfortable with a range of strategies for mathematical justification, including both formal justification and informal arguments. Secondary school mathematics teachers need to be able to understand and formulate different levels and types of mathematically and pedagogically viable justifications and proofs (e.g., proof by contradiction and proof by induction). They also need to recognize the need to specify assumptions in an argument, and they must be able to state assumptions on which a valid mathematical argument depends. Teachers' arguments often need not be as elegant as those for which mathematicians typically strive, and teachers need to be able to create proofs that explain as well as proofs that convince (Hersh, 1993).

Examples of justifying/proving include:

1. Constructing an array of justifications for why the sum of the first n natural numbers is $\frac{n(n+1)}{2}$, including appealing to cases, making strategic choices for pair-wise grouping of numbers, and appealing to arithmetic sequences and properties of such sequences.
2. Arguing by contradiction (excluded middle): To prove that if the opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle, one can construct a circumcircle through three vertices of a quadrilateral and argue that if the fourth vertex can be in neither the interior nor the exterior of the circle, then it must be on the circumcircle, and therefore the quadrilateral can be inscribed in a circle.

Reasoning when conjecturing and generalizing. In school mathematics, students (and teachers) engage in a similar activity when they develop conjectures based on their observations and data they have generated. Given a plausible conjecture—generated by the teacher or generated by students—a teacher must be able to test the conjecture with different domains or sets of objects.

Generalizing is the act of extending the domain to which a set of properties apply from multiple instances of a class or from a subclass to a larger class of mathematical entities, thus identifying a larger set of instances to which the set of properties applies. When generalizing, students may develop a formal argument that establishes the generalization as being true, and that argument must be evaluated by the teacher. In some instances, the teacher needs to produce an argument that convinces students of a generalization's truth or explains some aspect related to the statement or its domain of applicability.

Teachers also engage in mathematical reasoning in the context of conjecturing and generalizing when they create and use counterinstances of generalizations. The creation of counterinstances requires one to reason about the domain of applicability of a generalization and what results when that domain is constrained or extended. For example, the generalization that multiplication is commutative can be shown to be false when considering matrix multiplication.

Examples of reasoning when conjecturing and generalizing include:

1. Generating an example of a situation for which multiplication is not commutative.

2. Analyzing the extent to which properties of exponents generalize from natural number exponents to rational number or real number exponents,
3. Recognizing that a graphical approach to solving polynomial equations is far more generalizable than the usual set of polynomial factoring techniques.

Constraining and extending. Teachers engage in mathematical reasoning when they consider the effects of constraining or extending the domain, argument, or class of objects for which a mathematical statement is or remains valid while preserving the structure of the mathematical statement. They need to recognize when it is useful to relax or constrain mathematical conditions. The mathematical reasoning involved in constraining and extending enables teachers to create extensions to given problems and questions.

With secondary school mathematics as the bridge between prealgebra mathematics and collegiate mathematics, secondary mathematics teachers are often challenged to explore the consequences of imposing or relaxing constraints. To *constrain* in mathematics means to define the limits of a particular mathematical idea. Constraints can be removed or replaced to explore the resulting new mathematics. When mathematicians tinkered with the constraint of Euclid's fifth postulate, new geometries were formed. When one removes the constraint of the plane in using Euclidean figures, the mathematics being used changes as well. Secondary mathematics teachers regularly encounter situations in which to provide a suitable response, they must tailor a generalization so that it can reasonably be extended to a larger domain of applicability. Teachers with an understanding of the mathematics their students will encounter in further coursework can structure arguments so that they extend to a more general case.

Examples of constraining and extending are the following:

1. When finding the inverse of a function, one must sometimes constrain the domain if one wants the inverse to be a function as well. The inverse of $f(x) = \sin x$ is a function only if the new domain is restricted.
2. Extending the concept of absolute value to a modulus definition as the domain is extended from real to complex numbers.
3. Extending the object "triangle" from Euclidean to spherical geometry.
4. Being cautious of extending the rules of exponents, developed and proved for natural number exponents, to negative, rational, real, or complex exponents.

Mathematical Creating

The essence of mathematical creating is the production of new mathematical entities through the mathematical activities of representing, defining, and transforming.

Representing (new way to convey). Inherent in mathematical work is the need to represent mathematical entities in ways that reflect given structures, constraints, or properties. The creation of representations is particularly useful in creating and communicating examples, nonexamples, and counterexamples for mathematical objects, generalizations, or relationships. Teachers need to be fluent in the rapid construction of representations that underscore key features of the represented entity.

Each representation affords different views of the mathematical object, but several different representations can highlight the same feature. Teachers need to be able to assess what features of the mathematical entity each form captures and what features it obscures. Representing involves choosing or creating a useful form that conveys the crucial aspects of the mathematical entity that are needed for the task at hand.

Some types of mathematical representations are common. Teachers need to be able to create representations of those common forms in ways that reflect conventions. Teachers also need to be able to create representations of less common and even novel forms. In this activity, attention to structures, constraints, and properties is critical.

Defining (new object). The mathematical activity of defining is the creation of a new mathematical entity by specifying its properties. Generating a definition requires identifying and articulating a combination of a set of characteristics and the relationships among these characteristics in such a way that the combination can be used to determine whether an object, action, or idea belongs to a class of objects, actions, or ideas.

Teachers of secondary mathematics need to be able to appeal to a definition to resolve mathematical questions, and they need to be able to reason from a definition. Less frequently, teachers need to create definitions and to assess the definitions that students create or propose.

Modifying/transforming/manipulating (new form). Perhaps the most recognizable form of transforming is symbolic manipulation. Teachers need to see these transformations as purposeful activities undertaken to produce a symbolic form that conveys particular information. Transformations of graphs (e.g., window changes, bin sizes) to create more meaningful representations are similarly important. Whether technology supported or completed by hand, transforming one representation to another representation (of a similar or different form) is fundamental in solving problems.

Integrating Strands of Mathematical Activity

Mathematical modeling provides one example of how the strands of mathematical activity intertwine in mathematical work. A popular description of the modeling process starts with a real-world problem that is translated into a formal mathematical system. Mathematical noticing occurs as the modeler specifies the conditions and assumptions that matter in the real-world setting. Devising the model requires mathematical creating informed by mathematical reasoning. After a potential model has been generated, mathematical creating takes over as the model is manipulated until a solution is found. The solution is mapped back to the real world to be tested with the problem through mathematical noticing. If a real-world conclusion does not align with the modeling goal, aspects of the model, such as initial conditions that are assumed, may be constrained, expanded, or altered to form a new model. It is important to note that the issue is one of fit and utility rather than absolute correctness.

Mathematical modeling activities in secondary school might involve authentic modeling tasks that involve the generation of novel models or more restricted modeling work that is done in the service of students learning curricular mathematics, the mathematics that is the focus of classroom lessons (Zbiek & Conner, 2006). A close analysis of mathematical modeling across these contexts suggests that mathematical modeling is a nonlinear process that incorporates the three strands of mathematical activity (for an elaboration of modeling activities, see Zbiek & Conner, 2006).

Mathematical Work of Teaching

Not only should teachers of secondary mathematics be able to know and do mathematics themselves, but also their proficiency in mathematics must prepare them to facilitate their students' development of mathematical proficiency. In Ryle's (1949) terminology, the mathematical work of teaching requires both knowing how and knowing that. It moves beyond the goal of establishing a substantial and continually growing proficiency in mathematics for oneself as a teacher to include the goal of effectively helping one's students develop mathematical proficiency. Possessing proficiency in the mathematical work of teaching mathematics enables teachers to integrate their knowledge of content and knowledge of processes to increase their students' mathematical understanding.

Analyze Mathematical Ideas

Analyzing mathematical ideas requires investigating and pulling apart mathematical ideas. Mathematics is dense. One goal in doing mathematics is to compress numerous complex ideas into a few succinct, elegant expressions. Although mathematical efficiency and rigor are essential if one is to engage in complex mathematical thinking, they can also cause confusion, especially for those just being initiated into the culture of mathematics.

Analyzing mathematical ideas also requires a broad knowledge of mathematical content and associated mathematical activities such as defining, representing, justifying, and connecting. Teachers need mathematical knowledge that will help them pull apart mathematical ideas in ways that allow the ideas to be reassembled as students mature mathematically. They need to recognize and honor the conventions and structures of mathematics and recognize the complexity of elegant mathematical ideas that have been compressed into simple forms.

Examples of analyzing mathematical ideas are as follows:

1. Understanding the role of the domain in determining the values for which a function is defined.
2. Recognizing the similarities and differences between multiplying real numbers and matrices.

3. Exploring the standard deviation of a set of data in terms of an average distance each value is from the mean of the set of data.
4. Exploring the various meanings of division—partitive and quotitive—to recognize that division is more than just the inverse of multiplication.

Access and Understand the Mathematical Thinking of Students

Mathematics teachers should be proficient in understanding how their students are thinking about mathematics. A proficient teacher uncovers students' mathematical ideas in a way that helps them see the mathematics from a learner's perspective. Teachers can gain some access to students' thinking through the written work they do in class or at home, but much of that information is highly inferential. Through discourse with students about their mathematical ideas, the teacher can learn more about the thinking behind the students' written products. Classroom interactions play a significant role in teachers' understanding of what students know and are learning. It is through a particular kind and quality of discourse that implicit mathematical ideas are exposed and made more explicit.

Students often discuss mathematics using vague explanations or terms that have a colloquial meaning different from their mathematical meaning. A teacher needs the proficiency to interpret imprecise student explanations, help students focus on essential mathematical points, and help them learn conventional terms. Success in such endeavors requires understanding the nuances and implications of students' understanding and recognizing what is right about their thinking as well as features of their thinking that lead them to unproductive conceptions. Achieving such a balance requires the teacher to have an extensive knowledge of mathematical terminology, formal reasoning processes, and conventions, as well as an understanding of differences between colloquial uses and mathematical uses of terms.

Examples of mathematical knowledge needed for accessing students' mathematical knowledge are as follows:

1. Determining whether a student means *face* or *edge* (or something else) in using the word *side* in a discussion of Platonic solids.
2. Using a collection of useful representations (e.g., graph, table, drawing, or set of examples) that may help a student share mathematical ideas.
3. Designing mathematical tasks that expose students thinking and maintain a high level of cognitive demand as the tasks are discussed.

Know and Use the Curriculum

Teachers use the curriculum to help students connect mathematical ideas and progress to a deeper and better grounded mathematics. How mathematical knowledge is used to teach mathematics in a specific classroom or with a specific learner or specific group of learners is influenced by the curriculum that helps organize teaching and learning. A teacher's mathematical proficiency can help make that curriculum meaningful, connected, relevant, and useful.

Proficiency in knowing and using the mathematics curriculum requires a teacher to identify foundational or prerequisite concepts that enhance the learning of a concept as well as how the concept being taught can serve as a foundational concept for future learning. The teacher needs to know how the concept fits each student's learning trajectory. The teacher also needs to be aware of common mathematical misconceptions and how those misconceptions may sometimes arise at particular points in this trajectory. Proficient mathematics teachers understand that there is not a fixed or linear order for learning mathematics but rather multiple ways to approach a mathematical concept and to revisit it. Mathematical concepts and processes evolve in the learner's mind, becoming more complex and sophisticated with each iteration. Mathematical proficiency prepares a teacher to enact a curriculum that not only connects mathematical ideas explicitly but also develops a disposition in students so that they expect mathematical ideas to be connected and an intuition so that they see where those connections might be (Cuoco, 2001).

A teacher proficient in the mathematical work of teaching understands that a curriculum contains not only mathematical entities but also mathematical processes for relating, connecting, and operating on those entities (National Council of Teachers of Mathematics, 1989, 2000). A teacher must have such proficiency to set appropriate curricular goals for his or her students (Adler & Davis, 2006).

Examples of knowing and using the curriculum include the following:

1. Understanding the concept of area in a way that includes ideas about measure, descriptions of two-dimensional space, measures of space under a curve, measures of the surface of three-dimensional solids, infinite sums of discrete regions, operations on space and measures of space, foundations of the geometric properties of area, and useful applications involving area.
2. Selecting and teaching functions in a way that helps students build a basic repertoire of functions (Even, 1990).

Assess the Mathematical Knowledge of Learners

Assessing the mathematical knowledge of learners is an integral component of the mathematical work of teaching. During each class, teachers must exhibit a mathematical proficiency that enables them to assess or evaluate students' mathematical understanding. Such assessment is crucial not only for recognizing student error but also in determining where students are mathematically for purposes of developing tasks and planning lessons. Assessing students' mathematical knowledge involves much more than assessing a student's ability to follow a procedure. Teachers should possess a mathematical proficiency for teaching that helps them identify the essential components of mathematical concepts so they can in turn assess a student's ability to use and connect these essential ideas. Determining how students are progressing in class is at the heart of assessing the mathematical knowledge of learners. To determine the mathematical progress of their students, teachers must be attentive to the errors students frequently make.

Examples of assessments that teachers frequently perform are as follows:

1. Realizing that a mathematical error may be located in how the student is using and understanding mathematical language, or colloquial language such as "canceling out."
2. Choosing examples for finding solutions of quadratic equations graphically so that the set of examples includes equations with no, one, and two solutions.
3. Using open-ended questions to draw out the source of the student's confusion about the difference between the area and circumference of a circle.
4. Recognizing the common error of finding the reciprocal or multiplicative inverse of a function when asked to find its inverse.

Reflect on the Mathematics in One's Practice

Teachers should be proficient in analyzing and reflecting on their mathematics teaching practice in a way that enhances their mathematical proficiency. There are many ways to reflect on one's practice, and one of the most important is to use a mathematical lens. How did the mathematical complexity of the problem in this lesson change when the students were given a hint? Which of several equivalent definitions is most appropriate when this term is introduced? How was the topic of that test question connected to a topic treated earlier in the course? Thoughtful reflection on problems of practice can be reconsideration of a lesson just taught, or it can be part of the planning for a future lesson. It may occur as the teacher interprets the results of a formal assessment, or it may be prompted by a textbook treatment of a topic.

Teachers are often reflecting about their teaching as they teach—as they are making split-second decisions. A teacher's decisions about how to proceed after accessing student thinking depend on many factors, including the mathematical goals of the lesson. It is valuable to revisit these quick reflections and decisions when there is time to think about the mathematics one might learn from one's practice.

Examples of reflection on the mathematics in one's practice are as follows:
1. Identifying an unconventional notation that students are using and contrasting its properties with those of conventional notation.
2. Analyzing how the topic of a lesson might be presented so as to show mathematics as culturally situated.
3. Modifying a mathematical conjecture so that it could be proved in other ways.

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