Situations Project

Facilitators' Guide

Sampler:
Division Involving Zero

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D R A F T

The MAC-CPTM Situations Project

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Every day mathematics teachers encounter incidents that call on their mathematical understandings in the course of the work of planning and teaching mathematics. A given incident can evoke a range of mathematical understandings, but not all of those understandings are immediately available to teachers. Recognizing the need to understand what mathematics secondary teachers can productively use, mathematics educators at Pennsylvania State University and the University of Georgia, over the past several years, have collected descriptions of incidents they had witnessed in the work of teaching and developed descriptions of the mathematical ideas that a teacher might productively use in incidents like the ones described.

Each of these paired descriptions of incidents and related mathematical ideas constitutes what the creators called a Situation. Each Situation consisted of a Prompt (a description of the incident), several Mathematical Foci (descriptions of different mathematical ideas), and a Commentary (a discussion of overarching ideas in and connections among the Mathematical Foci). The Situations Project created, refined, and revised over 50 of these Situations that address algebra, number, geometry, and statistics. Each Situation delves more deeply into mathematical ideas on which school mathematics focuses.

They used an in-depth analysis of these Situations to create a framework for Mathematical Understanding for Secondary Teaching. The framework, to appear in a book now under development, identifies and elaborates on several categories of mathematical understanding that secondary mathematics teachers could productively use. The categories focus on three arenas: the mathematical proficiencies that secondary teachers develop (e.g., conceptual understanding, strategic competence), mathematical activities, and the settings in which teachers need to draw on mathematical understandings.

NCSM is collaborating with the two universities to create a facilitator's guide for the use of the Situations. The purpose of this draft facilitator's guide sample is to provide an example of a Situation and ideas for how professional developers or teacher educators might use such a Situation in a session for prospective or practicing teachers. The Situation selected for this document is one that resonates with the experience of many teachers. Every mathematics teacher at one time or another is faced with the dilemma of interpreting expressions such as $\frac{n}{0}$ or $\frac{0}{0}$. Students may ask about it, textbooks may introduce the idea, or the complete development of an idea requires dealing with interpretation of such indicated calculations. In the next few pages are ideas for exploring in a professional development or teacher preparation context a Situation about division involving 0. Although the Situations were developed form teaching dilemmas in secondary classrooms, some of them are dilemmas for elementary teachers as well, as can be seen in Dividing Involving Zero.

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MAC-MTL, CPTM, and NCSM leaders who worked on this facilitator's guide are Glen Blume, Diane Briars, M. Kathleen Heid, Suzanne Mitchell, Connie Schrock, James Wilson, and Rose Mary Zbiek.

Division Involving Zero

Overview

Situation

This situation addresses the possible values that result when zero is the dividend, the divisor, or both the dividend and the divisor in a quotient, i.e.,

- 0 ÷ n = ?
- n ÷ 0 = ? , n ≠ 0
- 0 ÷ 0 = ?

Relevance

- Preservice and inservice teachers at all levels have misconceptions about division involving zero, including that 0 ÷ 0 = 0, or 1.
- Even when teachers know correct answers for division with 0 problems, their understanding may be limited, i.e., they cite rules as the reason and may not be able to provide a valid mathematical explanation.

Goals

- Increase teachers' understanding of why dividing zero by a non-zero number is 0 and division by zero is undefined or indeterminate.
- Clarify common misconceptions about division involving zero.
- Consider how and when to address this issue with their students.

Key Mathematical Topics

- When zero is the dividend, the divisor, or both the dividend and the divisor in a quotient, the value of such a quotient would be zero, undefined, or indeterminate, respectively.
- The difference between undefined and indeterminate.
- Connections of division involving zero to ratios, factor pairs, Cartesian product (Focus 3), area of rectangles (e.g., Focus 4), and the real projective line (Focus 5).

Common Core Connections

CCSSM Standards for Mathematical Content	3.OA.4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = □ ÷ 3, 6 × 6 = ?.* ***What happens when 0 is one of the three numbers?*** 5.NF. 7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. *For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3.* ***Why non-zero?*** c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. *For*

	example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins? ***We have ½, 1/3, ¼, and so on. Why not 1/0?*** **7.NS.2.** Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers. b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real world contexts. ***Why non-zero divisor?*** **N.NR.3.** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. ***Is the product of a whole number and an irrational number an irrational number? Explain.*** **A-APR** 7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions. ***How does understanding division by 0 with rational numbers help us work with rational expressions?***
CCSSM Standards for Mathematical Practice	SMP2. Reason abstractly and quantitatively SMP3. Construct viable arguments and critique the reasoning of others SMP6. Attend to precision

Suggestions for Using This Situation	
Tools Calculators and/or other computing technologies (e.g., Excel, smart phone) Graphing application (e.g., graphing calculator, dynamic geometry system, Core Math Tools) Poster papers, markers Copies of the Prompt (separate from the Foci)	**Time** 2 to 3 hours, can be done in a single session or across multiple sessions

Division Involving Zero

Copies of the Foci	

Outline of Activities

Participants:

1. Read the prompt, think about their own answers for each problem in the prompt ($2 \div 0 = ?$; $0 \div 0 = ?$, $0 \div 2 = ?$), then discuss their answers to each problem with their group members or a partner.
2. Explore answers for each problem obtained from different computing technologies.
3. Re-evaluate original responses in light of answers obtained from technologies, and consider mathematical ideas that could be used to support the answers they now think are correct.
4. Analyze and discuss the Foci, then structure an argument to present to the entire group.
5. Closure and assessment

Division Involving Zero

MAC-CPTM Situations Project

Situation 46: Division Involving Zero[1]

Prompt

On the first day of class, preservice middle school teachers were asked to evaluate $\frac{2}{0}$, $\frac{0}{0}$, and $\frac{0}{2}$ and to explain their answers. There was some disagreement among their answers for $\frac{0}{0}$ (potentially 0, 1, undefined, and impossible) and quite a bit of disagreement among their explanations:

- Because any number over 0 is undefined;
- Because you cannot divide by 0;
- Because 0 cannot be in the denominator;
- Because 0 divided by anything is 0; and
- Because a number divided by itself is 1.

[1] Written and edited by Bradford Findell, Evan McClintock, Glen Blume, Ryan Fox, Rose Mary Zbiek, and Brian Gleason.

Division Involving Zero

Commentary

The mathematical issue centers on the possible values that result when zero is the dividend, the divisor, or both the dividend and the divisor in a quotient. The value of such a quotient would be zero, undefined, or indeterminate, respectively. The foci use multiple contexts within and beyond mathematics to represent and illustrate these three possibilities. Connections are made to ratios, factor pairs, Cartesian product, area of rectangles, and the real projective line.

Mathematical Foci

Mathematical Focus 1
An expression involving real number division can be viewed as real number multiplication, so an equation can be written that uses a variable to represent the number given by the quotient. The number of solutions for equations that are equivalent to that equation indicate whether the expression has one value, is undefined, or is indeterminate.

We can think of a rational number as being the solution to an equation. If division expressions involving zero also represent rational numbers, we should have consistent results when we examine equations involving these expressions. To find the solution of the equation $\frac{0}{2} = x$, we consider the equivalent statement $2x = 0$, which yields the unique solution $x = 0$. To see the impossibility of a numerical value for a rational number with a 0 in the denominator, we consider the equation $\frac{0}{0} = x$, and its potentially equivalent equation, $0x = 0$. Because any value of x is a solution to this equation, there are infinitely many solutions; hence, no unique solution, and so the expression $\frac{0}{0}$ is indeterminate. With the same thinking, if $\frac{2}{0} = x$, then $0x = 2$. No real number x is a solution to this equation, and so the expression $\frac{2}{0}$ is undefined.

Mathematical Focus 2
One can find the value of whole-number division expressions by finding either the number of objects in a group (a partitive view of division) or the number of groups (a quotative view of division).

In partitive division, we take a total number of objects and divide the objects equally among a number of groups. A non-zero example would be $\frac{12}{3}$, where we share 12 objects equally among 3 groups and ask how many objects would be in one group. Similarly, $\frac{0}{2}$ can be thought of as 0 objects in 2 groups, which means 0 objects in each group. Additionally, the expression $\frac{0}{0}$ is a model for dividing 0 objects among 0 groups. In other words, "If 0 objects are shared by 0 groups, how many objects are in 1 group?" There is not enough information to answer this question, and so the expression $\frac{0}{0}$ is indeterminate. If the number of objects in a group is 3, or 7.2, or any size at all, 0 groups would have 0 objects. Similarly, $\frac{2}{0}$ is a model for the example: "If 2 objects are shared by 0 groups, how many objects are in 1 group?" In this case the number of objects in the group is undefined, because there are 0 groups.

Using a quotative view of division, we interpret the expression, $\frac{12}{3}$ as a model of splitting 12 objects into groups of 3 and asking how many groups can be made. So $\frac{0}{2}$ can be thought of as splitting 0 objects in groups of 2, which means 0 groups of size 2. The expression $\frac{0}{0}$ models the splitting of 0 objects into groups of size 0, and asks how many groups can be made. Because there could be any number of groups, there are an infinite number of solutions, and so the expression is indeterminate. Lastly, the expression $\frac{2}{0}$ models the splitting of 2 objects into groups of 0, and asking how many groups can be made. Regardless of how many groups of 0 we remove, no objects are removed. Therefore, the number of groups is undefined.

Mathematical Focus 3

The mathematical meaning of $\frac{a}{b}$ *(for real numbers* a *and* b *and sometimes, but not always, with* $b \neq 0$*) arises in several different mathematical settings, including: slope of a line, direct proportion, Cartesian product, factor pairs, and area of rectangles. The meaning of* $\frac{a}{b}$ *for real numbers* a *and* b *should be consistent within any one mathematical setting.*

Division Involving Zero

There are mathematical situations in which ratios are necessary, and a quotient can be reinterpreted as a ratio. For example, the slope of a line between two points in the Cartesian plane can be defined as the ratio of the change in the *y*-direction to the change in the *x*-direction, or as the rise divided by run. In the case of two coincident points, the change in the *y*-direction and the change in the *x*-direction are both 0, which means that the rise divided by run is $\frac{0}{0}$. There are an infinite number of lines through two coincident points, and so the slope is indeterminate. In the case of two points lying on the same vertical line whose *y*-coordinates differ by *a*, the change in the *y* direction will be *a* and the change in the *x* direction is 0. It might be tempting to claim that since the slope of a vertical line is undefined, that $\frac{a}{0}$ is undefined. However, this claim is exactly what we are trying to show.

The model for direct proportion, $y = kx$, suggests a family of lines through the origin. For *y* and non-zero *x* as the coordinates of points on a line given by $y=kx$, the ratio $\frac{y}{x}$ equals *k*, which is constant. If this ratio held for the coordinates of the origin, it would be $\frac{0}{0} = k$. However, no one value of *k* would make sense as the value of $\frac{0}{0}$ because the origin is on every line represented by an equation of the form $y = kx$. Thinking about the equation $y = kx$ in terms of number relationships also leads to the conclusion that the value of $\frac{0}{0}$ cannot be determined: if $y = kx$ and $x = 0$, then $y = 0$ and *k* can be any real number, just as in Focus 1. It is important to note that in the case where $x=0$ and $y\neq 0$, such as $\frac{2}{0}$, it is difficult to explain via direct proportion; if $y = kx$, then $x=0$ and $y\neq 0$ is an impossible circumstance.

A different mathematical context for looking at division involving zero is the Cartesian product. A non-zero example is this: if 12 outfits can be made using 3 pairs of pants and some number of shirts, how many shirts are there? There must be 4 shirts, as this would give 12 pants/shirt combinations. Similarly, if 0 outfits can be made using 2 pairs of pants and some number of shirts, there must be 0 shirts. If 0 outfits can be made using 0 pairs of pants and some number of shirts, the number of possibilities for the number of shirts is infinite. Lastly, how many shirts are there if there are two outfits and 0 pairs of pants? No possible number of shirts can be used to make 2 outfits if there are 0 pairs of pants.

In the context of factor pairs, a division expression with an integral value represents an unknown factor of the dividend. For $\frac{12}{3}$, 3 and the quotient are a factor pair for 12. In this expression, 12 can be written as the product of 3 and the quotient: $12 = 3 \times 4$. For $\frac{0}{2}$, 2 and the quotient are a factor pair for 0. Therefore, the quotient must be 0, because $0 \times 2 = 0$. For $\frac{0}{0}$, 0 is part of an infinite number of factor pairs for 0 and so the expression is indeterminate. For $\frac{2}{0}$, 0 is not part of any factor pair for 2, thus the expression is undefined.

One side length of a rectangle is the quotient of the area of the rectangle and its other side length. Suppose we allow that rectangles can have side lengths of 0. If a rectangle has area 12 and height 3, what is its width? It would be a width of 4. If a rectangle has area 0 and length 2, its width is 0 and so 0 divided by 2 is 0. If a rectangle has area 0 and height 0, what is its width? Any width is possible and so 0 divided by 0 is indeterminate. If a rectangle has area 2 and height 0, what is its width? It is impossible for a rectangle to have area 2 and height 0 and so 2 divided by 0 is undefined.

Mathematical Focus 4
Contextual applications of division or of rates or ratios involving 0 *illustrate when division by* 0 *yields an undefined or indeterminate form and when division of* 0 *by a non-zero real number yields* 0.

If Angela makes 3 free throws in 12 attempts, what is her rate? If Angela makes 0 free throws in 2 attempts, her rate is 0. If Angela makes 0 free throws in 0 attempts, her rate could be any of an infinite number of rates. On the other hand, since it is not possible for Angela to make 2 free throws in 0 attempts, it is not possible to determine her rate.

Determining the speed of an object over a given period of time is another rate context. If one goes 12 miles in 3 hours, how fast is one going? The answer is 4 miles per hour. If one goes 0 miles in 2 hours, one is going 0 miles per hour. If one goes 0 miles in 0 hours, how fast is one going? An infinite number of speeds are possible. If one goes 1 mile in 0 hours, how fast is one going? This travel situation is an impossible setting. [Note that there is a sense of infinite speed here, so it might be tempting to define $\frac{1}{0}$ as infinity. However, this leads to further complications, as in Focus 5.]

Additionally, the idea of rate is prevalent when discussing the unit price, as when purchasing multiple quantities of an item in a store. If $12 buys 3 pounds of tomatoes, how much is 1 pound? If $0 buys 2 pounds of tomatoes, then 1 pound can be bought for $0. If $0 buys 0 pounds of tomatoes, there is an infinite number of possible costs for 1 pound. If $2 buys 0 pounds of tomatoes, it is not possible to determine the number of dollars needed to buy 1 pound.

Mathematical Focus 5

Slopes of lines in two-dimensional Cartesian space map to real projective one-space in such a way that confirms that the value of $\frac{a}{b}$ *when* b = 0 *is undefined if* a ≠ 0 *and indeterminate if* a = 0.

In the Cartesian plane, consider the set of lines through the origin, and consider each line (without the origin) to be an equivalence class of points in the plane.

Except when $x = 0$, the ratio of the coordinates of a point gives the slope of the line that is the equivalence class containing that point. The origin must be excluded because it would be in all equivalence classes, which is rather like saying that $\frac{0}{0}$ would be the slope of any line through the origin [see Focus 3]. Note that the slope of a line through the origin is equal to the y-coordinate of the intersection of that line and the line $x = 1$. This way, we can use slope to establish a natural one-to-one correspondence between the equivalence classes (except for the equivalence class that is the vertical line, since it does not intersect the line $x = 1$) and the real numbers. Thus, the real numbers give us all possible slopes, except for the vertical line.

When $x = 0$, all the points in the equivalence class lie on the vertical line that is the y-axis. (Again the origin must be excluded from this equivalence class.) The ratio of the coordinates is undefined, so the slope is undefined. As positively sloped lines approach vertical, their slopes approach ∞, suggesting the slope of the vertical line to be ∞. As negatively sloped lines approach vertical, their slopes approach −∞, suggesting the slope should instead be −∞.

However, there is only one vertical line through the origin, so it cannot have two different slopes. To resolve this ambiguity, we can decide that ∞ and –∞ are the same "number" because they should represent the same slope. So now, if we think about all possible slopes, we have all real numbers and one more number, which we will call ∞. Imagine beginning with the extended real line, $\Re \cup \{\infty, -\infty\}$, and gluing together the points ∞ and –∞ so that they are the same point. This is the real projective one-space, $\Re \cup \{\infty\}$.

Post-Commentary

For situations involving division with zero, there are three types of forms: 0, undefined, and indeterminate. The indeterminate form has particular importance in a calculus setting in that: given a function, *f*, that would be continuous everywhere except that *f*(*a*) is indeterminate, we can select a functional value to make a related function that is continuous everywhere. For all of its domain values except *a*, the new function would have the same values as the given function. For example, in the case of the function $f(x)=\frac{\sin x}{x}$, the function is continuous for all real numbers except 0, for the functional value at $x = 0$ is the indeterminate form $\frac{0}{0}$. The piecewise-defined function, $f(x)=\begin{cases}\frac{\sin x}{x}, x \neq 0 \\ 1, x=0\end{cases}$ is continuous for all real numbers. In this case, we used the fact that the limit of interest was 1: $\lim_{x\to 0}\frac{\sin x}{x}=1$. However, in other cases, limits related to $\frac{0}{0}$ do not have to be 1, or even an integer. For example, $\lim_{x\to 0}\frac{2\sin x}{3x}=\frac{2}{3}$. These are but two examples that show that, depending on the function, we would find it useful to assign two different numerical values for a limit involving $\frac{0}{0}$. The very ambiguity of this form suggests the need for L'Hôpital's rule.

Division Involving Zero

Facilitator Notes

About the Mathematics

Distinguish undefined and indeterminate. Include an example where undefined and indeterminate are clearly different—i.e., something is indeterminate but still defined.

Often when values are excluded from notation, log (base b) x, b cannot be negative, assumption is that function is undefined for the excluded values.

Reference: Division involving 0; School Science and Mathematics, Volume 75, Issue 7, Reys & Grouws

Launch:

In the launch, establish the importance of this idea and the need for teachers to understand the underlying mathematics so that they can help students understand division involving zero and not just give students "the rules." There may be a subset of participants who do not think this is important to clarify, that they are currently addressing it sufficiently, or that giving the rule is sufficient. To address this, ask participants why understanding this is important and where this idea occurs in the mathematics curriculum. Occurrences include:

- Basic division facts: What is $0 \div 3$? $3 \div 0$? $0 \div 0$?
- Definition of rational numbers as a/b where a and b are integers, $b \neq 0$.
- Slope of a vertical line is undefined.
- Product of a non-zero rational number and an irrational number is an irrational number.
- Throughout mathematics, statements often have exclusions involving zero.

The table at the beginning of this section identifies where division involving zero occurs in the Common Core Mathematics Standards.

Be sure participants understand the notation in the prompt, e.g., that 2/0 indicates division, i.e., $2 \div 0$.

1. **Read the prompt, think about their own answers for each problem in the prompt ($2 \div 0 = ?$, $0 \div 0 = ?$, $0 \div 2 = ?$), then discuss their answers to each problem with their group members or a partner. Include whether these questions have ever come up, or might come up, in their classroom and how they have or would address them with students.**

   Time: 7 - 10 minutes

   Participants' Anticipated Responses: In addition to the responses given in the prompt, participants may think there are multiple answers, e.g., $0 \div 0$ is sometimes 0, sometimes 1; or that anything divided by zero is zero (Ball, 1990).

   Facilitating the Activity: Let participants discuss in their groups; however, do not have groups report out. Chances are at least some of the participants' responses will be incomplete or incorrect. Purpose is for participants to reflect on their current knowledge of division involving zero, make different ideas public, including misconceptions, rather than to reach conclusions. This is an opportunity for you to assess the mathematics background of different groups and determine which Focus might be most useful for each group to analyze in #4 below.

2. **Explore answers for each problem obtained from different computing technologies.**

   Time: 10 minutes

   Anticipated Responses from various computing technologies: See Table 1 at the end of the Facilitator Notes.

   Facilitating the Activity: Purpose is for participants to recognize that different technologies provide different results for these different problems, so that technology is not going to resolve the issue for us. Consequently, this is a problem worth analyzing in depth. This also illustrates that results obtained from computing technologies are not always correct. Technology is not the focus of the activity, so do not spend time trying to explain how the technology arrives at different results, etc.

   Do not spend much time on this activity. In fact, ***this activity may be unnecessary if participants use technology to determine their answers in #1.***

   Assign each pair or group of participants to get the answers to the three problems from one or two computing technologies, depending on the size of the group and number of technologies to investigate. We recommend looking at the results from at least five (reviewers: ***is that enough?***) different technologies. ***Be sure to try out the technologies yourself***; even though we found the different results in the table above, technology programs change so current results may differ from those listed.

   Another approach is to have a ***technology scavenger hunt***, i.e., challenge each pair or group to find as many different answers to these problems as possible. Be prepared with sources that provide different answers (such as those listed in the table) in case participants do not find a range of answers.

   Create a list of the different results for each problem, similar to the table above.

   Issue for reviewers: To what extent does technology help or hinder investigation.

3. **Re-evaluate original responses in light of answers obtained from different technologies and consider mathematical ideas that could be used to support the answers they now think are correct.**

   Time: 5 minutes

   Participants' Anticipated Responses: Interpretation in terms of:

   - A rule; that's the way it is
   - Concrete situations (Foci 2 & 3)
   - Equations (Focus 1)
   - Division as partitioning (Focus 2)
   - Rational numbers as division (Focus 3)
   - Ratio and rate (Focus 4)
   - Slope (Focus 5)

   Facilitating the Activity: Purpose is to learn what mathematics each group is thinking about and with which they are familiar. This may aid in assigning a focus to each group, if that is the approach to be taken.

   Ask each group to discuss the mathematics they would use to support their solutions. Collect ideas from each group on a poster.

4. **Analyze and discuss the Foci, then structure an argument to present to the entire group.**

   Time: 1.5 to 2 hours

   Facilitating the activity: This activity may be structured in a number of ways, depending on the mathematical understanding of the participants, their instructional level (elementary, middle or high school), and the amount of time available. The amount of emphasis on particular foci will also depend on participants' understanding and background.

   Prompt for group:

   - Analyze your assigned focus or foci.
   - Create a 3 to 7 minute explanation using this mathematical focus (or foci) to explain the three cases involving division with zero—$0 \div 2$, $2 \div 0$, $0 \div 0$ -- to the whole group ***or*** to the preservice middle school teachers in the Situation prompt.

Option 1: All groups analyze all foci

Groups should be prepared to present any of the foci. Consider asking participants to compare/contrast the foci (what's similar, what's different), and/or rank the foci in terms of which they find most compelling or transparent for students that they teach. (Note: Asking participants to rank items often produces richer discussion than simply asking them to discuss or analyze each one.) Also consider asking how each focus advances their own understanding of division involving zero.

Option 2: Each group analyzes only one focus or a subset of foci

Assign a difference focus or foci to each group. In doing so, take into consideration what you know about their mathematics background and analysis of division involving zero in parts 1, 2 and 3. Each group should prepare a presentation about their focus. While listening to presentations of other groups about their foci, participants should be asked to compare and contrast their focus with that of others, and also consider how each focus advances their own understanding of division involving zero.

Key Points About the Foci

- Note the difference between $n \div 0$ and $0 \div 0$. $n \div 0$ is ***undefined***—there is no solution to the corresponding multiplication equation; $0 \div 0$ is ***indeterminate***—there are infinitely many solutions the corresponding multiplication equation. Even though textbooks typically refer to division by zero as being "undefined", use of this term to describe the case $0 \div 0$ is mathematically imprecise.
- Focus 1 addresses division involving zero through corresponding multiplication equations.
- Focus 2 addresses division involving zero through the two different inpretations of whole number division: finding the number of objects in each equal-sized group (partitive division) or finding the number of equal-sized groups (quotative division). The point is not the vocabulary associated with the two interpretations of division, but to use both interpretations to analyze division involving zero.

Division Involving Zero

- Focus 3 addresses division involving zero through five interpretations of a/b as a rational number: slope, direct proportion, Cartesian products (combinations), factor pairs (if $ab = c$, then a and b form a factor pair for c), and area of rectangles. Each interpretation is applied to address division involving zero. Depending on time and size of your group, you might assign different groups to analyze selected interpretations in this foci.

- Focus 4 uses rates as a context for analyzing division involving zero. Three specific rates are used: success rate, speed over time (e.g., mph), and unit price. Although participants may have different definitions of *rate* and *ratio,*[2] the distinction between these terms is not the point of this focus. Keep participants' attention on the use of rates and ratios to analyze division involving zero. You might ask participants to create their own rate example, in addition to analyzing the three provided.

- Focus 5 maps the slopes of lines in two-dimensional Cartesian space to real projective one-space to analyze division involving zero. This focus requires that participants know what a real projective space is, as well as understand equivalence classes. Consequently, this focus may not be accessible to all teachers. On the other hand, this focus highlights mathematics that secondary teachers may want to further explore.

- Post-Commentary provides a discussion that connects this topic to limits in calculus and the need for L'Hôpital's rule.

Key Points for Discussion

- After presentations, ask participants to compare and contrast the ways that division involving zero is addressed in the different foci. E.g., How is the idea of indeterminate form brought out in the contexts presented in the different foci?

  Likely connections include those between Foci 1 and 3 (viewing division through a multiplicative lens)—*more to come based on connections made by participants in PD sessions using this situation.*

- Engage participants in a reflective discussion about their learning from this activity:
  - Which foci you considered before?
  - Which were new to you?
  - Which ones were most compelling?
  - What new insights did you gain, if any, from looking at division involving zero from these different foci?
  - What confusion remains about division involving zero?

5. **Closure and Assessment:** The purpose is for participants to connect the session's activities to their own classroom practice, and to assess what participants learned from the session. Specific activities depend on the setting for the professional development session, i.e., whether it is a stand-alone session or part of an ongoing series of sessions.

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[2] Ratios can be used to make part-to-whole, part-to-part and whole-to-whole comparisons. Rates are typically defined as comparisons of quantities with different units (e.g., miles per hour, number of attempts vs. number of completions). For more information, see Lobato & Ellis, 2010.

Suggested assessment activities:

- Ask participants to determine when division involving zero might come up in their curriculum, then design and/or adapt a lesson/task/explanation to ensure that their students understand division involving zero in this context. Participants work in grade level groups, make a poster of their lesson/task/explanation. If this session is part of an ongoing series of professional development sessions, participants could be asked to try their task/lesson/explanation in their classroom, then bring classroom artifacts to the next session, as a lesson study or modified lesson study.

  One way to debrief and evaluate participants' lessons is to do a modified gallery walk, in which, as participants read/review each lesson, they write questions about the lesson on post-it notes. As the facilitator, you might want to provide "expert commentary" on the lessons in addition to participants' comments. Participants could be asked to revise their lesson based on the feedback they received.

  Another possibility is to structure a participant peer review process in which the most compelling arguments are identified. One option is a process of elimination: (1) lessons are paired, and subsets of participants select the stronger of each pair of lessons; (2) the selected lessons are then compared, etc., until one lesson (or one per grade band) emerges as the strongest explanation. Another option is to ask each participant to vote on the three explanations that they find the most compelling.

- Ask participants to revisit the situation prompt and determine how they would respond to students who gave each of the answers to $0 \div 0$, e.g., "How would you respond to a student who said $0 \div 0$ is equal to 1 because a number divided by itself is 1?"

  This activity could be an individual writing prompt, or different prompts could be given to different groups. The options for debriefing described above could be used here.

Also consider asking participants to reflect on which of the Standards for Mathematical Practice they were engaged in during the session.

Reflection questions:

1. Has our work today caused you to consider or reconsider any aspects of *your own thinking and/or practice* about division involving zero? Explain.
2. Has our work today caused you to reconsider any aspects of *your students' mathematical learning* about division involving zero? Explain.
3. What additional questions has our work today raised for you?

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Division Involving Zero

Values for 0 ÷ 2, 0 ÷ 0, and 2 ÷ 0 displayed by various technological tools

(Retrieved as of February 11, 2013)

Tool	0 ÷ 2	0 ÷ 0	2 ÷ 0
Excel spreadsheet	0	#DIV/0!	#DIV/0!
http://mathworld.wolfram.com/DivisionbyZero.html	(not addressed by this site)	The uniqueness of division breaks down when dividing by zero, ... division by zero is undefined for real numbers ...	The uniqueness of division breaks down when dividing by zero, ... division by zero is undefined for real numbers ...
Wolfram Alpha (entering 0 ÷ 0)	0	(indeterminate)	Complex infinity "Complex infinity is an infinite number in the complex plane whose complex argument is unknown or undefined. Complex infinity may be returned by Mathematica, where it is represented symbolically by ComplexInfinity. The Wolfram Functions Site uses the notation" infinity overscored by ~ to represent complex infinity.
Mac computer (calculator application)	0	Division by zero	Division by zero
Mr. Meyers' website (www.mrmyers.org/Math_Mania/div**facts**.html)	0	0	0
http://en.wikipedia.org/wiki/Division_%28mathematics%29 see *Division (mathematics)* entry, *By zero* section	(not addressed by this site)	Division of any number by zero (where the divisor is zero) is undefined.	Division of any number by zero (where the divisor is zero) is undefined.
http://en.wikipedia.org/wiki/Division_by_zero	(not addressed by this site)	... division by zero is undefined. Since any number multiplied by zero is zero, the expression 0/0 has no defined value and is called an indeterminate	... division by zero is undefined.

Division Involving Zero

Tool	0 ÷ 2	0 ÷ 0	2 ÷ 0
		form.	
Graphing Calculator 4.0 (PacificT.com)	0	∞	∞
iPad app (HD calculator)	0	0	0
iPad app (pocketCAS)	0	undef	∞
iPhone	0	error	error
TI-89	0	undefined	undefined
Mac Calculator app in dashboard	0	error	error
Calculator on Android phone	0	Invalid operation	Invalid operation
Geometer's Sketchpad	0	undefined	∞
Ask Dr. Math, Dr. Robert's answer to "Why is 0÷0 'indeterminate' and 1/0 'undefined'?" ()		Division by zero is an operation for which you cannot find an answer, so it is disallowed.	
HandwritingForKids.com/ handwrite/math/division/m athfacts.htm (see table of values)	0	0	(not addressed by this site)
Google scientific calculator	0	error	∞
Scientific calculator at Math.com	0	error	error
www.picalc.com	0	Syntax/math error	∞
www.calculator.com (beta version)	0	NAN (presumably an abbreviation for *not a number*)	∞