Situation: Summing the Natural Numbers

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Prompt

The course was a mathematical modeling course for prospective secondary mathematics teachers.

The discussion focused on finding an explicit formula for a sequence expressed recursively. During the discussion, a need to sum the integers from 1 to n was expressed. After several attempts to remember the formula, a student hypothesized that the formula contained n and $n + 1$. Another student said he thought it was $n(n + 1)/2$, but was not sure. During the ensuing discussion, a third student asked, "How do we know that $n(n + 1)/2$ is the sum of the integers from 1 to n, and won't this formula sometimes give fractions?"

Commentary

This prompt illustrates a reason to make a relationship between two concepts: an explicitly defined formula that could effectively determine the sum of a series with actually working the entire sum. In creating this closed from, the question arises on the development of such a formula and its effectiveness for appropriate values.

This prompt offers a variety of approaches. There are seven different foci that can be taken from this prompt. Focus 1 uses the sum of any arithmetic series to develop the sum of the first n natural numbers. The most common approach to showing this formula is the basis for the second focus: writing the sum of the natural numbers and then reversing the order to create sums of the same value, $n + 1$. Focus 3 involves triangular numbers; a triangular number is a number that can represent the perimeter of an equilateral triangle of natural number length. The fourth focus represents each number as its corresponding number of square units then derives the sum of the series using areas of geometric shapes, particularly triangles and squares. Mathematical induction proves that the given formula is indeed correct for the fifth focus. In Focus 6, the terms of the series are paired starting from the first and last terms and then increasing from the first term and decreasing to the last term to complete the pairings, as much as possible. The final focus shows the sum of the first n natural numbers can be found by using binomial expansions and then how binomial expansions can be used to find the sum of the first n squares, cubes and so forth.

To answer the second part of the question given in the prompt, it is helpful to remember that this problem deals strictly with natural numbers. Since the problem deals with consecutive natural numbers in the numerator, it will always be the case that one number will be even and one odd. Dividing an even number by two will always yield a natural number and multiplying by another natural number will still have a natural number product.

Mathematical Foci

Mathematical Focus 1

The series for the first n natural numbers form an arithmetic series, since there exists a common difference, 1, between any two consecutive terms. The formula for the sum of any arithmetic series is $S_n = \frac{n(a_1 + a_n)}{2}$. For the sum of the first n natural numbers, it will always be the case that $a_1 = 1$ and $a_n = n$. Substituting the appropriate values will yield the desired formula.

Mathematical Focus 2

Let S be the sum of the first n natural numbers. Then $S = 1 + 2 + 3 + ... + n-1 + n$. By the commutative and associative properties of addition,

$$S = 1 + 2 + 3 + ... + n$$

$$S = n + (n-1) + (n-2) + ... + 1$$

$$2S = (n+1) + (2+n-1) + (3+n-2) + ... + (n+1)$$

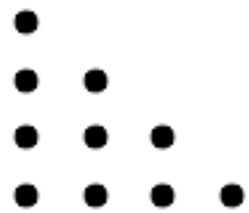
$$2S = (n+1) + (n+1) + (n+1) + ... + (n+1)$$

$$2S = n(n+1)$$

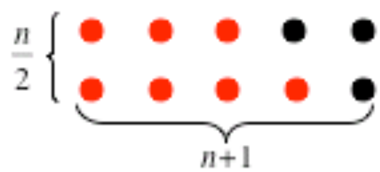
$$S = \frac{n(n+1)}{2}$$

Mathematical Focus 3

We can represent the first n natural numbers with a triangular array of dots, with each row of dots representing a number. The following array represents the sum of the first four natural numbers.

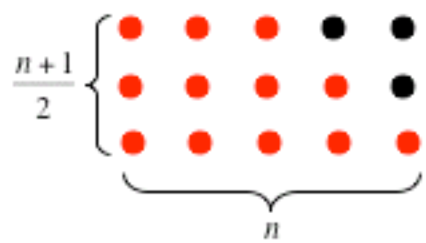


One way to verify the formula would be to rearrange the dots into a rectangle. If *n* is even, we would have *n*/2 rows, as shown in the first figure below.



The remaining rows will be rotated and moved to complete a rectangle having dimensions *n*/2 by *n* + 1 as shown in the second figure above.

If n is odd, we would have (*n* + 1)/2 rows, as shown in the first figure below.

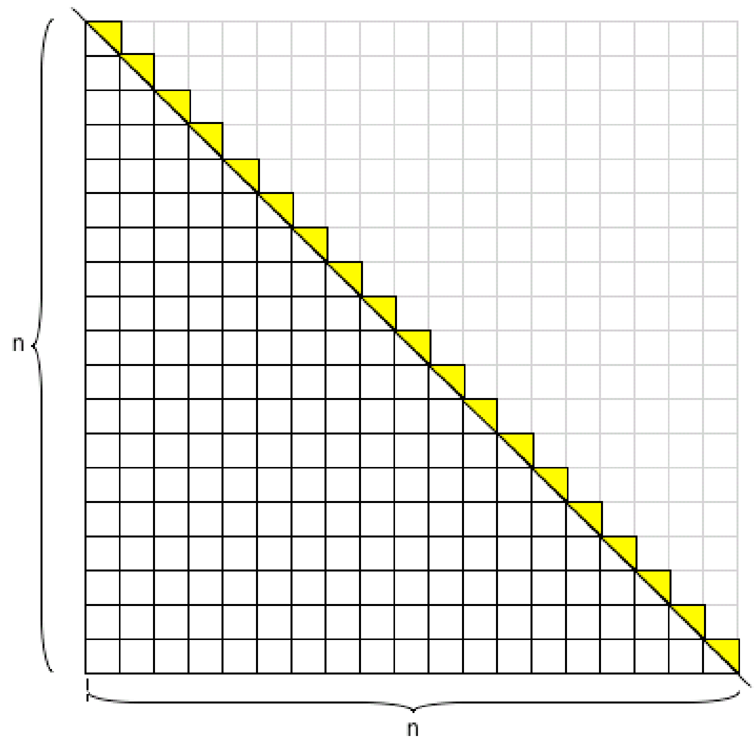


The remaining rows will be rotated and moved to complete a rectangle having dimensions (*n* + 1)/2 by n as shown in the second figure above.

Since this method only requires rearranging the dots, it is clear that it will always yield a natural number result.

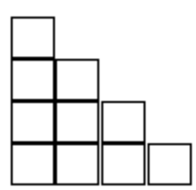
Mathematical Focus 4

We can represent the first *n* natural numbers by unit squares arranged in *n* rows with each row of squares representing a number. The sum of the first *n* natural numbers is equal to the number of squares in the region, which, since the squares are unit squares, is equal to the area of the region.



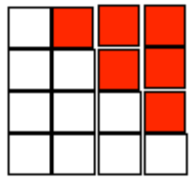
The area of this region can be found by finding the area of the right triangle with legs of length n and adding the area of each of the small triangles left out of this triangle (highlighted above). The area of the large right triangle is $n^2/2$, and the area of each of the n small triangles is 1/2, so the combined area is $n^2/2 + n/2 = n(n + 1)/2$.

A second way to visualize this is to use unit squares for the corresponding natural numbers. Each number's squares will be arranged in ascending order from top to bottom…



So far there are $1 + 2 + 3 + \ldots + n$ squares, which is the sum of the first n natural number, S.

Add $n - 1$ blocks to the first row of blocks, $n - 2$ blocks to the second row of blocks and continue the pattern of adding $n - r$ number of blocks to the r^{th} row of blocks. This will create a square with an area of n^2 square units.



Now there are S + (*n* – 1) + (*n* – 2) + … + 3 + 2 + 1 number of square units. Since the two quantities represent the same area, we will set the equal to each other.

$$S + (n-1) + (n-2) + .. + 3 + 2 + 1 = n^2$$

Adding *n* to both sides of the equation yields

$$S + (n-1) + (n-2) + .. + 3 + 2 + 1 + n = n^2 + n$$

which can be rewritten as

$$S + S = n^2 + n$$

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

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Mathematical Focus 5

We can use mathematical induction to prove that our formula will always work. This proof does not derive the formula, but proves that the formula works once we have it.

Let sum be $\sum_{i=1}^{n} i$.

Our inductive hypothesis is that $S_n = \frac{n(n+1)}{2}$.

When *n* = 1, we have 1 = 1(2)/2 = 1, which establishes our base case.

We assume that, for all numbers less than or equal to *m*, $S_m = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$. We must show that this also holds for *m* + 1, that is, we must show that $\sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}$.

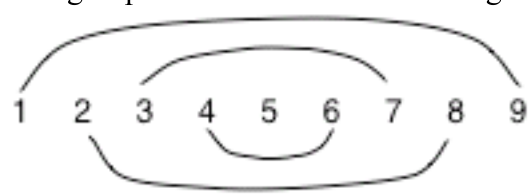
$\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + m + 1 = \frac{m(m+1)}{2} + m + 1 = \frac{m(m+1)}{2} + \frac{2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$ as required.

Thus the sum of the first *n* natural numbers is *n*(*n* + 1)/2.

Mathematical Focus 6

Considering these questions could lead to an exploration of the sum of the first n natural numbers. For example, suppose that $n = 16$. One could begin by adding the numbers (i.e. $1 + 2 + 3 + \ldots + 16$). However, one way to add the numbers, using the both the commutative and associative laws of addition, could be to change the order and groupings of the numbers. In our example, the first grouping could be the largest number with the smallest number (i.e. $1 + 16$), next grouping the second largest number with the second smallest number (i.e. $2 + 15$), then grouping the third largest number with the third smallest number ($3 + 14$), and so forth until all possible groupings are created. We will have 8 groups that sum to 17, since 8 is half of sixteen. Thus the sum of the numbers from 1 to 17 is $8*17 = 136$. We see that this is in line with the given formula of $n(n+1)/2$. However, this only shows that the formula seems to work and give a natural number when adding an even number of numbers.

Now suppose that $n = 9$. One could begin by adding the numbers (i.e. $1 + 2 + 3 + \ldots + 9$). However, we could also group the numbers as above using the following diagram.



Notice that since 9 is odd, 5 is not paired with another number. Thus we have 4 groups that sum to 10. Notice that 5 is half of our group sum of 10, so we have 4 1/2 groups of 10. 4 1/2 is half of 9, and 10 is $9 + 1$, so our formula $n(n + 1)/2$ seems to hold. When we are adding an odd number of numbers, grouping as above will always result in even sums since we add odd to odd and even to even. Even though we will always have to deal with half of a group sum (as in the 5 in our example above), that sum is even, so the result is a natural number. We have obtained a natural number result of the formula when adding an odd number of numbers.

While these examples work for a specific number of terms, the following proofs are explanations for any number of terms, depending on whether the number of terms is even or odd.

I want to use the same grouping technique that used with $n = 9$. This time it would be $1 + n$, $2 + (n-1)$, $3 + (n-2)$, and so on. Each one of those sums is $n+1$. There are always $n/2 – ½$ of these groupings. However, there is always one term left out in the grouping: $n/2 + ½$. To add up all of these terms, I want to add all of the groupings and the leftover term. A little commutative property of multiplication and factoring will give me the desired formula.

$$\left(\frac{n}{2}-\frac{1}{2}\right)(n+1)+\left(\frac{n}{2}+\frac{1}{2}\right)$$
$$\left(\frac{n-1}{2}\right)(n+1)+\left(\frac{n+1}{2}\right)$$
$$(n-1)\left(\frac{n+1}{2}\right)+\left(\frac{n+1}{2}\right)$$
$$(n-1+1)\left(\frac{n+1}{2}\right)$$
$$n\left(\frac{n+1}{2}\right)=\frac{n(n+1)}{2}$$

Now to show that this answer would yield a natural number answer. If I look at the formula developed in this solution

$$\left(\frac{n}{2}-\frac{1}{2}\right)(n+1)+\left(\frac{n}{2}+\frac{1}{2}\right)$$

We could simplify the fractions to $\left(\frac{n-1}{2}\right)(n+1)+\left(\frac{n+1}{2}\right)$. Since n is an odd number then both n – 1 and n + 1 are even numbers. Dividing an even number by 2 will yield a natural number answer. The multiplying and adding of natural numbers will still result in a natural number.

Now to show a general case for an even number of numbers.

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

If we add the first and last terms then the second and second to last terms and continuing inward through the terms of the series. Each sum of pairs is n + 1. There are a total of n/2 groups of sums that are added together. Therefore the sum of the series is $(n+1)\frac{n}{2}$, and through the commutative property we get $\frac{n(n+1)}{2}$.

Mathematical Focus 7 & Extension

Polya (1981) describes how Pascal solved this problem and extended it to the general case of finding the sum of the k^{th} powers of the first n natural numbers. The method depends on expanding $(n+1)^{k+1}$.

Let S_1 be the sum of the first n natural numbers. To find S_1, expand $(n+1)^2$ and write out its values for several numbers to see the pattern:

$(n+1)^2 = n^2 + 2n + 1$ so $(n+1)^2 - n^2 = 2n + 1$.

$$(n+1)^2 - n^2 = 2n+1$$

(1) $2^2 - 1^2 = 2 \bullet 1 + 1$

(2) $3^2 - 2^2 = 2 \bullet 2 + 1$

(3) $4^2 - 3^2 = 2 \bullet 3 + 1$

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(n) $(n+1)^2 - n^2 = 2n+1$

Then we can add all the equations (1) through (n) together. The sum of these equations will be $(n+1)^2 - 1 = 2(S_1) + n$. Then

$$S_1 = \frac{(n+1)^2 - n - 1}{2} = \frac{(n+1)^2 - 1(n+1)}{2} = \frac{(n+1-1)(n+1)}{2} = \frac{n(n+1)}{2}.$$

To find the sum of the first n squares, S_2, expand $(n + 1)^3$ and write out its values for several numbers to see the pattern:

$(n+1)^3 = n^3 + 3n^2 + 3n + 1$, so $(n+1)^3 - n^3 = 3n^2 + 3n + 1$

$(n+1)^3 - n^3 = 3n^2 + 3n + 1$

$2^3 - 1^3 = 3 \bullet 1^2 + 3 \bullet 1 + 1$

$3^3 - 2^3 = 3 \bullet 2^2 + 3 \bullet 2 + 1$

$4^3 - 3^3 = 3 \bullet 3^2 + 3 \bullet 3 + 1$

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$(n+1)^3 - n^3 = 3n^2 + 3n + 1$

Again add all the equations together. The sum of these equations will be $(n+1)^3 - 1 = 3 \bullet S_2 + 3S_1 + n$. Substituting for S_1 and solving for S_2 gives us

$$S_2 = \frac{(n+1)^3 - 3\frac{n(n+1)}{2} - n - 1}{3} = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{6} = \frac{(n+1)[2(n+1)^2 - 3n - 2]}{6}$$

$$= \frac{(n+1)(2n^2 + n)}{6} = \frac{n(n+1)(2n+1)}{6}$$

This method can be extended to find the sum of the first n cubes, and then the first n fourth powers and so on, as long as the sums of all the previous powers have been found.

References

Polya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving.* John Wiley & Sons: New York.

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