

MAC-CPTM Situations Project

Situation 49: Similarity

Prepared at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
15 July 2005 – Evan McClintock, Sue Peters, Donna Kinol

Edited at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
10 May 2006 – Evan McClintock, Shari Reed, Heather Godine

Edited at University of Georgia
Center for Proficiency in Teaching Mathematics
10 February 2007 – Erik Tillema

Edited at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
14 February 2007 – Heather Godine
23 February 2007 – Rose Mary Zbiek, M. Kathleen Heid

Edited at University of Georgia and Penn State University
3 May 2007 – Sarah Donaldson, Eileen Murray, Glen Blume

Prompt

In a geometry class, students were given the diagram in Figure 1 depicting two acute triangles, $\triangle ABC$ and $\triangle A'B'C'$, and students were told that $\triangle ABC \sim \triangle A'B'C'$. From this, a student concluded that $m\angle B' = 150^\circ$.

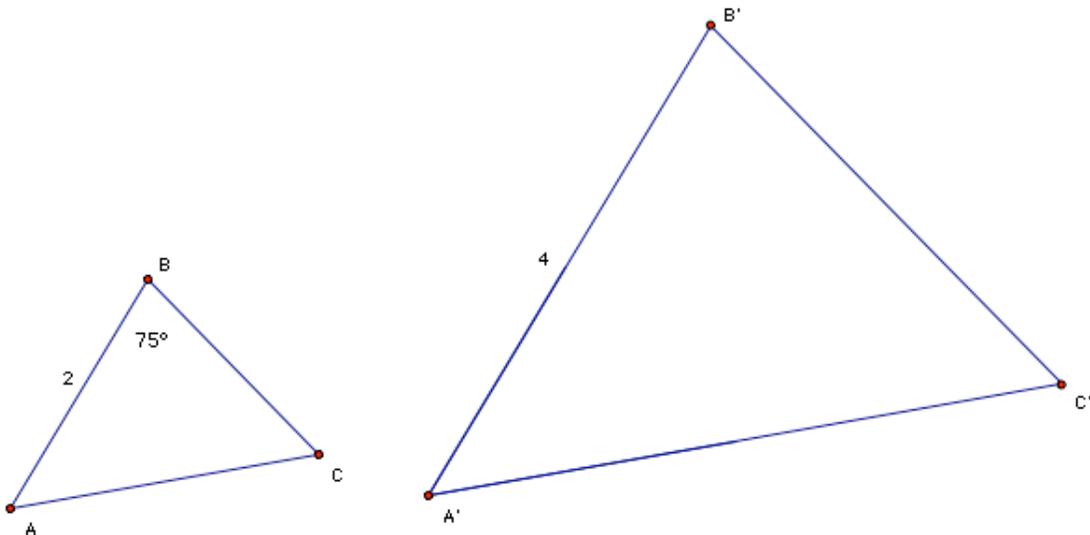


Figure 1.

Commentary

By definition, two polygons are similar if and only if their corresponding angles are congruent and their corresponding side lengths are proportional. Thus, similar figures may have different sizes, but they have the same shape. The following foci incorporate a variety of approaches (geometric, graphical, and symbolic) to shed light on the concept of similarity. The first focus refutes the claim made in the Prompt by appealing to the definition of similar triangles, and the second focus refutes the claim using an indirect proof that considers the impact of doubling the measures of each of the angles of the original triangle. In Foci 3 and 4, similarity is examined in terms of transformations in general, and dilations in particular. Under a geometric similarity transformation, angle measure is preserved and the ratio of the measures of corresponding distances is constant. [Finally, a geometric construction and proof is provided which lends further insight into the definition of similarity.] In each focus, the concept of ratio is emphasized, because common ratio lies at the heart of why size, but not shape, may change for similar figures.

Mathematical Foci

Mathematical Focus 1

Length measures of corresponding parts of similar figures are related proportionally with a constant of proportionality that may be other than 1, but angle measures of corresponding angles of similar figures are equal.

By definition, similar triangles have corresponding angles that are congruent and corresponding sides that are proportional. Figure 1 in the prompt depicts two similar triangles, $\triangle ABC$ and $\triangle A'B'C'$, with $AB = 2$ and $A'B' = 4$. A ratio of the lengths of the sides AB and $A'B'$ can be used to determine the corresponding lengths of the sides of $\triangle A'B'C'$ as scaled sides of $\triangle ABC$ or vice versa. In particular, since $A'B' = 2AB$ and $\triangle ABC$ and $\triangle A'B'C'$ are similar, it must be true that $A'C' = 2AC$, and $B'C' = 2BC$. Although the constant of proportionality, 2, can be used to find lengths of corresponding sides of these similar triangles, it does not apply to the measures of the angles, since the angles must, by definition, be congruent.

Mathematical Focus 2

Using the constant of proportionality (ratio of similitude) to increase the angle measures of a triangle to determine the angle measures of a similar triangle results in a contradiction of the theorem stating that the sum of the degree measures of the angles of a triangles is 180.

OR, A MORE GENERAL ITALICIZED STATEMENT:

The degree measures of a triangle cannot be some non-unit constant times the corresponding degree measures in a similar triangle.

OR

The degree measure sum for the angles of a triangle must be 180.

Suppose that one doubled the degree measures of each of the angles of $\triangle ABC$. This would result in the sum of the degree measures of the angles of $\triangle A'B'C'$ being 360. But this is not possible because the sum of the degree measures of the angles of any triangle is 180. So, the degree measures of the angles A' , B' , and C' cannot each be double the corresponding degree measure in $\triangle ABC$.

More generally, consider $\triangle ABC \sim \triangle A'B'C'$ with $A'B' = k * AB$ where $k > 0$ and $k \neq 1$. If one supposed that each degree measure in $\triangle A'B'C'$ were k times the corresponding degree measure in $\triangle ABC$, then the sum of the degree measures of the angles of $\triangle A'B'C'$ would be $180k$. This would contradict the fact that the sum of the degree measures of a triangle must be 180. Therefore, the degree measures of the angles of $\triangle A'B'C'$ cannot be k times the degree measures of the corresponding angles of $\triangle ABC$, where $k \neq 1$.

Mathematical Focus 3

When comparing corresponding parts of figures and their images under similarity transformations, angle measure is preserved and the length of an image is the product of the ratio of similitude and the length of the original figure. Isometries, for which the ratio of similitude is 1, are a subset of similarity transformations.

Transformations in which shape is preserved but size is not necessarily preserved are similarity transformations. Given $\triangle ABC$, consider $\triangle A'B'C'$ to be the image of a similarity transformation of $\triangle ABC$. A geometric similarity transformation is an angle-preserving function such that all distances are scaled by a constant ratio, $k \neq 0$. For the given similarity transformation, the lengths of sides of $\triangle A'B'C'$ are double the length of the corresponding sides of $\triangle ABC$, and, because angles are preserved by the similarity transformation, the angles of $\triangle A'B'C'$ are congruent to the corresponding angles of $\triangle ABC$.

Dilations whose center is the origin in a coordinate plane are similarity transformations of the form $F((x, y)) = (kx, ky)$ for some constant ratio $k \neq 0$. In general, if $|k| < 1$, the mapping results in a contraction, for which the resulting image is smaller than its pre-image. If $|k| > 1$, the mapping results in an expansion, for which the resulting image is larger than its pre-image. If $k = 1$, the mapping is the identity transformation under composition of transformations, for which the resulting image is the same size and shape as its pre-image.

Transformations in which shape and size are preserved are known as isometries, for which the resulting image is congruent to the pre-image. There are five distinct types of isometries: identity, reflection, non-identity rotation, non-identity translation, and glide reflection. In the coordinate plane, any figure may be mapped to a similar figure by a composition of dilations and isometries. The constant of proportionality is the product of the ratios of similarity. When one figure is mapped to its image by only isometries, the product of the ratios is a power of 1 and the figures are congruent.

Mathematical Focus 4

For a triangle inscribed in a circle and its dilation through the center of the circle, the relationship between the inscribed angle and the length of the intercepted arc shows that angle measure is preserved under dilation.

Using GSP, a dynamic diagram (see Figure 2) can be created to illustrate that shape and angle measure are preserved for similar triangles for which one can be represented as the expansion or contraction of the other triangle (with the center of the circle that inscribes that triangle as the center of dilation). Consider a triangle inscribed in a circle centered at the origin. Using polar coordinates, the coordinates of any point on the circle are (r, θ) , where r is the radius of the circle and θ is the measure of the angle in standard position formed by the x -axis and a ray from the origin to a point on the circle. For any angle in the triangle, the measure of the angle is equal to half the length of the intercepted arc divided by the radius. Thus, angle measure is a function of arc length and radius, namely,

$$m(a, r) = \frac{a}{2r}. \text{ Since the length of an arc is equal to the product of the radian}$$

measure of the arc angle and the length of radius of the circle, the ratio of half the length of the arc to the radius of the circle will be equal to the radian measure of the angle in the triangle intercepting that arc, regardless of the length of the radius. Therefore, as a circle centered at the origin is expanded or contracted, the measure of any angle in the inscribed triangle remains constant. Thus, angle measure is preserved.

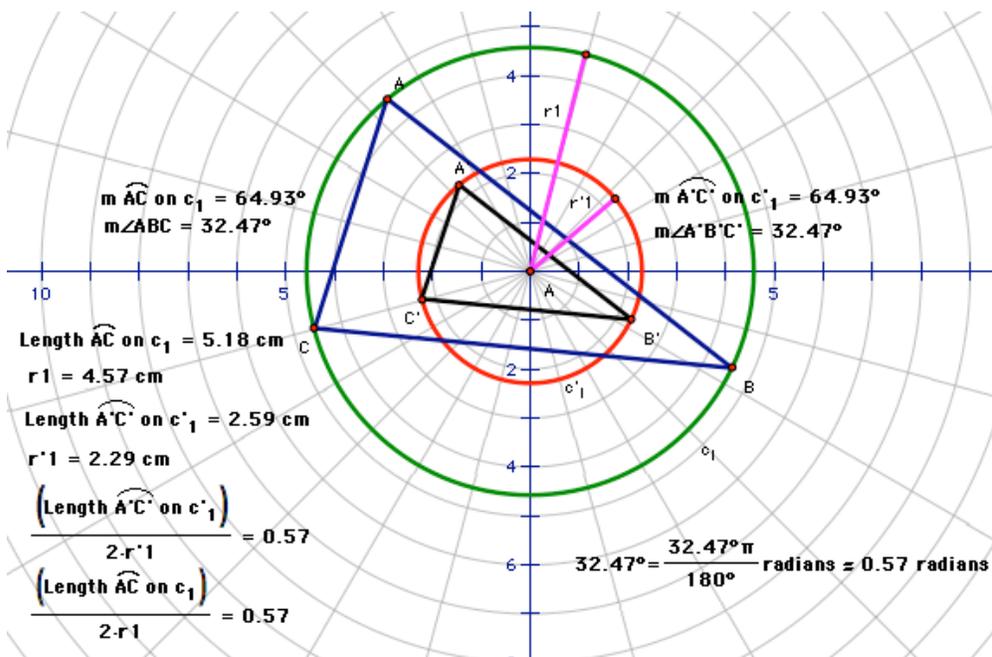


Figure 2.

Mathematical Focus 5

Question for the CPTM/MAC group: This Focus seems to prove something that is not directly related to the prompt, namely, that if three angles of one triangle are congruent to three angles of another triangle, then the corresponding sides must be proportional. One suggestion was that proving the converse, namely, that “given proportional sides, the angles must be congruent” might be more pertinent to the prompt. The question is, should what is proved be 1) only the original statement in this Focus (essentially leave it as is), 2) the original statement in this Focus and its converse, or 3) only the converse of the original statement?

Similar triangles have corresponding angles that are congruent and have corresponding sides that are proportional in length.

Question for the CPTM/MAC group: *IF WE KEEP THIS FOCUS AS IS, DOES THE FOLLOWING ITALICIZED STATEMENT BETTER CAPTURE THE MATHEMATICAL IDEA: If the measures of the angles of one triangle are equal to the measures of the corresponding angles of another triangle, then the corresponding sides of the triangles are proportional.*

Consider two triangles, $\triangle ABC$ and $\triangle DEF$ such that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$ (Figure 3).

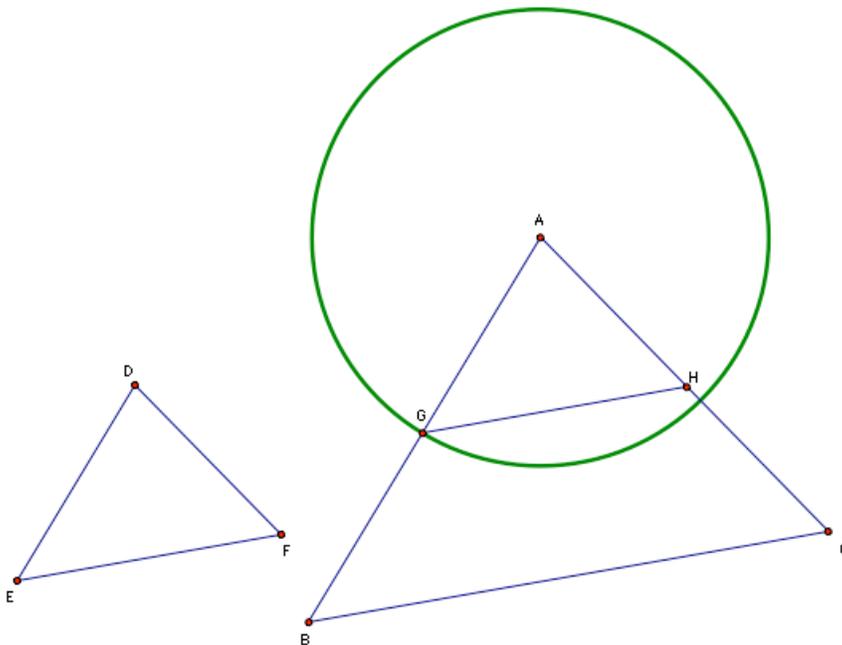


Figure 3.

Construct a circle centered at A with radius \overline{DE} . Let G be the point of intersection of the circle and \overline{AB} . Construct a line through G parallel to \overline{BC} . Let

H be the point of intersection of the parallel line and \overline{AC} . Since $\overline{GH} \parallel \overline{BC}$, corresponding angles are congruent, so $\angle AGH \cong \angle E$. Therefore, $\triangle AGH \cong \triangle DEF$ by the ASA theorem of congruence. Since $\overline{GH} \parallel \overline{BC}$, and parallel lines divide transversals proportionally, $\frac{GB}{AG} = \frac{HC}{AH} \Rightarrow \frac{AB}{AG} = \frac{AC}{AH} \Rightarrow \frac{AB}{AC} = \frac{AG}{AH}$, which is equivalent to $\frac{AB}{AG} = \frac{AC}{AH}$. Since $AG = DE$ and $AH = DF$, by substitution, $\frac{AB}{DE} = \frac{AC}{DF}$.

Using similar reasoning, it can be shown that $\frac{BC}{EF} = \frac{AB}{DE}$. Therefore, by the

transitive property of equality: $\frac{BC}{EF} = \frac{AC}{AH}$. Thus, it can be concluded that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Post Commentary

Question for the CPTM/MAC group: We blended this Post Commentary into the Commentary, suggesting that this Post Commentary could be removed. Is that acceptable?

In all of the foci, we find some type of ratio. It might be a ratio of similarity, a ratio among side lengths, or a ratio of proportionality. This common ratio lies at the heart of why size but not shape may change for similar figures. When the ratio corresponds to 1, congruence emerges as a special case of similarity.