MAC-CPTM Situations Project

Situation 09: Perfect Square Trinomials

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Prompt

A teacher is teaching about factoring perfect square trinomials and has just gone over a number of examples. Students have developed the impression that they need only check that the first and last terms of a trinomial are perfect squares in order to decide how to factor it. They are developing the impression that the middle term is irrelevant. The teacher needs to construct a counterexample on the spot, and he wants one whose terms had no common factor besides 1.

Commentary

Each of the following Foci address particular aspects of the mathematical knowledge a teacher needs in order to fully master the concept presented in this Prompt. Not only must the teacher know what a perfect square trinomial is, and how to factor it, but he/she must have the ability to recognize and product terms that are, and are not, perfect squares. Additionally, the teacher must be able to recognize and produce terms that are, and are not, in the form $2ab$. Foci 3 and 4 present two different models for understanding $(a + b)^2$: Focus 3 includes a geometric model, and Focus 4 a graphical (function) model.

Mathematical Foci

Mathematical Focus 1: Perfect square trinomial: $a^2 + 2ab + b^2$

A trinomial, by definition, consists of 3 terms. A perfect square trinomial is a specific kind of trinomial that is the result of multiplying a binomial (two terms) by itself. That is, when a binomial is squared, the result is a perfect square trinomial:

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

Factoring a term or group of terms means breaking it up into parts that, when multiplied, result in the original term or group of terms. Factoring a trinomial can be understood as breaking the trinomial into binomials such that, when the binomials are multiplied, the result is the original trinomial. A perfect square trinomial is the result of multiplying a binomial by itself. Factoring a perfect square trinomial, then, means finding the binomial which, when multiplied by itself, yields the given trinomial.

Mathematical Focus 2: Recognizing perfect squares and $2ab$.

Perfect squares result from multiplying a rational term (or group of terms) by itself. Therefore a perfect square can be recognized by noticing that its square root is a rational term. Perfect squares in this Situation are monomials.

The middle term of a perfect square trinomial is $2ab$ (see Focus 1). This indicates that the middle term is twice the product of "a" and "b," the square roots of the first and last terms of the trinomial. On the other hand, a term is <u>not</u> $2ab$ if it is <u>not</u> twice the product of a and b.

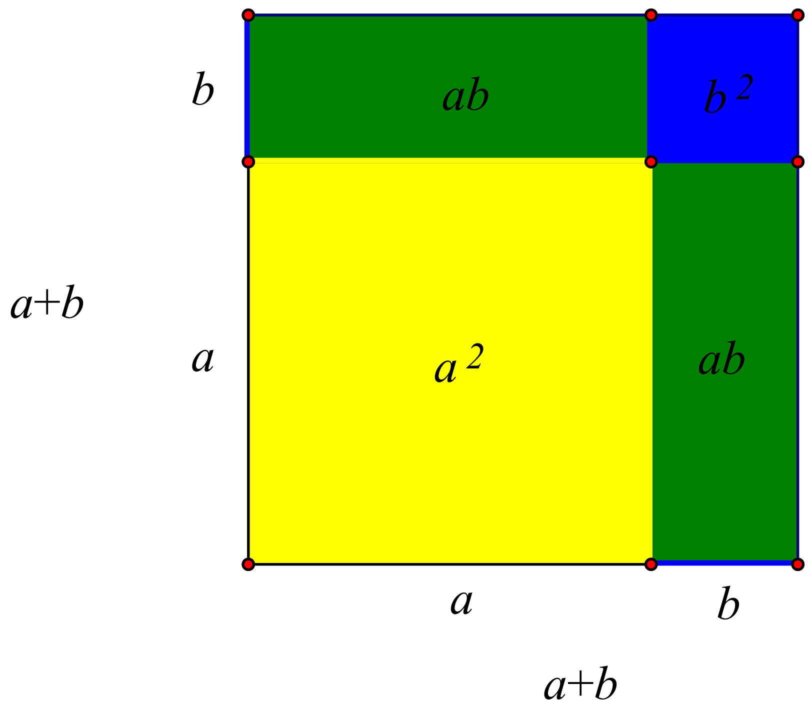
For example, consider $16x^2 + 24xy + 9y^2$.
In this case $16x^2 = (4x)^2$, $9y^2 = (3y)^2$, and $24xy = 2(4x)(3y)$.
Therefore if we say $4x = a$ and $3y = b$, then $16x^2 + 24xy + 9y^2 = a^2 + 2ab + b^2$, a perfect square trinomial and can be factored as $(a + b)^2$, or, in this case $(4x + 3y)^2$.

However, a trinomial such as $16x^2 + 23xy + 9y^2$ is <u>not</u> a perfect square trinomial because although the first and last terms are perfect squares, the middle term ($23xy$) is not twice the product of the square roots of the first and last terms of the trinomial. In order to produce a trinomial to demonstrate that the middle term matters when factoring, the teacher must simply write a trinomial in the form $a^2 + x + b^2$ in which $x \neq 2ab$.

Mathematical Focus 3: Geometric model of $(a+b)^2$

The model below shows a square whose sides are made up of two lengths, a and b. The sum of these lengths is one side of the square ($a+b$). Therefore the area of the square is $(a+b)^2$. It can be seen in this model that the square has 4 sections, each with dimensions involving the lengths a and b. There is a small square with area a^2, a small square with area b^2, and two rectangles with area ab. Therefore the expansion of $(a+b)^2$ is $a^2 + 2ab + b^2$.

This kind of model can be used when factoring perfect square trinomials as well. In that case, one would be given an area and have to determine what values, a and b, would result in a square with these 4 sections: a^2, b^2, and two ab.

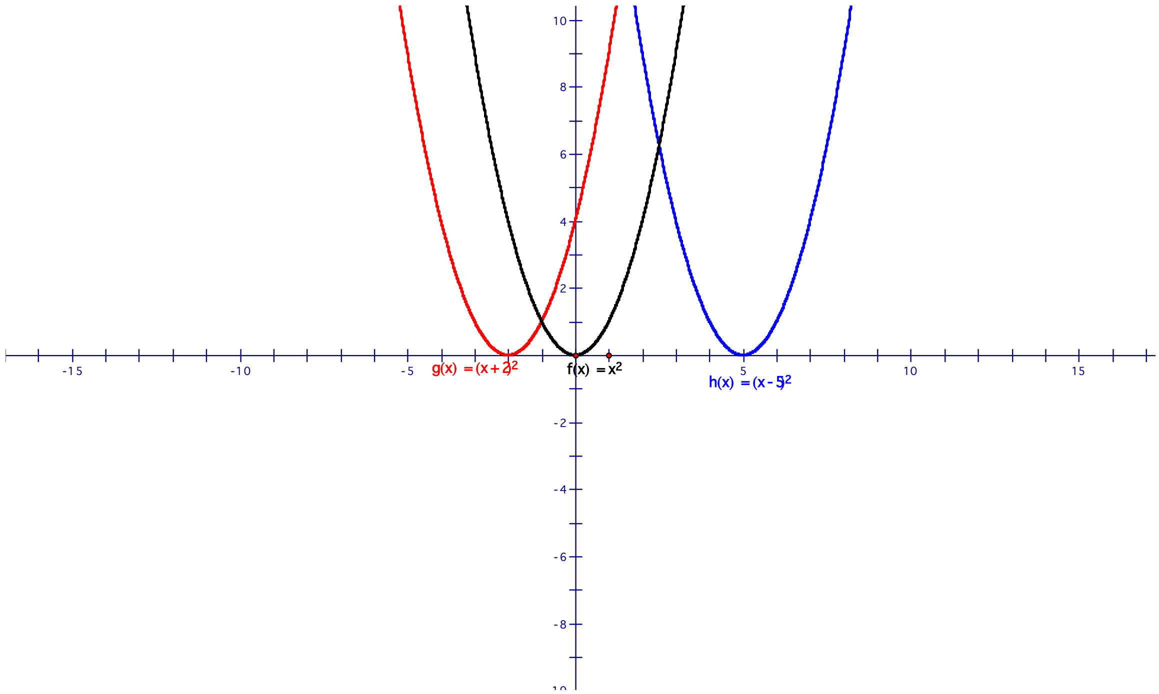


$$(a+b)^2 = a^2 + 2ab + b^2$$

Mathematical Focus 4: The function $f(x) = (x + b)^2$.

One application of factoring a quadratic trinomial is to identify the solutions of the quadratic function. The perfect square trinomial $x^2 + 2xb + b^2$, in its factored form, is $(x + b)^2$. When $f(x) = (x + b)^2$ is graphed, it can be seen to have a single solution (x-intercept), and this solution is $-b$.

In the graphs below, specific b-values (2, 0, and -5) have been used as examples. In each case, the parabola intersects the x-axis only once: at -2, 0, and 5 respectively.


g(x) = (x + 2)²
f(x) = x²
h(x) = (x - 5)²