Situation: Slopes of Perpendicular Lines
Prepared at UGA
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1 Prompt

In a tutoring session, a rising ninth-grader was asked to find the slope of a line perpendicular to $y = 2x - 3$. She wrote down $m = \dfrac{1}{2}$ and said "I think you just flip it over."

Question

What are some proofs, examples, or explorations that a teacher could do with the student to show her the reasons why the slopes of perpendicular lines are negative reciprocals of each other[1]?

2 Mathematical Foci

Mathematical Focus 1: A Geometric Construction

A student can see this relationship using a geometric construction which leads to congruent triangles.

1. Pick a general point on the line. In the case of our line, it is point A. Drop a line from A that's perpendicular to the x-axis. Let B be the y-intercept of our line. Drop a line from B to our new line to make a right triangle. Call this new point C.

   We now have a right triangle with a right angle at $\angle ABC$. See Figure 1 on the following page.

2. Construct a line that's perpendicular to line AB. Construct a point D such that $AB \cong BD$. Like in the first case, drop two perpendiculars that meet at point E.

   We now have a right triangle with a right angle at $\angle DBE$. See Figure 2 on the following page.

[1]Except in the case of a vertical and horizontal line, where slope is not even defined in the case of the vertical one.

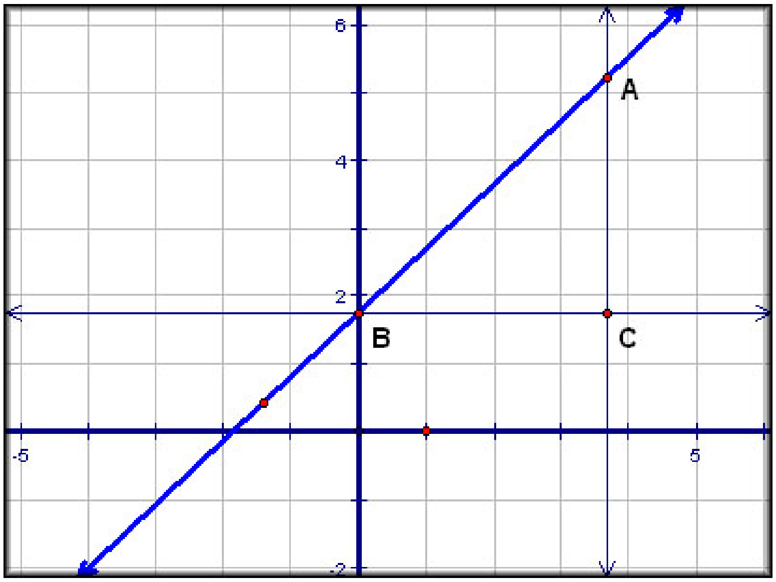

Figure 1: The first construction.

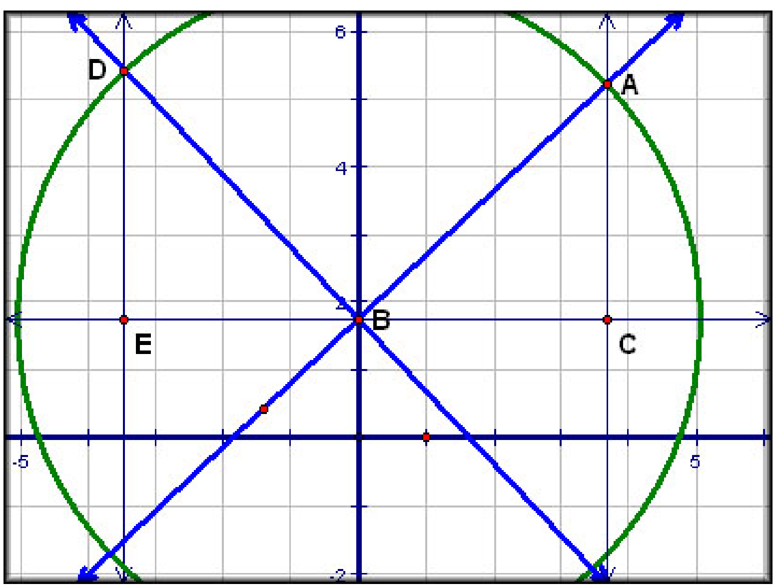

Figure 2: Completing the construction.

Now we want to show that $\triangle EBD \cong \triangle CAB$. Notice that:

$$\angle EBD + \angle DBA + \angle ABC = 180^o$$

Since $\angle DBA = 90^o$ (by construction),

$$\angle EBD + 90^o + \angle ABC = 180^o$$

$$\angle EBD + \angle ABC = 90^o$$

This implies that $\angle EBD$ and $\angle ABC$ are complementary. Since $\angle EBD$ and $\angle EDB$ are also complementary, we get $\angle EDB \cong \angle ABC$. It follows that $\angle EBD \cong \angle CAB$. Since $AB \cong BD$ by construction,

$$\triangle EBD \cong \triangle CAB$$

... by Angle-Side-Angle. Thus, since corresponding sides of congruent triangles have equal length, $AC \cong BE$ and $BC \cong DE$. Finally, the slope of the line AB is its ratio of rise over run. This can be expressed as $\frac{AC}{BC}$. The slope of the line BD is $\frac{DE}{BE} = \frac{BD}{AC}$.

It is clear from the pictures that the slope of the perpendicular has a slope which is negative to our original line. Thus, the slopes are negative reciprocals of one another.

Mathematical Focus 2: Dot Product

An understanding of a linear algebra makes this proof trivial. Two vectors are perpendicular if their *dot product is 0.* Given a vector (a, b) which has a slope of $\frac{b}{a}$, take its dot product with another vector (x, y):

$$(a, b) \bullet (x, y) = 0$$

$$a \cdot x + b \cdot y = 0$$

$$a \cdot x = -b \cdot y$$

$$\frac{x}{y} = -\frac{b}{a}$$

Thus the slopes are negative reciprocals of one another.

Mathematical Focus 3: Law of Sines

In this approach we will use trigonometry. Our setup is similar to the one in Focus 1.

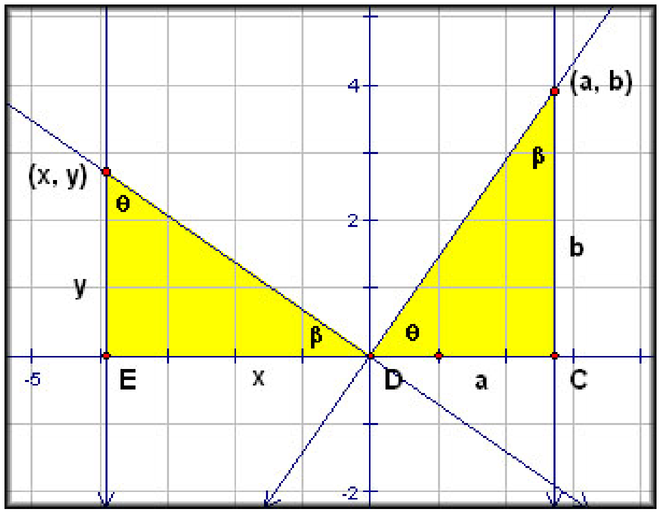

Figure 3: The two lines are perpendicular.

Using the Law of Sines, we get $\dfrac{\sin\theta}{b} = \dfrac{\sin\beta}{a}$ and $\dfrac{\sin\theta}{x} = \dfrac{\sin\beta}{y}$. So $\dfrac{\sin\theta}{\sin\beta} = \dfrac{b}{a}$ and $\dfrac{\sin\theta}{\sin\beta} = \dfrac{x}{y}$, and we get our desired result:

$$\frac{b}{a} = \frac{x}{y}$$