

Situation 23: Simultaneous Equations

Prepared at the University of Georgia

EMAT 6500 class

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Nicolina Scarpelli

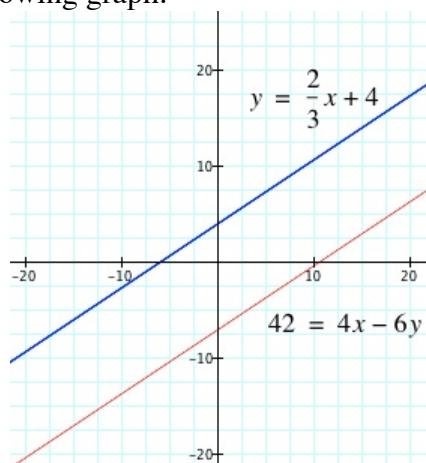
Prompt:

A mentor teacher and student teacher are discussing a student teacher's lesson after it has been taught and the mentor is encouraging the student teacher to probe student thinking and to ask good questions. The class was solving simultaneous equations and the student teacher had chosen the pair of equations below to discuss.

$$y = \frac{2}{3}x + 4$$

$$42 = 4x - 6y$$

A high school student constructed the following graph:



The student teacher asked how many solutions the system of equations had, and the student raised his hand and said, “There is no solution because the lines are parallel, so they will never cross.”

Commentary:

The beginning of this document provides a quick history of where simultaneous equations arose within mathematics and a couple definitions that are needed to understand the concept. The first focus highlights the properties of parallel lines since the situation deals specifically with the graph of parallel lines. Focus 2, 3, 4, and 5 emphasize different techniques to solve systems of equations. Focus 2 highlights the concept of graphing simultaneous equations to find a solution, and the misconceptions that may arise. Focus 3 emphasizes a more algebraic way to solving systems of equations: substitution and elimination. Focus 4 deals with solving simultaneous equations by using matrices and matrix operations. Focus 5 underlines Cramer's rule, which uses the determinants of square matrices to solve simultaneous equations. Finally, focus 6 gives a few examples of real – world applications of simultaneous equations. In order to present the concept of solving simultaneous equations to a high school student, a teacher may offer a real – life contextual illustration. For example, saying two men are backpacking through the mountains in Switzerland. Both men start from their homes and travel in the path of a straight line. If they want to meet at a certain point and backpack the rest of the way together, we will need to see at what point their paths will cross. Solving a system of equations can help identify when their paths will cross.

History of Simultaneous Equations (Retrieved from: <http://hom.wikidot.com/cramer-s-method-and-cramer-s-paradox>)

As early as 200 B.C. the Chinese had devised a clever method for solving systems of two linear equations with two unknowns. Following the Chinese, in 1750, Gabriel Cramer (1704-1752), a Swiss mathematician, published the famous rule for solving systems of linear equations in his manuscript *Introduction to the Analysis of Algebraic Curves*. His method of solving systems of equations dealt with finding the equation of a plane curve passing through a certain number of fixed points. Following Cramer, in 1873, Wilhelm Jordan (1842-1899), a German geodesist, published a book titled *Handbuch der Vermessungskunde* in which he discussed Gauss' method of elimination to

convert a given system of linear equations to a triangular system. In this book he explained how the triangular system could then be further reduced to a diagonal one, which could be solved directly. This algorithm is called the Gauss-Jordan Elimination method, which can still be found in the high school curriculum we use today.

Important Definitions:

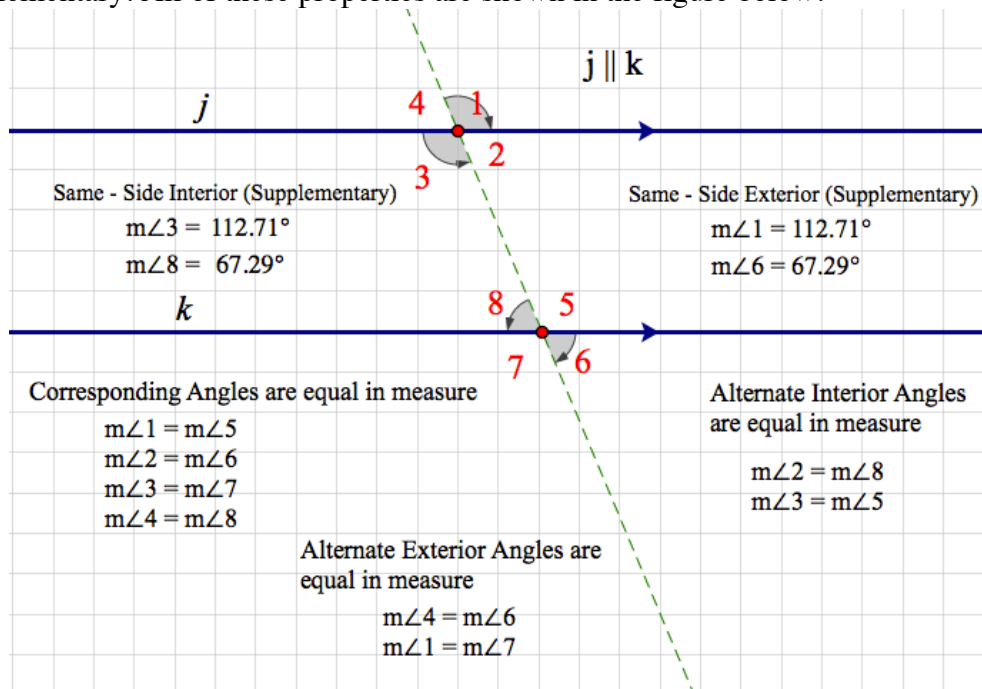
- 1) **Linear Equation:** A linear equation is an equation of the first degree in any number of variables. This means there are no variables higher than the first degree such as x^2 , nor any product of variables such as xy . A linear equation is often in the form $y = ax + b$, where a and b are any real number; however, any linear equation can be put in the form $c_1x_1 + c_2x_2 + \dots + c_nx_n = c_0$, where n is the number of variables, the variables are x_1, x_2, \dots, x_n and c_0, c_1, \dots, c_n are constants. (i.e. linear equations may have two variables, three variables, or n variables). A linear equation makes a straight line when graphed in the Cartesian coordinate system.
- 2) **Simultaneous Linear Equations (system of equations):** A system of equations is a set or collection of linear equations.

Mathematical Foci:

Focus 1: Properties of Parallel Lines in Algebra and Geometry

Parallel lines are very important throughout algebra and geometry. Two lines are parallel if they both lie in the same plane, are always equidistant, have the same slope, and never intersect (i.e. the lines will continue forever without crossing). A contextual example of parallel lines in real-life would be a football field. The sidelines are parallel to each other; in addition, the yard lines are parallel to each other. In Algebra, it is useful to use the properties of parallel lines to prove simple geometric proofs within the coordinate plane. For example, proving a rectangle is a rectangle in a coordinate plane, one must prove that opposite sides are parallel using the concept of slope.

In geometry, when two lines are cut by a transversal, any two of the eight angles formed will be supplementary (add up to 180 degrees). Two lines are parallel if a pair of alternate interior angles have the same measures, a pair of corresponding angles have the same measures, or a pair of alternate exterior angles have the same measures. Same side interior angles and same side exterior angles that are formed when two parallel lines are cut by a transversal are supplementary. All of these properties are shown in the figure below.

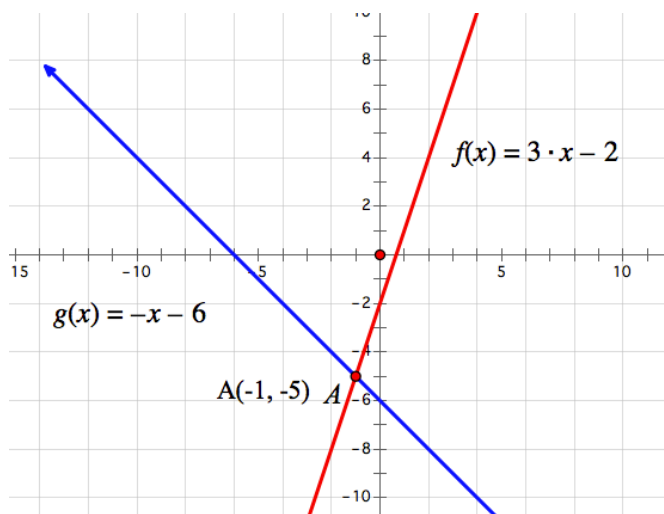


Focus 2: Solving Linear Simultaneous Equations by Graphing

In order for students to grasp the idea of solving simultaneous equations using a graphical display, they must have a solid understanding of graphing functions including how to plot the y – intercept and use the slope properly to graph the lines correctly. In addition, if the linear equation is given in standard form, $ax + by = c$, students will need to know how to convert it to slope – intercept form in order to graph the equation effectively. When dealing with a system of two equations, we must graph both equations on the same coordinate system. At whatever coordinate point the lines intersect at is the solution to the system. However, a system of equations may have no solution (i.e. lines are parallel and will never intersect), or a system of equations may have infinitely many solutions. An example of each is shown below:

I. A system of equations with a unique solution (the lines intersect at an exact coordinate point):

$$\begin{cases} y = 3x - 2 \\ y = -x - 6 \end{cases}$$



A solution for a single equation is any point that lies on the line for that equation. However, a solution for simultaneous equations is any point that lies on both lines (i.e. an intersection point). You can confirm this intersection point is a solution to the system by substituting the x and y – values into both equations, and it will hold true in both equations.

$$\begin{cases} -5 = 3(-1) - 2 \\ -5 = -(-1) - 6 \end{cases}$$

For example, substituting $x = -1$, and $y = -5$ into the system of equations above holds true. The coordinate point must satisfy BOTH equations, if it holds true in one equation but not the other, it is not a solution to the system. A system of equations that has a unique solution is called a **consistent** system of equations.

II. A system of equations with no solution (i.e. the lines are parallel to each other):

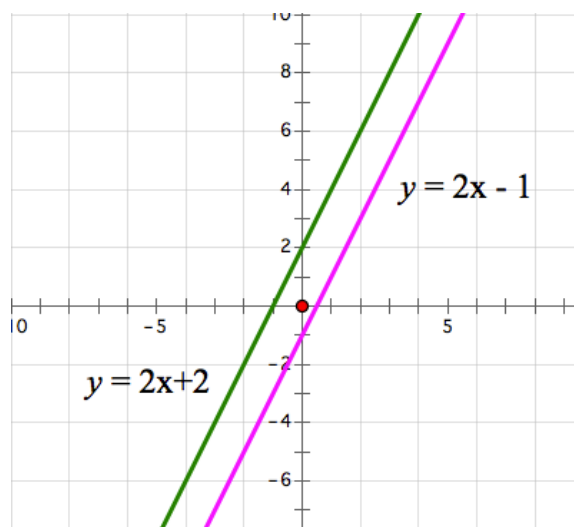
$$\begin{cases} y = 2x - 1 \\ 4x = 6x - y + 2 \end{cases}$$

In order to graph these two equations on the same coordinate plane, one must put the second equation into slope – intercept form.

$$\begin{aligned} 4x &= 6x - y + 2 \\ y &= 6x - 4x + 2 \\ y &= 2x + 2 \end{aligned}$$

Now, we have the following system of equations:

$$\begin{cases} y = 2x - 1 \\ y = 2x + 2 \end{cases}$$



Since both lines have the same slope, $m = 2$, but different y – intercepts they are parallel lines and will never intersect. Therefore, since parallel lines never cross, there is no intersection point; thus, for a system of equations that graphs as parallel lines, there is no solution and the system of equations is called **inconsistent**.

III. A system of equations with infinitely many solutions (i.e. the lines are the same or collinear):

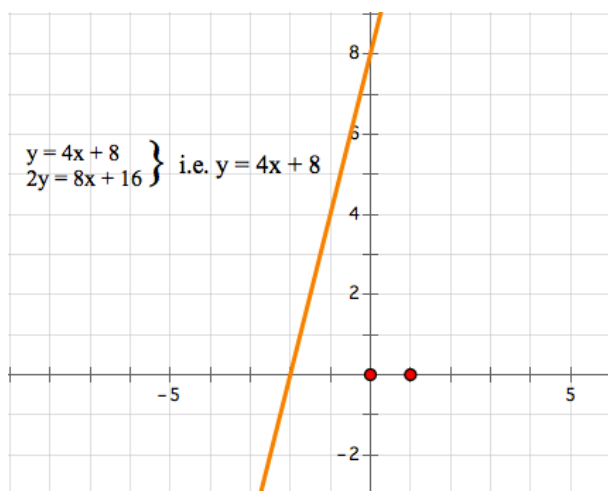
$$\begin{cases} y = 4x + 8 \\ 2y = 8x + 16 \end{cases}$$

In order to graph these two equations on the same coordinate plane, one must put the second equation into slope – intercept form.

$$\begin{aligned} 2y &= 8x + 16 \\ y &= 4x + 8 \end{aligned}$$

Now we have the following system of equations:

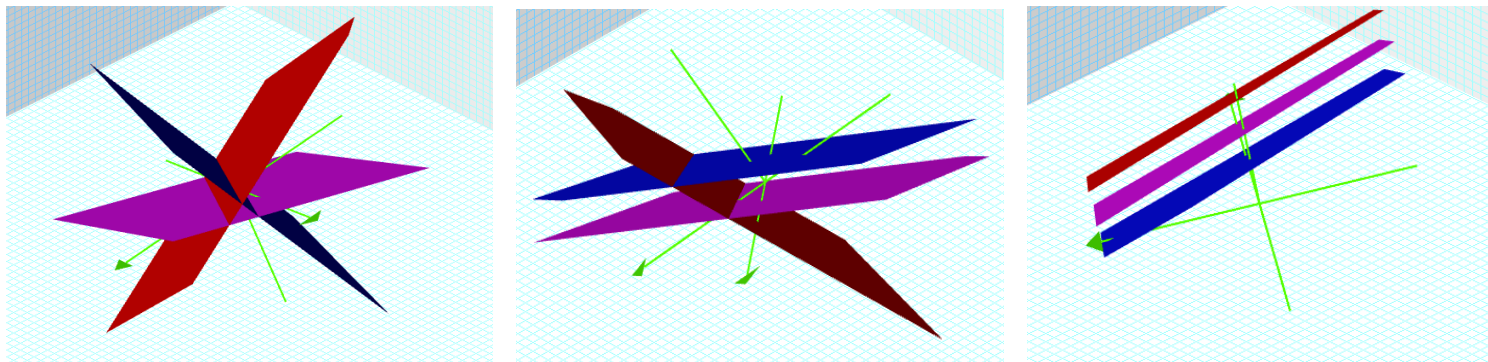
$$\begin{cases} y = 4x + 8 \\ y = 4x + 8 \end{cases}$$



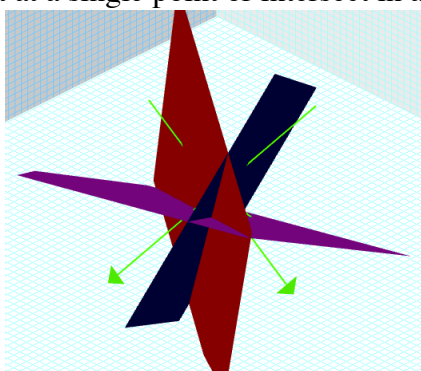
This time the equations represent the same line, since both can be written in the form $y = 4x + 8$. Both equations have the same slope and same y – intercept, so any point of this line has coordinates that will satisfy both equations, so there are an infinite number of solutions. In general, two equations represent the same line if one equation is a

multiple of the other. This type of system of equations is called dependent and **consistent**. In conclusion, lines are either inconsistent or consistent. Inconsistent lines have no solution (i.e. are parallel) and consistent lines have at least one solution. If they are consistent, they are either independent meaning the lines intersect or they are dependent meaning the lines coincide and have infinitely many solutions.

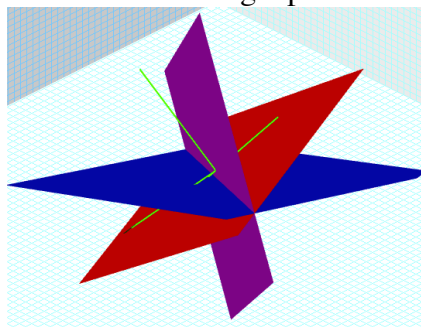
Some of the same properties hold for systems of equations with three variables, the only difference is that you will obtain a solution that is an ordered triple (x, y, z) that is a solution to all three equations rather than an ordered pair (x, y) . The graph of a linear equation in three variables is a plane in three – dimensional space. The graphs of three equations that form a system are three planes whose intersection determines the number of solutions of the system, as shown in the picture below.



The above images are examples of systems of equations in 3 unknowns with No Solutions. All three planes never meet at a single point or intersect in a line.



The above image is an example of a system of equations in 3 unknowns having one unique solution – the three planes meet at a single point



The above image is an example of a system of equations in 3 unknowns having infinite solutions – the three planes intersect in a line.

The system may have infinitely many solutions, meaning the planes intersect in a line or are the same plane; or the system may have exactly one solution, meaning the planes intersect in a single point; or the system may have no solutions, meaning the planes have no common point of intersection.

Often times, a solution obtained with using a graphical display is not exact, it may be an approximation. When graphing a system of equations, one must be extremely precise with his or her graphing techniques (i.e. constructing a straight axis system and drawing exact straight lines). In addition, being able to determine whether two lines are

parallel within the coordinate plane can be tricky. By inspection, the two lines may appear to be parallel, but mathematically they are not. Further, an intersection point may not be a nice pair of whole numbers, they might be fractions or decimals. Students may use a T- chart to avoid making these errors; however, teachers may encourage students to become more familiar with substitution and elimination methods for solving simultaneous linear equations shown in the next focus.

Focus 3: *Solving Simultaneous Equations Algebraically using Substitution and Elimination*

There are two basic methods that are used to solve linear simultaneous equations: substitution and elimination. In this focus, we will highlight how each method is used to solve a system of two equations with two unknowns; however, each method works for systems with more than two equations and more than two unknowns.

In order to use substitution, we ultimately solve for one of the variables in one of the equations in terms of the other variable and substitute that value in the other equation. After substituting the one variable, you will have an equation in one variable only, which you may solve and then substitute back into the original equation to get the solution for the other variable.

Steps to take to solve a system of equations by Substitution:

- 1) Solve one of the equations for one of the variables.
- 2) Substitute the expression found in Step 1 into the other equation (this will yield an equation of a single variable)
- 3) Solve the equation in Step 2 for the variable
- 4) Substitute the value found in Step 3 into one of the original equations and solve for the other variable.

Example: Solve the system of equations by substitution

$$\begin{aligned}3y - 2x &= 11 \\ y + 2x &= 9\end{aligned}$$

Step 1: Solve one of the equations for one of the variables

$$y = 9 - 2x$$

Step 2: Substitute the expression found in Step 1 into the second equation

$$3(9 - 2x) - 2x = 11$$

Step 3: Solve the equation in Step 2 for the variable x .

$$3(9 - 2x) - 2x = 11$$

$$27 - 6x - 2x = 11$$

$$-6x - 2x = -16$$

$$-8x = -16$$

$$x = 2$$

Step 4: Substitute the x – value found in Step 3 into the original equation to solve for the y – variable

$$y + 2(2) = 9$$

$$y + 4 = 9$$

$$y = 5$$

When using substitution and elimination it is essential to stress the importance of checking your solutions. Thus, we must substitute the (x, y) coordinate pair that we found into both equations to make sure it satisfies both equations.

$$3(5) - 2(2) = 11$$

$$(5) + 2(2) = 9$$

Therefore, the coordinate pair $(2, 5)$ is a solution for the system of equations.

The Elimination method is also called the addition method. In order to use Elimination to solve a system of equations, we will modify the equations so that for one variable, its coefficients in the two equations cancel out by being

opposites of one another. We only need to modify the equations if we are given equations where we can not eliminate a variable by addition or subtraction directly (i.e. $2x$ and $-2x$).

Steps to take to solve a system of equations by Elimination:

- 1) Add or subtract a multiple of one equation to or from the other equation, in such a way that either the x -terms or the y -terms cancel out
- 2) Solve for whichever variable is left (either x or y)
- 3) Solve for the other variable by substituting back into the original equation.

Example: Solve the system of equations by elimination

$$4x + 3y = -2$$

$$8x - 2y = 12$$

Step 1: Multiply the top equation by a multiple of 2 and the bottom equation by a multiple of 3 in order to get the y - variables to cancel out. We now have the following system of equations:

$$2(4x + 3y = -2)$$

$$3(8x - 2y = 12)$$

$$8x + 6y = -4$$

$$24x - 6y = 36$$

Step 2: Solve for x by adding the two equations together and cancelling out the y - term.

$$8x + 6y = -4$$

$$+ \quad 24x - 6y = 36$$

$$\hline 32x = 32$$

$$x = 1$$

Step 3: Solve for y by substituting x back into one of the original equations

$$4(1) + 3y = -2$$

$$4 + 3y = -2$$

$$3y = -6$$

$$y = -2$$

Thus, we have the coordinate pair $(1, -2)$ as the solution to this system of equations. Just like we did above, it is important to check the solution to make sure it satisfies both equations

$$4(1) + 3(-2) = -2$$

$$8(1) - 2(-2) = 12$$

It does satisfy both of the equations, thus, the coordinate pair $(1, -2)$ is in fact a solution to the system of equations.

Focus 4: *Solving simultaneous equations by using matrices (Gauss – Jordan Elimination Method)*

Gauss – Jordan Elimination (also known as Gaussian Elimination) is a method of solving a linear system of equations. This method solves a system of equations by transforming the system's augmented matrix into reduced row – echelon form by means of elementary row operations.

Key words:

- 1) Matrix – A rectangular array of numbers – numbers are grouped into rows and columns. The dimensions of a matrix are the number of rows by the number of columns in the matrix. We can use matrices to represent and solve systems of linear equations.
- 2) Augmented Matrix: A matrix form of a linear system of equations obtained from the coefficient matrix.

$$\begin{array}{l} \text{System of Equations} \rightarrow \\ a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}$$

$$\text{Augmented Matrix} \rightarrow \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right]$$

- 3) Reduced Row – Echelon Form – A matrix is said to be in Reduced Row – Echelon Form is the following conditions are satisfied:
 - The first non – zero number in a row is a 1 (i.e. leading 1)
 - All rows of zero are together at the bottom of the matrix
 - Each column that contains a leading 1 has only zeros below it
 - Each column that contains a leading 1 has zeros everywhere else

The following matrix is in reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

- 4) Elementary Row Operations for Matrices – There are three types of elementary row operations
 - Row Interchange: Switching any two rows of the augmented matrix
 - Row Scaling: Multiplying a row by a non – zero scalar (i.e. any row can be multiplied by any number except 0 – When a row is multiplied by a scalar, every element in that row must be multiplied by the same scalar).
 - Row Addition: adding a multiple of one row of the matrix to another row
 -

Expressing Systems of Equations as Matrices

Given the following system of equations:

$$\begin{array}{l} ax + by = c \\ dx + ey = f \end{array}$$

The above system of equations can be expressed as a matrix system as follows:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

In order to solve for the variables x and y, we want to put the system of equations into an augmented matrix and convert it into reduced row echelon form. To put a system into an augmented matrix, you use the coefficients associated with each variable and the constants. The above matrix can be put into the following augmented matrix:

$$\left[\begin{array}{cc|c} a & b & c \\ d & e & f \end{array} \right]$$

The same can be done for a system of three equations with three unknowns. Take a look at the examples shown below.

Example 1: Solve the following system of equations using Gaussian Elimination:

$$\begin{aligned}x + 5y &= 7 \\ -2x - 7y &= -5\end{aligned}$$

Convert the system of equations into an augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array} \right]$$

Now, use elementary row operations to get the augmented matrix into reduced row echelon form.

Step 1) Add twice row 1 to row 2

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array} \right]$$

Step 2) Multiply row 2 by $\frac{1}{3}$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array} \right]$$

Step 3) Add -5 times row 2 to row 1

$$\left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array} \right]$$

Step 4) Solve for x and y using the reduced augmented matrix, then check the solutions by substituting the values back into the original equations.

$$\begin{aligned}x &= -8 \text{ and } y = 3 \\ -8 + 5(3) &= 7 \\ -2(-8) - 7(3) &= -5\end{aligned}$$

Since the coordinate $(-8, 3)$ holds true in both equations, it is a solution to the system of equations. If you did not want to complete step 3, we could have solved the system from step 2. From the augmented matrix in step 2, we know that $y = 3$. Thus, we could have created a new set of equations from the augmented matrix, and solved for x. (i.e.

$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array} \right] \rightarrow \begin{cases} x + 5y = 7 \\ y = 3 \end{cases}$). Solve for x by substituting in $y = 3$ (i.e. $x + 5(3) = 7 \rightarrow x = -8$). Either way, we get the same solution.

Example 2: Solve the following system of equations using Gaussian Elimination:

$$\begin{aligned}x - 3y + z &= 4 \\ 2x - 8y + 8z &= -2 \\ -6x + 3y - 15z &= 9\end{aligned}$$

Convert the system of equations into an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array} \right]$$

Now, use elementary row operations to get the augmented matrix into reduced row echelon form.

Step 1) Replace row two with the sum of 2 times row 1 and row 2, and replace row three with the sum of 6 times row 1 and row 3.

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{array} \right]$$

Step 2) Multiply row two by $\frac{1}{2}$ and multiply row three by $\frac{1}{3}$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & -5 & -3 & 11 \end{array} \right]$$

Step 3) Replace row three with the sum of -5 times row two and row 3.

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -18 & 36 \end{array} \right]$$

Step 4) If we use the above matrix to create a system of equations we have:

$$\begin{aligned} x - 3y + z &= -4 \\ 0x - y + 3z &= -5 \\ 0x + 0y - 18z &= 36 \end{aligned}$$

Step 5) We can now solve for the variables x, y, and z by using back substitution in the above system of linear equations.

$$\begin{aligned} 18z &= 36 \rightarrow z = 2 \\ -y + 3(2) &= -5 \rightarrow y = 11 \\ x - 3(11) + 2 &= -4 \rightarrow x = 27 \end{aligned}$$

For a system of equations with a 3x3 matrix of coefficients, the goal of the process of Gaussian Elimination is to convert the augmented matrix into an equivalent matrix that is upper triangular. In simple terms, this means that we want to create a triangle of zeros in the lower left hand corner of the matrix below the diagonal. Notice how the diagonal and the numbers above it form a triangle, that is why it is called upper triangular. Then, we can use back substitution like we did above to solve the corresponding set of equations.

Steps to reduce a matrix via Gaussian Elimination:

Step 1: Construct the augmented matrix corresponding to the system of equations to be solved.

Step 2: Use elementary row operations to convert the augmented matrix into reduced row – echelon form. You ultimately want the resulting matrix to be in upper triangular form.

Step 3: Once the augmented matrix is in reduced row – echelon form, write down the corresponding system of linear equations and use back – substitution to solve for the unknown variables.

Interpreting Results:

When solving a system of linear equations with matrices, there are 3 possible situations that can arise.

1. Independent

When a system is independent, there is exactly one solution to the system. The resulting matrix will look like the one shown below, where the real numbers a , b , and c are the solutions to the equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$$

2. Dependent

When a system is dependent, there are infinitely many solutions to the system. The resulting matrix will look like the one shown below. A row of zeroes in the bottom row indicates that there are infinitely many solutions.

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

3. Inconsistent

When a system is inconsistent, there are no solutions to the system. The resulting matrix will look like the one shown below. The bottom row indicates there are no solutions because when the bottom row shows that a zero is equal to any number other than zero, there are no solutions to the system.

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 0 & | & 3 \end{bmatrix}$$

Focus 5: Cramer's Rule: The Method of Determinants

Every $m \times m$ matrix has a unique determinant. The determinant of a matrix is a real number. Determinants are an important mathematical concept because they tell you whether or not a matrix can be inverted. Finding the determinant depends on the dimensions of the matrix; however, determinants only exist for square matrices.

For a two by two matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is

$$D = \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a three by three matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant is

$$D = \det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

To solve a system of equations with two unknowns, Cramer's Rule states that the unique solution to the system of equations:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

is given by:

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

in which

$$D = \det \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

$$D_x = \det \begin{vmatrix} c & b \\ f & e \end{vmatrix} \text{ (replace the coefficients of } x \text{ with the solutions)}$$

and

$$D_y = \det \begin{vmatrix} a & c \\ d & f \end{vmatrix} \text{ (replace the coefficients of } y \text{ with the solutions)}$$

Cramer's Rule works if and only if $D \neq 0$ because you cannot divide by zero.

Example 1: Use Cramer's rule to solve the following simultaneous equations

$$\begin{aligned}2x + y &= 7 \\ 3x - 4y &= 5\end{aligned}$$

Step 1) Put the system of equations into matrix form, and calculate the determinant.

$$D = \det \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = (2)(-4) - (1)(3) = -8 - 3 = -11$$

Step 2) Calculate D_x by replacing the coefficients of x (i.e. 2 and 3 which are in column one of D) with the terms or solutions on the right hand sides of the equations (i.e. 7 and 5).

$$D_x = \det \begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix} = (-4)(7) - (1)(5) = -28 - 5 = -33$$

Step 3) Calculate D_y by replacing the coefficients of y (i.e. 1 and -4, which are in column two of D) with the terms or solutions on the right hand sides of the equations (i.e. 7 and 5).

$$D_y = \det \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} = (2)(5) - (7)(3) = 10 - 21 = -11$$

Step 4) Calculate x and y

$$\begin{aligned}x &= \frac{D_x}{D} = \frac{\det \begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix}}{\det \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}} = \frac{-33}{-11} = 3 \\ y &= \frac{D_y}{D} = \frac{\det \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}}{\det \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}} = \frac{-11}{-11} = 1\end{aligned}$$

Thus we have found that $x = 3$ and $y = 1$. We must check these solutions by substituting both values back into the original equations:

$$\begin{aligned}2(3) + 1 &= 7 \\ 3(3) - 4(1) &= 5\end{aligned}$$

The coordinate point $(3, 1)$ holds true for both equations; thus, it is a solution to the system.

To solve a system of equations with three unknowns, Cramer's Rule states that the unique solution to the system of equations:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3\end{aligned}$$

is

$$x_1 = \frac{D_{x_1}}{D}, x_2 = \frac{D_{x_2}}{D}, x_3 = \frac{D_{x_3}}{D}$$

in which

$$D = \det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_{x_1} = \det \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, D_{x_2} = \det \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, D_{x_3} = \det \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

As stated above, Cramer's Rule works if and only if $D \neq 0$.

Example 2: Use Cramer's rule to solve the following simultaneous equations

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 3 \\2x_1 + x_2 - x_3 &= 5 \\3x_1 - x_2 + 2x_3 &= 12\end{aligned}$$

Step 1) Put the system of equations into matrix form, and calculate the determinant.

$$\begin{aligned}D &= 1 * \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) * \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 * \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\D &= 1(2 - 1) + 2(4 + 3) + 1(-2 - 3) \\D &= 1 + 14 - 5 = 10\end{aligned}$$

Step 2) Find the value of D_{x_1} .

$$\begin{aligned}D_{x_1} &= \det \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = \det \begin{vmatrix} 3 & -2 & 1 \\ 5 & 1 & -1 \\ 12 & -1 & 2 \end{vmatrix} = 3 * \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) * \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} + 1 * \begin{vmatrix} 5 & 1 \\ 12 & -1 \end{vmatrix} \\D_{x_1} &= 3(2 - 1) + 2(10 + 12) + 1(-5 - 12) \\D_{x_1} &= 3 + 2(22) + 1(-17) \\D_{x_1} &= 3 + 44 - 17 \\D_{x_1} &= 47 - 17 = 30\end{aligned}$$

$$\begin{aligned}D_{x_2} &= \det \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} = \det \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & -1 \\ 3 & 12 & 2 \end{vmatrix} = 1 * \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} - 3 * \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 * \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix} \\D_{x_2} &= 1(10 - (-12)) - 3(4 - (-3)) + 1(24 - 15) \\D_{x_2} &= 22 - (3 * 7) + 9 = 10\end{aligned}$$

$$\begin{aligned}D_{x_3} &= \det \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} = \det \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & 12 \end{vmatrix} = 1 * \begin{vmatrix} 1 & 5 \\ -1 & 12 \end{vmatrix} - (-2) * \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix} + 3 * \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\D_{x_3} &= 1(12 - (-5)) + 2(24 - 15) + 3(-2 - 3) \\D_{x_3} &= 17 + 18 - 15 = 20\end{aligned}$$

Now, we can solve for x_1 , x_2 , and x_3 using the determinants we found.

$$\begin{aligned}x_1 &= \frac{30}{10}, x_2 = \frac{10}{10}, x_3 = \frac{20}{10} \\x_1 &= 3, x_2 = 1, x_3 = 2\end{aligned}$$

Thus, we have solved the system of equations using Cramer's rule, but we must check our solutions in the original equations.

$$\begin{aligned}3 - 2(1) + 2 &= 3 \\2(3) + 1 - 2 &= 5 \\3(3) - 1 + 2(2) &= 12\end{aligned}$$

The solutions hold true in all three equations, therefore, $x_1 = 3, x_2 = 1, x_3 = 2$ is a solution to the system.

Cramer's rule is quite laborious but it is an effective technique for calculating a unique solution for a given systems of equations; however, students must have a strong understanding of how to calculate determinants of different square matrices.

Focus 6: *Real – world example problems using simultaneous equations*

In real – world application problems, systems of linear equations may not have solutions that are exact integers; they may be incredibly complex. However, most of the example problems that students will face in the high school curriculum are much simpler. A few examples of the real – world application problems that can be seen in the high school curriculum are shown below.

Example 1: A newsstand owner sold a total of 96 newspapers and magazines, which brought in a total of \$117.54 in a day. He sells newspapers for \$0.55 and magazines for \$2.21. Write the system of equations in two variables to represent the scenario. Solve the system and explain your method. What does the solution mean in the context of the problem?

Step 1) Label your variables, and set up the system of equations using the context of the word problem. Let n = newspapers and m = magazines.

$$\begin{aligned}n + m &= 96 \\0.55n + 2.21m &= 117.54\end{aligned}$$

Step 2) Solve the system of equations using substitution, elimination, graphing, or matrices. In this example, I will show how to solve the system of equations using substitution. Solve for n or m in the first equation, then substitute the variable in the second equation.

$$\begin{aligned}n &= 96 - m \\0.55(96 - m) + 2.21m &= 117.54\end{aligned}$$

Step 3) Solve for m .

$$\begin{aligned}52.8 - 0.55m + 2.21m &= 117.54 \\1.66m &= 64.74 \\m &= 39\end{aligned}$$

Step 4) Solve for n by substituting m into one of the original equations.

$$\begin{aligned}n + m &= 96 \\n + 39 &= 96 \\n &= 57\end{aligned}$$

Step 5) Check your solutions by substituting the value of n and m into both of the original equations.

$$\begin{aligned}57 + 39 &= 96 \\0.55(57) + 2.21(39) &= 117.54\end{aligned}$$

The values hold true in both equations; thus, $n = 57$ and $m = 39$ is a solution to the system of equations. It is important for students to understand what the solution means in the context of the problem. Therefore, in the context of this problem, this means that the newsstand owner sold 57 newspapers and 39 magazines.

Example 2: The admission fee at a baseball game is \$3.50 for children and \$7.50 for adults. On a certain night, 1,500 people enter the baseball game and \$6,050 is collected. How many children and how many adults attended the baseball game?

Step 1) Label your variables, and set up the system of equations using the context of the word problem. Let c = number of children and a = number of adults.

$$\begin{aligned}3.50c + 7.50a &= 6,050 \\a + c &= 1,500\end{aligned}$$

Step 2) Solve the system of equations using substitution, elimination, graphing, or matrices. In this example, I will show how to solve the system of equations using substitution. Solve for a or c in the first equation, then substitute the variable in the second equation.

$$c = 1,500 - a$$

Step 3) Solve for a by substituting in c .

$$\begin{aligned}3.50(1,500 - a) + 7.50a &= 6,050 \\5,250 - 3.50a + 7.50a &= 6,050 \\5,250 + 4a &= 6,050 \\4a &= 800 \\a &= 200\end{aligned}$$

Step 4) Solve for c by substituting a into one of the original equations.

$$\begin{aligned}a + c &= 1,500 \\200 + c &= 1,500 \\c &= 1,300\end{aligned}$$

Step 5) Check your solutions by substituting the value of a and c into both of the original equations.

$$\begin{aligned}200 + 1,300 &= 1,500 \\3.50(1,300) + 7.50(200) &= 6,050\end{aligned}$$

The values hold true in both equations; thus, $a = 200$ and $c = 1,300$ is a solution to the system of equations. It is important for students to understand what the solution means in the context of the problem. Therefore, in the context of this problem, this coordinate pair means that there were 200 adults and 1,300 children that attended the baseball game on that certain night.

Example 3: A landscaping company placed two orders with a nursery. The first order was for 13 rose bushes and 4 dogwood trees, and totaled \$487. The second order was for 6 rose bushes and 2 dogwood trees, and totaled \$232. The owner of the landscaping company would like to know how much each item was; however, the receipts from the nursery do not list the price of each item. What is the cost of one rose bush and of one tree?

Step 1) Set up the system of equations and label your variables.

Let r = price of an individual rose bush
Let d = price of an individual dogwood tree

$$\begin{aligned}13r + 4d &= 487 \\6r + 2d &= 232\end{aligned}$$

Step 2) Solve the system of equations using any technique. We will use elimination to solve the system:

Multiply the second equation by (-2) so the d variable will cancel out.

$$\begin{aligned}13r + 4d &= 487 \\-2(6r + 2d = 232)\end{aligned}$$

Simplify the equations to get the following system:

$$\begin{aligned}13r + 4d &= 487 \\-12r - 4d &= -464\end{aligned}$$

Add the equations together to cancel out the d variable and solve for r .

$$\begin{aligned}13r + 4d &= 487 \\-12r - 4d &= -464\end{aligned}$$

$$r = 23$$

Step 3) Substitute r into one of the original equations and solve for d .

$$13(23) + 4d = 487$$

$$299 + 4d = 487$$

$$4d = 188$$

$$d = 47$$

Step 4) Check your solutions by substituting the value of r and d into both of the original equations.

$$13(23) + 4(47) = 487$$

$$6(23) + 2(47) = 232$$

The values hold true in both equations; thus, $r = 23$ and $d = 47$ is a solution to the system of equations. It is important for students to understand what the solution means in the context of the problem. Therefore, in the context of this problem, this solution means that the price for one rose bush was 23 dollars and the price for one dogwood tree was 47 dollars.

Post Commentary:

In mathematics, linear systems serve as a fundamental basis for linear algebra. Having a strong understanding of how to manipulate variables and simplify equations is needed for higher-level mathematics. A few topics that were not emphasized throughout this document were the inverses of matrices, eigenvalues, eigenvectors, and how the determinant is related to each of these concepts. Each of these concepts is stressed within higher mathematics; thus, these topics may serve as extensions to this situation. Further, not only are systems of equations taught in the high school curriculum but also systems of inequalities are an important concept. Thus, having solid background knowledge of the properties of systems of inequalities may be helpful in teaching this situation as well. Additionally, a concept that was not covered within this document is the concept of equivalent systems. Two linear systems using the same set of variables are equivalent if each of the equations in the second system can be derived algebraically from the equations in the first system, and vice-versa. In addition, two linear systems are equivalent if and only if they have the same solution set. Therefore, these topics could be discussed in further detail.

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