Circle Inversions and Applications to Euclidean Geometry

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Chapter 0

Introduction

We have seen that reflections and half turns are their own inverses, that is $M_l^{-1} = M_l$ and $H_O^{-1} = H_O$, or equivalently, $M_l^2 = I$ and $H_0^2 = I$, where I is the identity function on the plane. No other isometries or similarity transformations have this property. However, a new transformation — *inversion in a circle* — which will be introduced in this supplement, also is its own inverse. This particular transformation was probably first introduced by Apollonius of Perga (225 BCE – 190 BCE). The systematic investigation of inversions began with Jakob Steiner (1796-1863) in the 1820s [**?**], who made many geometric discoveries using inversions by the age of 28. During the following decades, many physicists and mathematicians independelty rediscovered inversions, proving the properties that were most useful for their particular applications. For example, William Thomson used inversions to calculate the effect of a point charge on a nearby conductor made of two intersecting planes [**?**]. In 1855, August F. Möbius gave the first comprehensive treatment of inversions, and Mario Pieri developed the subject axiomatically and systematically in *New Principles of the Geometry of Inversions*, memoirs I and II in the early 1910s, proving all of the known results as its own geometry independent of Euclidean geometry [**?**].

An inversion in a circle, informally, is a transformation of the plane that flips the circle inside-out. That is, points outside the circle get mapped to points inside the circle, and points inside the circle get mapped outside the circle.

Definition 0.1. Let C be a circle with radius r and center O. Let T be the map that takes a point P to a point P' on the ray OP such that $OP \cdot OP' = r^2$. Then, T is an inversion in the circle C.

Often, an inversion in a circle is referred to simply as an inversion. Notice that points inside the circle and close to the center, O, get sent to points that are very far away from the circle. In addition, it can be easily verified that points on C are fixed by inversion in C. However, there is one point in the plane that does not have an image under inversion, O. As P gets closer to O, P' get farther away from O, so in some sense, we can think of O as mapping to a point at infinity. We will explore this in more detail in Chapter **??**, with a geometric interpretation on the sphere. For now, we will say that the center of the circle of inversion, O, is mapped to a "point at infinity" on the extended plane, and O will be called the *center* of the inversion. Note that if P maps to P', then P' also maps to P. We say that P and P' are *inverses* with respect to the inversion in C.

Problem 1. *Given a circle C with center O and a point P, we would like to be able to construct the image of P under inversion in C. We will do this for a point P outside the circle C.*

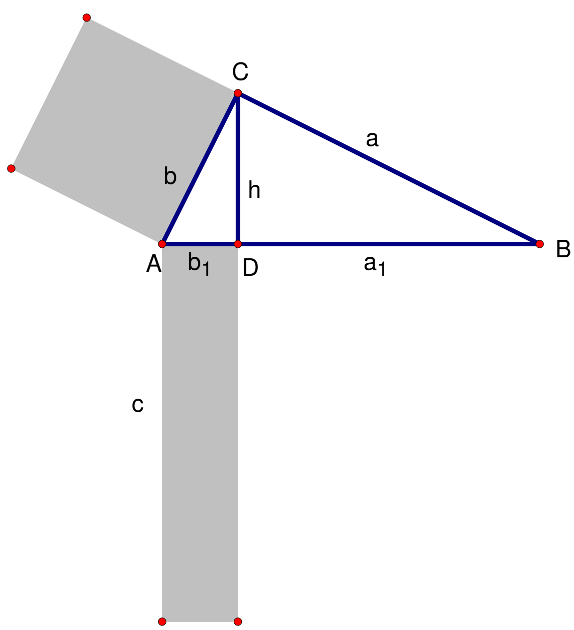


Figure 1: A right triangle ABC with $b^2 = b_1 c$.

Investigation: Given a point P, we would like to construct a point P' such that $OP \cdot OP' = r^2$. This brings to mind the result from Euclid's proof of the Pythagorean theorem (pages 132-134 in the text). In Figure **??**, $b^2 = b_1 c$ where c is the hypotenuse and b_1 the projection of leg b onto the hypotenuse.

Consequently, we have the following construction.

Construction: Construct a circle through O centered at the midpoint of $\overline{OP}$. Let A be one of the two points of intersection of this circle with C. Construct the perpendicular to $\overline{OP}$ that passes through A. Then P' is the intersection of this perpendicular with $\overline{OP}$ (see Figure **??**).

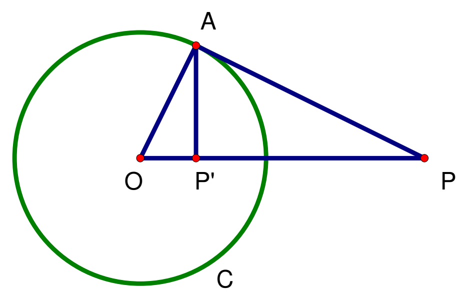


Figure 2: Construction of the inverse of a point P outside the circle C of inversion.

Proof. By construction, $\angle AP'O$ is a right angle, since $\overline{AP'} \perp \overline{OP}$. Also, $\angle OAP$ is a right angle since it is the inscribed angle of a semicircle. Because $\angle AOP = \angle AOP'$, the two triangles $\triangle AOP'$ and $\triangle POA$ are similar.

Consequently, the ratios of corresponding sides are equal. Therefore,

$$\frac{OP}{OA} = \frac{OA}{OP'}.$$

Letting $OA = r$, we have that $OP \cdot OP' = r^2$. So P' is the inverse of P. $\square$

We will now introduce some notation. If C is a circle, we will write the inversion with respect to C as I_C, so that if P' is the inverse of P, then $I_C(P) = P'$. Notice that $I_C(P') = P$ also, so that $I_C(I_C(P)) = I_C(P') = P$. Thus, $I_C^2 = I_C \circ I_C$ is the identity function on the plane. Recall that reflection M_l in a line l also has the same property, that is M_l^2 is the identity transformation.

In fact, inversion in a circle is related to reflection in a line in another way. Let l be a line, with $P' = M_l(P)$ the reflection of P in l. Take circles $C_1, C_2, C_3 \ldots$ tangent to l with centers $O_1, O_2, O_3, \ldots$ such that $\overleftrightarrow{O_1P} \perp l$ and P between O_i and l for each i as in Figure **??**. Construct the inverse of P

in circle C_i and call it $P_i = I_{C_i}(P)$. Notice that P_i appears to approach the reflection P' of P as i gets bigger (for an interactive demonstration, see [**?**]).

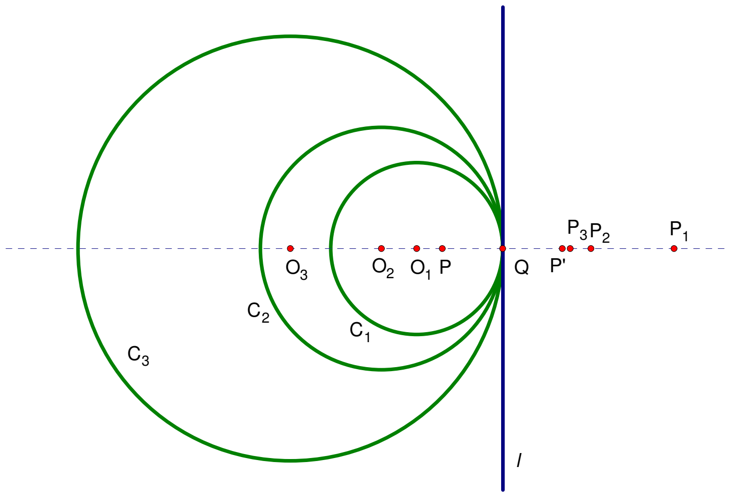


Figure 3: As the circles C_i get bigger, the inverses $P_i = I_{C_i}(P)$ approach the reflection P'.

Now, let's prove that P_i in fact does approach P'. First, notice that by definition of inversion, $O_iP \cdot O_iP_i = r_i^2$ where r_i is the radius of C_i. Thus,

$$O_iP_i = \frac{r_i^2}{O_iP}.$$

but $O_iP = O_iQ - QP = r_i - QP$. Hence,

$$O_iP_i = \frac{r_i^2}{r_i - QP}.$$

We want to see what happens to QP_i as $i \to \infty$.

$$\begin{aligned} QP_i &= O_iP_i - O_iQ \\ &= O_iP_i - r_i \\ &= \frac{r_i^2}{r_i - QP} - r_i \\ &= \frac{r_iQP}{r_i - QP} \\ &= \frac{QP}{1 - \frac{QP}{r_i}}. \end{aligned}$$

So as i increases, so does r_i, and QP_i approaches QP. Hence, P_i approaches P'.

Problem Set ??

1. Construct the inverse of a point P inside the circle of inversion C, and prove that the construction is correct.

2. An alternative construction of the image of P under inversion can be based on the fact that in Figure **??**, $h^2 = a_1 b_1$ (see Theorem 4.10 on page 182 of the text). If P is in the interior of the circle, find its inverse under inversion by first finding its image P^* under a half turn in O, and then finding the point Q such that $OP^* \cdot OQ = r^2$. Complete and describe this approach.

3. **Inversion on the Complex Plane.** Another way of viewing inversions is by defining a function on the complex plane.

   (a) Show that the mapping $F(z) = \frac{1}{z}$ satisfies the equation
   $$|z| \cdot |z'| = 1,$$
   where $z' = F(z)$ is the image of z under F.

   (b) Show that there exists a point $z \in \mathbb{C}$ such that $z' = F(z)$ is not on the ray from the origin to z.

   (c) Notice that the previous part shows that $F(z)$ is not an invesion map since z and z' do not lie on the same ray emanating from the origin. We can fix this by defining
   $$z' = \bar{F}(z) = \frac{1}{\bar{z}}$$
   as our inversion mapping. Show that z and z' lie on the same ray emanating from the origin and that $\bar{F}$ still satisfies the equation $|z| \cdot |z'| = 1$. Thus, $\bar{F}$ is inversion in a circle of radius 1 centered at the origin.

   (d) Show that inversion in a circle of radius r centered at the origin of the complex plane can be written as $D_r \circ \bar{F} \circ D_r^{-1}(z)$, where $D_r(z) = rz$ is dilatation by a factor of r.

(e) Use translations and the previous exercise to write the equation for inversion in a circle of radius r centered at $z_0 = a_0 + b_0 i$.

Chapter 1

Properties of Inversions

Now that we know what happens to points under an inversion in a circle, it is of interest to know what happens to basic geometric objects under inversions. This will turn out to be helpful in using inversions to solve problems in geometry.

First, let us consider the image of lines under an inversion.

Theorem 1.1. *The image of a line through the center of the inversion is itself.*

Proof. Let O be the center of the inversion, and l a line through O. Let P be a point on l and P' the image of P under the inversion. Then, P' is on the ray OP, so P' is also on l. Thus, every point on l maps to a point on l. Moreover, P' maps to P under the inversion, so every point on l is in the image of l. Hence, it must be that the image of a line through the center is itself. □

What about a line not through the center of the inversion? Consider a line l that intersects the circle of inversion but does not pass through its center O (see Figure **??**). Let's see what happens by constructing the inverse of a few points on the line. First, take the perpendicular to l through O, and let P be the point of intersection of the perpendicular and l. Construct P' using the construction outlined previously. Observe that the image of a point on l farther and farther from O is closer and closer to O. This suggests that the image of l cannot be a line.

To figure out exactly what the image of l looks like, take any other point Q on l. Construct Q' as in Figure **??**. By definition of the inversion map, we know that

$$OP \cdot OP' = r^2$$
$$OQ \cdot OQ' = r^2$$

where r is the radius of the circle of inversion. Hence, $OP \cdot OP' = OQ \cdot OQ'$, or equivalently,

$$\frac{OP}{OQ} = \frac{OQ'}{OP'}.$$

Since $\triangle OPQ$ and $\triangle OQ'P'$ have a common angle and the ratio of the two adjacent sides are the same, the two triangles are similar. By construction, $\angle OPQ$ is a right angle, so $\angle OQ'P'$ is also a right angle.

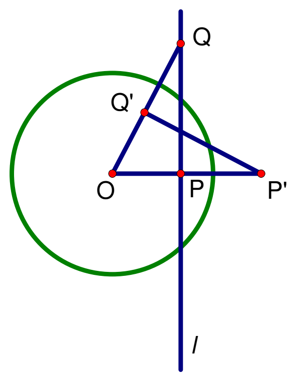


Figure 1.1: The image of the point P on the line closest to O and the image of any other point Q.

Thus, $\angle OQ'P'$ is inscribed in a circle with OP' as diameter so that Q' must lie on this circle. As a consequence, the image of l is contained in a circle with diameter OP'. By reversing the construction, every point on the circle has a point on l as its preimage, so the image of l is in fact the entire circle with OP' as its radius. The same argument holds even if l does not intersect the circle of inversion. Conversely, it can be shown that the image of a circle through O is a line by reversing the above construction. We have proved the following theorem:

Theorem 1.2. *Let C be a circle with center O. The image under inversion in C of a line that does not pass through O is a circle through O. Conversely, the image of a circle that passes through O is a line that does not pass through O.*

The above theorem tells us that circles through O map to lines under inversion in a circle centered at O. It is natural to wonder what happens to circles not through O when they are inverted. If our inversion is an inversion in the circle C, we already know that points of C are their own inverses, so C is fixed under the inversion. That is, the circle C maps to a circle (namely, the circle C). If C_0 is a circle with radius r_0 that is concentric with C — that is C_0 has O as its center — then each point of C_0 is equidistant from O, so their images under inversion will also be equidistant from O. The image of C_0 is a circle with radius $\frac{r^2}{r_0}$ since if P is a point on C_0, then

$$\begin{aligned} OP \cdot OP' &= r^2 \\ \Rightarrow r_0 \cdot OP' &= r^2 \\ \Rightarrow OP' &= \frac{r^2}{r_0}. \end{aligned}$$

In these special cases, we have seen that circles not through O map to circles. Surprisingly, this fact is true for all circles that do not pass through O, as we shall see in the next theorem.

Theorem 1.3. *The image under inversion of a circle not through the center of the inversion is itself a circle.*

Investigation: Let C be the circle of inversion with center O and C_1 be a circle not through O. Let O_1 be the center of C_1 If we take the point P on C_1 closest to O and the point R on C_1 farthest from O, the segment $\overline{PR}$ is a diameter of C_1. Furthermore, under the inversion I_c, P will map to a point P' in the image of C_1 that is farthest from O and R will map to a point R' in the image of C_1 that is closest to O. So we can expect that $P'R'$ will also be a diameter of the image circle.

Proof. There are three cases, when C_1 is inside C, when C_1 intersects C, and when C_1 lies outside of C. We will prove the first case, and as a consequence, the third case. Take the ray OO_1, and label the intersections of the ray and C_1 as P and R. Then, if P' is the inverse of P and R' is the inverse of R, both P', R' lie on the ray OO_1. Let Q be another point on C_1, and take Q' to be the image of Q under inversion in C. We know that $\angle PQR$ is a right angle because it is inscribed in a semicircle. If we can show that $P'Q'R'$ is also a right angle, then $\angle P'Q'R'$ is inscribed in a circle with $P'R'$ as a diameter, so Q' lies on a circle with diameter $P'R'$.

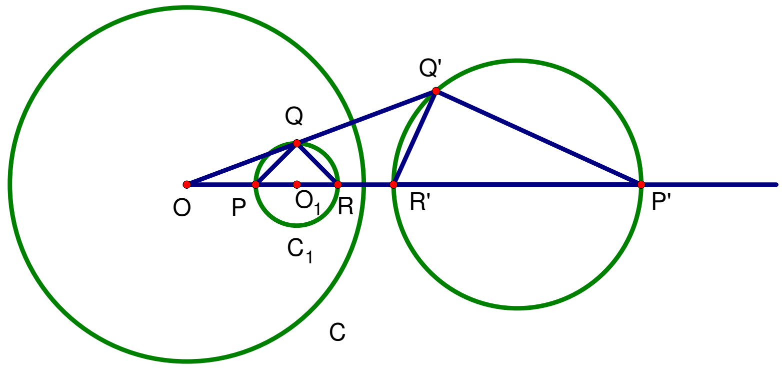


Figure 1.2: A circle mapping to a circle under inversion in C.

By definition of inversion in a circle, $OP{\cdot}OP' = OQ{\cdot}OQ' = OR{\cdot}OR' = r^2$, where r is the radius of C. Thus,

$$\frac{OP}{OQ} = \frac{OQ'}{OP'}$$
$$\frac{OR}{OQ} = \frac{OQ'}{OR'}.$$

Since $\triangle OPQ$ and $\triangle OQ'P'$ share the angle $\angle POQ$ and the ratios of the adjacent sides are equal, they are similar. Likewise, $\triangle ORQ$ and $\triangle OQ'R'$ are similar.

Therefore, $\angle OPQ = \angle OQ'P'$. But $\angle OPQ$ is an exterior angle, so it is the sum of the two opposite interior angles. Thus, $\angle OPQ = \angle PRQ + \angle PQR$. We also know that $\angle ORQ = \angle OQ'R'$ since $\triangle OQR$ and $\triangle OR'Q'$ are similar. Hence,

$$\begin{aligned}
\angle OPQ &= \angle OQ'P' \\
\angle PRQ + \angle PQR &= \angle OQ'R' + \angle P'Q'R' \\
\angle ORQ + \angle PQR &= \angle ORQ + \angle P'Q'R' \\
\angle PQR &= \angle P'Q'R'.
\end{aligned}$$

Angle $\angle PQR$ is a right angle, so it must be that $\angle P'Q'R'$ is also a right angle.

So far, we have shown that the image of every point on C_1 is a point on the circle with diameter $R'P'$. To show that image of C_1 is the entire circle with diameter $R'P'$, we need only to show that the preimage of every point S on th e circle with diameter $R'P'$ is a point on C_1. For that purpose, consider $I_C^{-1}(s)$. Because $I_C^{-1} = I_C$, $I_C^{-1}(s) = I_C(s) = w$. It suffices to show that if $\angle R'SP'$ is a right angle, then so is $\angle PWR$. This can be shown in a way similar to what was done for $\angle PQR$ and $\angle P'Q'R'$.

Since C_1 is inside C, the image of every point of C_1 lies outside of C. So the image of a circle inside C that does not pass through O is a circle outside of C. Conversely, the image of a circle outside of C is a circle inside C that does not pass through O. The proof for a circle not through O that intersects C is similar. $\square$

So far, we have seen that under inversion, lines map to lines or circles and circles map to circles or lines, with the result depending on whether the original object passes through the center of the inversion. Clearly, inversion cannot be an isometry since point inside the circle of inversion will become spread out over the rest of the plane when inverted. The theorems we have seen so far also show that inversion is not a simple dilatation because lines can be inverted into circles and vice-versa.

The next best thing we can hope for is that angles are preserved by inversion. Since we know that straight lines do not necessarily invert to straight lines, we will have to clarify what we mean by an angle between two curves. For two curves that intersect at a point P, the angle δ between them at P is the angle between their tangent lines, as in Figure **??**. Since there are two such angles which are supplementary, we will always choose $0 \leq \delta \leq \frac{\pi}{2}$. The next theorem states that angles are, in fact, preserved by inversion.

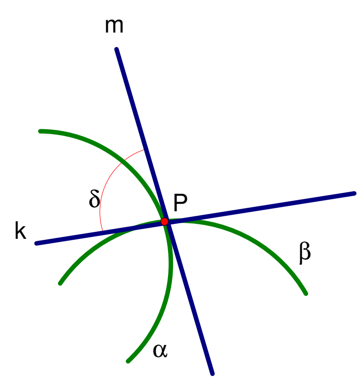


Figure 1.3: The angle between curves α, β is the angle between their tangents k, m at the point of intersection.

Theorem 1.4. *The magnitude of the angle between two intersecting curves is not changed by an inversion.*

Remark: Notice that there are two angles between the two curves α and β. If one of the angles is δ, the other is $180^o - \delta$, so we can always choose the angle δ to be an acute (or right) angle.

Proof. Let C be a circle with O as its center. Take two curves, α and β, which intersect at a point P. Let α', β' be the images of α, β, respectively, under inversion in C, with P' the inverse of P. Take a line through O that intersects both α and β, and denote the points of intersection as M and N, respectively. Let M' be the inverse of M and N' be the inverse of N under inversion in C. Note that M', N' also lie on the line OM.

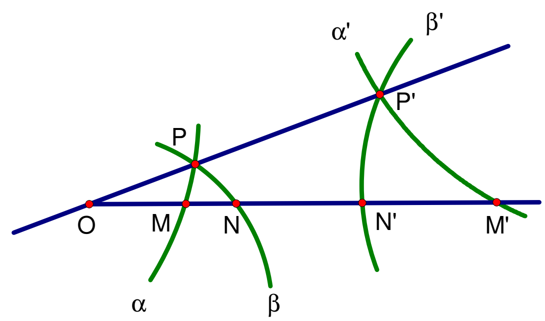


Figure 1.4: Two curves, α and β and their images under inversion about O.

We have shown earlier that $\triangle OPM$ and $\triangle OM'P'$ are similar. Therefore, $\angle OMP = \angle OP'M'$. Similarly, $\triangle ONP$ and $\triangle ON'P'$ are similar, so that $\angle ONP = \angle OP'N'$. Since $\angle OMP$ is an exterior angle in $\triangle OMP$, it is the sum of the two interior angles $\angle MPN$ and $\angle ONP$. Consequently,

$$\begin{aligned}\angle MPN &= (\angle MPN + \angle ONP) - \angle ONP \\ &= \angle OMP - \angle ONP \\ &= \angle OP'M' - \angle OP'N' \\ &= \angle M'P'N'.\end{aligned}$$

As we let the line OM approach the line OP, M and N will tend to P, and M' and N' will tend to P'. In addition, the secants PM and PN will limit to the tangents of α and β, respectively. Likewise, $P'M'$ and $P'N'$ will limit to the tangents of α' and β', respectively. The equality of the angle between the secants holds as we approach this limit, so that the angles between the tangent lines are also equal. □

Notice that although the magnitude of the angle is preserved, the direction of the angle is reversed. That is to say, inversion is an orientation-reversing transformation. A transformation that preserves angles is called a *conformal* transformation. It is customary to differentiate between such transformations that are orientation-preserving and those that are orientation-reversing by calling the latter an *anti-conformal* transformation.

As an interesting corollary of this result, consider two circles C and C_1 such that they intersect at P and Q at right angles, as in Figure **??**. Such circles are called orthogonal. Assume that C_1 does not go through the center of C. What is the image of C_1 under inversion in C? By Theorem **??**, C_1 maps to some circle C_1'. Since points on C are fixed under inversion, C_1' also intersects C at P and Q. Moreover, angles are preserved, so C_1' intersects C at right angles.

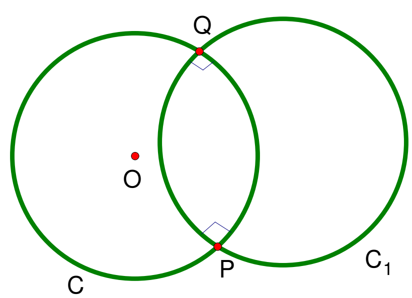


Figure 1.5: Two circles intersect so that the angles formed are right angles.

We claim that $C_1 = C_1'$. This can be proved by showing that there is a unique circle that intersects C perpendicularly at P and Q. Let C' be a circle orthogonal to C through P and Q. Notice that the tangent to C at P is perpendicular to both line OP and to the line tangent to C' at P. Therefore, line OP is the same line as the tangent to C' at P. Similarly, OQ is tangent to C' at Q. Therefore, if l is the perpendicular to OP through P and k is the perpendicular to OQ through Q, then the center of C' is at the intersection O' of k and l, which is unique. There is a unique circle with center O' passing through P, which proves our claim.

Conversely, suppose C_1 is a circle not through O such that $I_C(C_1) = C_1$. Consider the two supplementary angles δ_1 and δ_2 formed at a point of

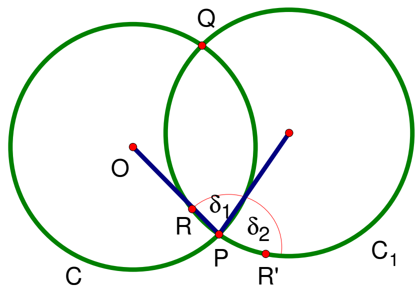


Figure 1.6: Two angles α and β are formed at the intersection of two circles.

intersection of C and C_1, as shown in Figure**??**. To find the image of δ_1 under I_C, take a point R on C_1 inside C. As R approaches P, the angle formed by line RP and the line tangent to C at P approaches δ_1. If we follow the image $R' = I_C(R)$, then the angle formed by line $R'P$ and the line tangent to C at P approaches δ_2. Thus, the image of δ_1 under inversion is δ_2. Since inversion preserves the size of angles, $\delta_1 = \delta_2$. So it must be that δ_1 and δ_2 are right angles. Hence, C and C_1 are orthogonal.

We have proved the following:

Theorem 1.5. *Let C be a circle with center O and C_1 a circle not through O. Then, the image of C_1 under inversion in C is itself if and only if C_1 is orthogonal to C.*

Problem Set ??

1. Prove that the distance between two points is not preserved by inversion.

2. Suppose an equilateral triangle is inscribed inside a circle C. What is the image of the triangle under inversion in C?

3. Prove that the image of a line that does not intersect the circle of inversion is a circle through O.

4. Prove that the image of a circle not through O that intersects the circle of inversion is itself a circle not through O that intersects the circle of inversion.

5. Let C be a circle with center O and C_1 a circle that does not intersect O. The image under inversion in C of C_1 is a circle C_1'. Show that the inverse of the center of C_1 is not necessarily the center of C_1'.

6. Let C be a circle with center O. Let P', Q' be the images of points P, Q under inversion in C. Show that $\triangle OPQ$ is similar to $\triangle OQ'P'$.

7. If A, B, C, D are points in the plane, the *cross ratio* is defined as

$$\frac{AC}{AD} \bigg/ \frac{BC}{BD} .$$

   Show that the cross ratio is invariant under inversion in a circle whose center is not any of the four points A, B, C, or D. That is, if A', B', C', D' are the respective images of A, B, C, D under inversion, that

$$\frac{AC}{AD} \bigg/ \frac{BC}{BD} = \frac{A'C'}{A'D'} \bigg/ \frac{B'C'}{B'D'} .$$

8. Let P' be the image of P under I_C. Prove that any circle through P and P' is orthogonal to C.

Chapter 2

Applications of Inversions

We have seen that the key properties of inversions are that circles map to either circles and lines and that inversion preserves the size of angles. Thus, inversions will prove to be most useful whenever we are dealing with circles.

Inversion in a circle provides another way of looking at geometric problems, sometimes making some problems much easier to solve. For example, consider four circles C_1, C_2, C_3, and C_4 such that each pair of is internally tangent at a single point P. Let C be any circle that passes through P, as in Figure **??**.

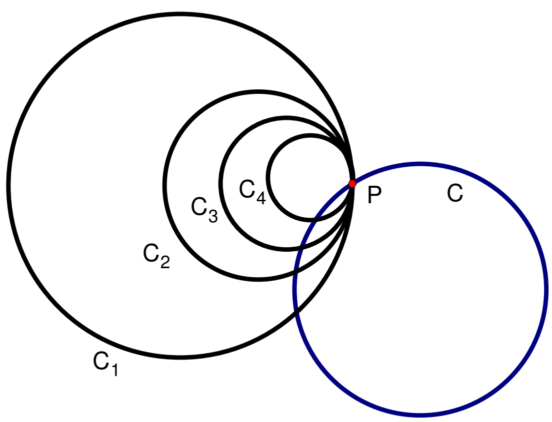


Figure 2.1: Four circles are internally tangent at P, with a fifth circle passing through P.

From the fact that the four circles are internally tangent, it follows that the tangents at P to each of the four circles is the same line. Thus, the angle between the circle C and each of the four circles is the same. When we invert all the circles in a circle centered at P, we get five lines. Since C_1, C_2, C_3, C_4 only intersect at P, their images under inversion only intersect at infinity. Thus, the four lines C_1', C_2', C_3', C_4' corresponding to the images of C_1, C_2, C_3, and C_4, respectively, form four parallel lines in the plane. Because C intersected each of the other four circles, its image C', which is a line, will also intersect the four lines C_1', C_2', C_3', and C_4'. Thus, any problem about the circles is transformed into a problem about parallel lines cut by a transversal. If, for instance, we wanted to know the relations between the angles formed by C and each of the C_i at the intersection not at P, the above shows that the four angles have equal measure.

The Shoemaker's Knife

Consider three semicircles that are mutually tangent at points on a line k, as in Figure **??**. Inscribe a chain of circles $C_1, C_2, \cdot, C_n$ as illustrated. This is called the shoemaker's knife. We would like to show that the center of the nth circle, C_n is at a distance nd_n from the line k, where d_n is the diameter of C_n.

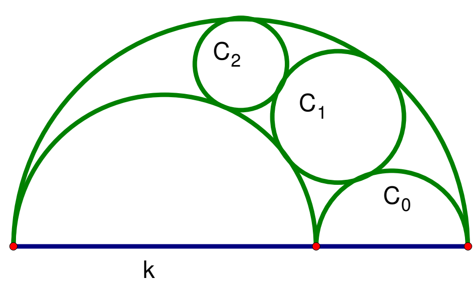


Figure 2.2: The shoemaker's knife.

A clever inversion makes this easy to show. We will demonstrate the proof for C_2. Label the small semicircle on the left A and the large semicircle B (see Figure **??**. Let O be the point at which they are tangent. Let C be a circle with center O that is orthogonal to C_2. Apply the inversion I_C. The result is shown in Figure **??**.

Since C_2 is orthogonal to C, the image of C_2 under inversion is itself. Notice that A and B are circles through the center of inversion, so their

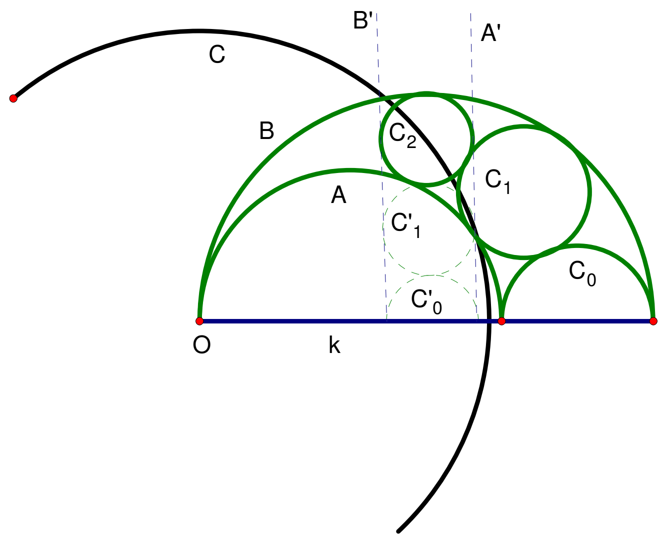


Figure 2.3: Inversion of the shoemaker's knife.

images are lines. By the proof of Theorem **??**, we know that the image line is parallel to the tangent of the circle at O. So A' and B', the images of A and B under inversion, are lines perpendicular to k. Moreover, since A and B were tangent to C_2, A' and B' are tangent to C_2 as well. Thus, the diameter of C_2 is the distance between A' and B'.

Now, let's consider C_1', the image of C_1. In the preimage, C_1 was also tangent to A and B, so the image C_1' is tangent to A' and B' since tangency is preserved as a consequence of angles being preserved. Hence, the diameter of C_1' is the distance between A' and B'. Moreover, C_1 was tangent to C_2, so C_1' is also tangent to C_2. Similarly, C_0', the image of C_0 under inversion, is tangent to A', B', and C_1'.

Thus, the result of the inversion is a sequence of circles all congruent to C_2, with the center of C_0' on k. Hence, the distance from the center of C_2 to k is $2d_n$. The proof for the nth circle proceeds similarly.

Apollonius' Problem

Inversions in circles can also be helpful in certain geometric constructions. One such example is Apollonius' problem: given three circles, construct a circle tangent to all three circles. In general, the three circles may or may not intersect, and they may have different radii. In some cases, the problem is impossible: for example, when the circles are concentric with different radii. When a solution does exists, there are generally up to eight different circles that are tangent to all three of the givern circles. The constructed circle may be externally tangent to all three circles, internally tangent to one circle and externally tangent to the other two, and so on. We will demonstrate the construction for the circle that is externally tangent to three non-intersecting circles.

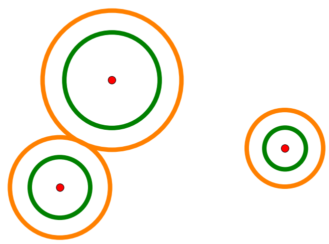


Figure 2.4: The original circles are in green, and the enlarged circles in orange.

Suppose we are given three non-intersecting circles. We can increase the radii of all three circles by a fixed amount, δ, so that the two closest circles are tangent, as in Figure **??**. Now, if we find a circle that is tangent to all three of the enlarged circles, we can increase the radius of that circle by δ, with the resulting circle being tangent to the three original circles.

Thus, we will simplify the problem to the case of three circles such that two are tangent at a point O. We apply an inversion with O as the center, as in Figure **??**. For simplicity, we will take an inversion that leaves the third circle invariant. In other words, let the circle of inversion be a circle with center O that intersects the third circle orthogonally. This is easy to do by finding the two lines through O that are tangent to the third circle. The circle of inversion passes through both of these points.

The result of the inversion is the third circle (which maps to itself) and two parallel lines which are the images of the two circles through O. Since circles not through O map to circles, we need to find a circle C tangent to the parallel lines and the third circle. Then, the preimage of C will be a circle tangent to the three orange circles. The center of C must be equidistant from the two parallel lines, so it lies on the line l that is exactly halfway between C_1' and C_2'. Also, the radius of C is half the distance d between the lines. Thus if the radius of the third circle is r, the center of C is at a distance

$r + \frac{d}{2}$ from the center of the third circle. There are two such points. We will take the point that is farther from O to be the center of C.

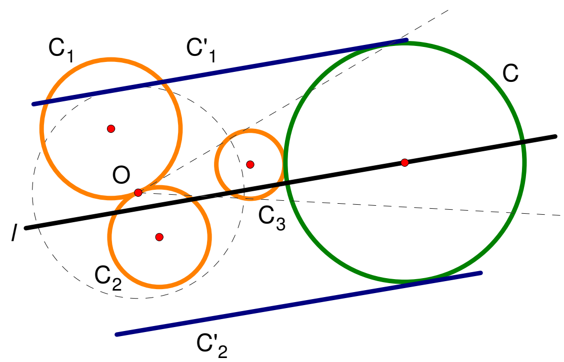


Figure 2.5: The result of inversion about O in the circle marked with dashed lines.

To find the preimage of C, take any three points on C and construct their inverses under the inversion. Then, construct the circle through these three points. The resulting circle is tangent to our three orange circles.

Peaucellier's Linkage

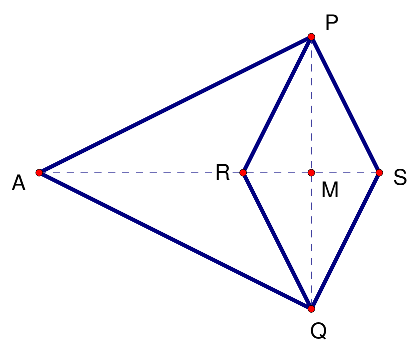


Figure 2.6: Peaucellier's linkage.

Another application of inversions is a solution to a problem that was of great interest in the nineteenth century [?]. At the time, it was thought that there was no way to construct a mechanical device that could take rotational motion and convert it into linear motion. In 1864, a French engineer and captain in the French army by the name of Charles-Nicolas Peaucellier used properties of inversion in a circle to show that such a device was indeed possible.

Peaucellier's linkage consists of six bars, as shown in Figure **??**. The four bars PR, PS, QR, QS are of equal length, and AP and AQ are also of equal length. The bars are free to pivot at each of the points A, P, Q, R, S, and point A is fixed so that it cannot move.

Then, if M is the center of $PSQR$, we have

$$\begin{aligned} AR \cdot AS &= (AM - MR)(AM + MS) \\ &= (AM - MR)(AM + MR) \\ &= AM^2 - MR^2 \\ &= (AP^2 - PM^2) - (PR^2 - PM^2) \\ &= AP^2 - PR^2. \end{aligned}$$

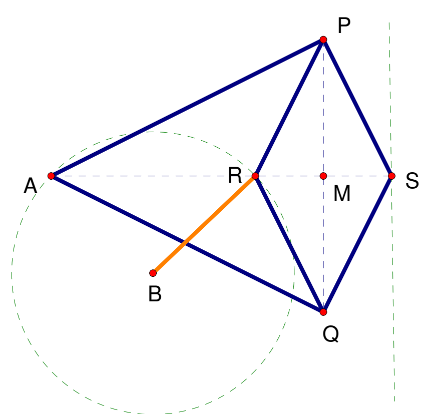


Figure 2.7: The path of R is a circle and the path of S is its image under inversion, a straight line.

Because the lengths AP and PR are fixed, S is the inverse of R in an inversion of a circle of radius $\sqrt{AP^2 - PR^2}$ through A. We know that the inverse of a circle through the center of inversion is a straight line. Thus, if we can make R travel in a circle that also passes through A, then S will travel in a straight line. This can easily be accomplished by adding a seventh bar BR and fixing the position of B such that $AB = BR$ (Figure **??**). Then, R rotates in a cicle that passes through A, so S will travel in a straight line.

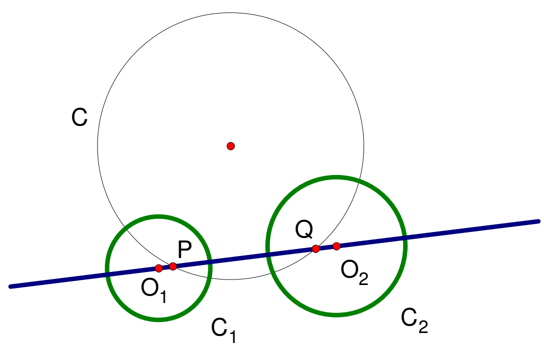
Problem Set ??

1. Given a circle C centered at O and two points A, B outside of C, prove that there exists a circle through A, B and orthogonal to C.

2. In the construction of Apollonius' problem, there were two possibilities for the center of the circle C. What would have happened if we chose the center of C to be the point closer to O? Why did we choose the point farther from O?

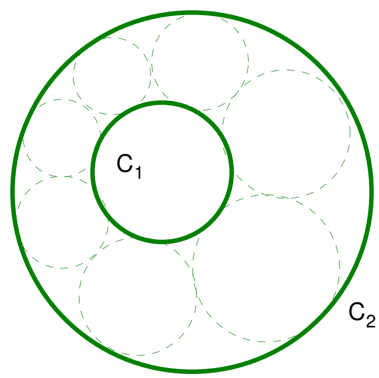
3. **Steiner Chains**

   (a) Let C_1 and C_2 be two circles with centers O_1 and O_2, respectively. Then, any circle C orthogonal to both C_1 and C_2 intersects the line O_1O_2 at two points (see 25.1 in [**?**] for an analytic proof).

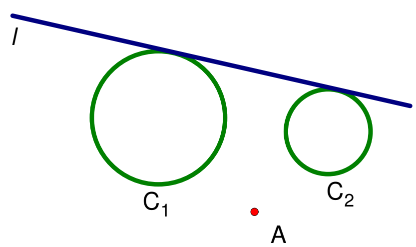
   Use this fact to show that given two non-intersecting circles, there exists an inversion such that the image of the two circles is two concentric circles.

   (b) Let C_1 be a circle lying within the interior of a second circle C_2. Suppose that there exists a chain of circles such that each circle is tangent to both C_1 and C_2, and such that adjacent circles are tangent, as shown below.



This is called a Steiner chain, named after the discoverer of the following property: *if one such chain exists, then no matter where we start the first circle, we will end up with a Steiner chain.* Prove this property.

4. The figure below suggest an alternative method for solving Apollonius' problem. Describe it.



Chapter 3

Stereographic Projection and the Riemann Sphere

We have previously remarked that in a sense, the inverse of the center of an inversion is a "point at infinity." Since as a point P gets closer to the center, it gets sent to a point farther from the circle of inversion, we can think of the center as being sent to a point that is very, very far away. We will attempt to give this intuitive idea a more rigorous treatment.

Stereographic projection is a way to map points on a sphere to the plane. In our particular case, we will find it convenient to let the sphere be a sphere with unit radius, and the plane of projection to be the complex plane: the set of all points (a, b) corresponding to the complex number $a + bi$. We begin by taking the sphere and placing it "on top" of the complex plane so that the plane is tangent to the sphere at O, where O is the origin of the complex plane and the south pole of the sphere.

For each point P on the sphere, we construct the straight line l that passes through P and the north pole N of the sphere, as in Figure **??**. The point P' at which l intersects the complex plane is the image of P under stereographic position. Notice that the image of the south pole is the point O of the complex plane, and the image of a point near the north pole is very far away from the origin. Since the mapping is one-to-one, there is an inverse mapping to stereographic projection. The inverse map takes complex numbers and maps them onto the sphere. We will write the stereographic projection as π, so that $\pi(P) = P'$.

What happens to the north pole, N? Stereographic projection is not defined on the point N, since there is no unique line through N. As remarked

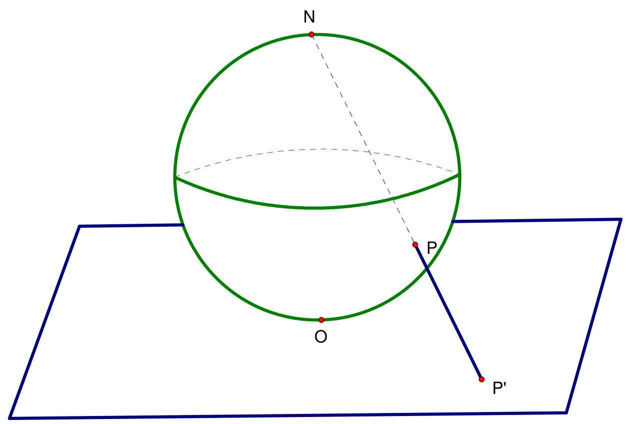


Figure 3.1: Stereographic projection of P to a point P'.

in the previous paragraph, as points get closer and closer to the north pole, their stereographic projections onto the plane get farther and farther away from where the sphere touches the plane. We make N on the sphere correspond to the point ∞ that we adjoint to the plane. In other words, the sphere provides a geometric representation of the extended plane, with the north pole corresponding to the point at infinity.

In general, stereographic projection does not have to be a projection onto the plane tangent to the sphere at the south pole. Often, the plane is taken to be the plane through the equator of the sphere. When the sphere is associated to the complex plane by stereographic projection, the sphere is called the *Riemann sphere*.

Let's see what happens to a basic geometric object on the sphere under stereographic projection. When we studied the properties of inversion in a circle, we first looked at the image of lines. On a sphere, there are no straight lines. If two people met at a designated spot on the Earth and began walking in opposite directions, they will eventually meet each other on the other side of the Earth. Unlike the situation on a plane, they cannot keep walking farther and farther away from each other in a line indefinitely. Thus, on a sphere, circles play a similar role to that of lines on a flat plane, so we will concentrate our efforts on seeing what happens to circles under stereographic projection. We will accomplish this by looking at two cases: for circles through N and circle not through N.

First, let's see what happens to circles through N under stereographic projection. Let C be a circle on the sphere through N. Then, C lies on some plane L, which cuts through the sphere (see Figure **??**). In particular, this plane cannot be tangent to the sphere at N, so it is not parallel to the complex plane. Thus, L intersects the complex plane. Moreover, for every point P on C, the line NP lies on the plane L since both N and P are on L. Consequently, the stereographic projection P' of P must also be on L. It follows that P' is in the intersection of L and the complex plane. But the intersection of two planes is a line, so the image of C is some subset of a line in the complex plane.

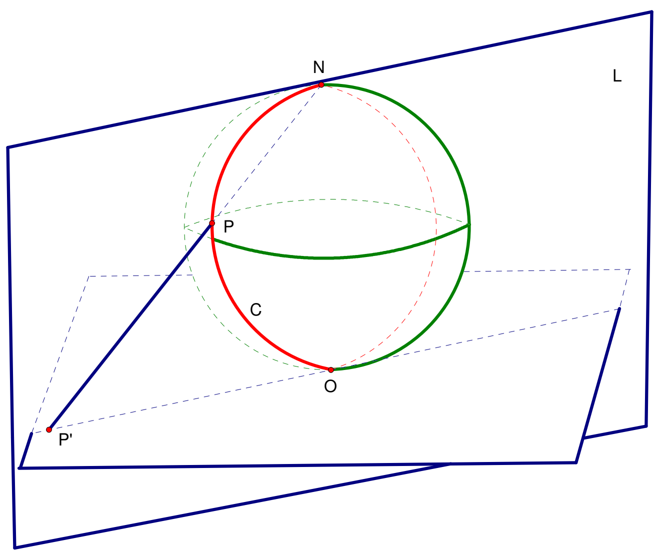


Figure 3.2: Stereographic projection of a circle C through N.

Moreover, the only line through N on L that does not intersect the complex plane is the line tangent to C at N. Hence, for every point P' on the line of intersection between the complex plane and L, there exists a point P on the circle C such that NP intersects P'. Every point on the line has a preimage under stereographic projection, so the image of C is precisely the line of intersection of L and the complex plane. We have proved the following theorem.

Theorem 3.1. *The image under stereographic projection of a circle through the north pole is a line.*

What about circles on the sphere not through N? Perhaps it is simplest to see what happens to stereographic projection of a great circle. If we take Q to be the center of the Riemann sphere and P a point on the equator, then $\triangle NQP$ forms an isosceles right triangle, where $\angle NQP$ is a right angle. If we extend the segment NP to the $P' = \pi(P)$ as in Figure **??**, then $OP'||QP$, so that $\triangle NQP$ and $\triangle NOP'$ are similar.

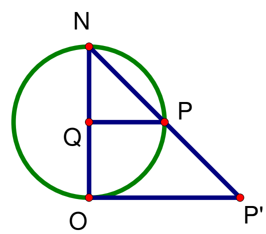


Figure 3.3: $OP'||QP$, so that $\triangle NQP$ and $\triangle NOP'$ are similar.

Consequently, $\triangle NOP'$ is also right isosceles, and $OP' = NO$. This is true for any point P' in the image of the equator under stereographic projection, so the image is contained in the circle of points that have distance NO from the origin O. By reversing the argument, every point in this circle has a point on the equator that maps to it, so the image of the equator is a circle.

A similar argument can be used to show that the image of any circle that is parallel to the plane is a circle. In the following theorem, we will see that this holds even if the circle is not parallel to the plane.

Theorem 3.2. *The image under stereographic projection of a circle on the sphere not through N is a circle in the plane.*

Before proving the theorem, we will need a few facts about geometry in three dimensions. Let C be a circle and V a point that is not in the plane of the circle. For each point on the circle, draw a line through the point and V. The set of all points that lie on such lines is called a *circular cone* with *vertex* V. If the line from V to the center of C is perpendicular to the plane of the circle, then the cone is a *right circular cone*. Otherwise, it is an *oblique circular cone*. If instead of a circle, we take the set of all points on lines from V to an ellipse, then we form an *elliptical cone*. The line through the center of the ellipse (or the circle) and V is called the *axis* of the cone.

One important fact about cones is that the intersection of any plane with a right or oblique circular cone is a conic section — a circle, ellipse, hyperbola, or a parabola (a beautiful illustration and geometrical proofs can be found in [**?**]). In particular, if a plane intersects a cone in all of the lines through its vertex but does not intersect the vertex itself, then the intersection forms an ellipse.

Let L be a plane through the axis of an elliptical cone. The other fact we will use is that if a plane perpendicular to L intersects the cone at an angle of δ with respect to that axis such that the intersection forms a circle, then the plane perpendicular to L that intersects the cone at an angle of $180^o - \delta$ with respect to the axis also intersects the cone in a circle (see Figure **??**).

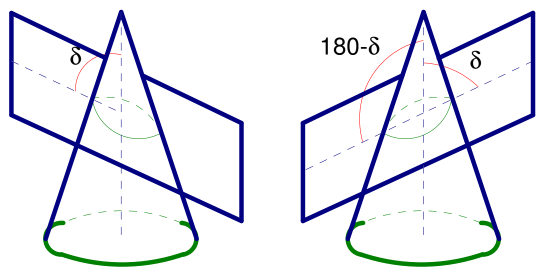


Figure 3.4: Planes cutting a cone at an angle of δ and $180^o - \delta$ intersect the cone in identical conic sections.

Proof of Theorem **??**. Let C be a circle not through N. Notice that we can make a circular cone through circle C with N as its vertex. Let L be a plane through the vertical axis of the sphere such that L divides C into two semicircles. We will look at the intersection of the sphere with L, shown in Figure **??**.

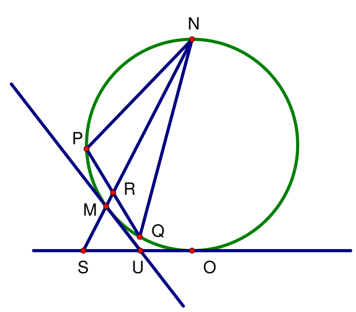


Figure 3.5: A cross-section of the sphere.

In the figure, $\overline{PQ}$ is a diameter of C, and R is the midpoint of the diameter. Then, the line RN bisects $\angle PNQ$. Moreover, any plane through PQ

that is perpendicular to L intersects the circular cone in an ellipse, so the cone is an elliptical cone with line RN as its axis.

Thus, C is the intersection of the cone with a plane through PQ which is perpendicular to L, and its projection $C' = \pi(C)$ is the intersection of the same elliptical cone with a plane through OS, also perpendicular to L. So if we can show that the two angles $\angle NRQ$ and $\angle NSO$ satisfy $m(\angle NSO) = 180^o - m(\angle NRQ)$, we would have that C' is also a circle.

Let M be the point at which RN intersects the circle in Figure **??**, and U the point at which the tangent to the circle at M intersects OS. Then, both lines MU and OU are tangent to the circle, so $m(\angle OMU) = m(\angle MOU)$. But $\angle NRQ = \angle NMU$ and $\angle NMU = \angle NMO + \angle OMU$. Since $\angle NMO$ is inscribed in a semicircle, $m(\angle NMO) = 90^o$. Thus, $m(\angle OMS) = 90^o$. Consequently, we have

$$\begin{aligned} m(\angle NSO) &= 180^o - (m(\angle OMS) + m(\angle MOU)) \\ &= 180^o - (m(\angle NMO) + m(\angle OMU) \\ &= 180^o - m(\angle NMU) \\ &= 180^o - m(\angle NRQ) \end{aligned}$$

as desired. Therefore, C' is a circle. □

So far, we have seen that circles through N map to lines under stereographic projection and that circles not through N map to circles. If instead, we take lines and circles in the plane and map them to the sphere by the inverse of stereographic projection, they will map to circles through N and circles not through N, respectively. Thus, both lines and planes in the plane correspond to circles on the Riemann sphere, and we can think of inversions on the extended plane as simply maps that take circles to circles on the Riemann sphere.

However, to make the analogy complete, we must first check that angles are preserved by stereographic projection. Notice that if two curves on the sphere intersect at N, then their projections intersect at the point at infinity in the extended plane, so it is tricky to define an angle of intersection between the two curves. In order to avoid this confusion, we will restrict ourselves to the angle between two curves intersecting at points other than N.

Theorem 3.3. *The angle between two curves intersecting at a point other than N is preserved by stereographic projection.*

Proof. Let α and β be curves on the sphere that intersect at a point P. Denote the line tangent to α at P by α' and the line tangent to β at P by β'. Let K be the plane through the line α' and the point N, and let L be the plane through the line β' and the point N.

The intersection of the plane K with the sphere is a circle C_1 through N. Similarly, the intersection of L with the sphere is a circle C_2 through N. Take points R on C_1 and S on C_2. Notice that as R, S approach P, $\angle RPS$ approaches the angle between α' and β'. Thus, to find the image of the angle between α and β at P, we need only to find the angle between the images of C_1 and C_2 under stereographic projection (notice that these images are lines that intersect at $P' = \pi(P)$).

The image of C_1 under stereographic projection is precisely the intersection of the complex plane with K, and the image of C_2 is the intersection of the complex plane with L. Call the angle between these two lines δ. Then, take the plane T tangent to the sphere at N. Notice that this plane is parallel to the complex plane. Thus, if we look at the angle δ' between the line of intersection of K with T and the line of intersection of L with T, we have $\delta = \delta'$.

Moreover, since C_1 and C_2 intersect at P and at N. It is not hard to see that if two circles intersect in two points such that the angle of intersection at one of the points is γ, the angle of intersection at the other intersection point is also γ. This is analogous to the situation for circles in two dimensions, and the proof follows by reflecting in a plane through the centers of both circles that takes one point of intersection to the other.

Therefore, the angle formed at N by the intersection of C_1 and C_2 is the same as the angle they form at P. But by construction, this angle is the angle between α' and β'. Consequently, δ, the image of the angle between α and β at P under stereographic projection, is equal to the angle between α' and β' at P'. $\square$

Since stereographic projection and inversions preserve angles, we can think of inversions as angle-preserving maps on the Riemann sphere. When we want to apply an inversion, we first use stereographic projection to map to the complex plane, apply an inversion, then use the inverse of stereographic projection to return to the Riemann sphere. Notice that circles on the sphere map to circles and lines on the plane, and inversions map circles and lines to other circles and lines. Then, when we return to the Riemann sphere, the circles and lines on the plane return to being circles on the sphere.

Thus, when in the context of the Riemann sphere, we can think of a line in the plane as a circle with infinitely large radius so that it closes up only at the point at infinity. An inversion is a transformation of the Riemann sphere that takes circles to circles while also preserving the size of angles. In conjunction with the other familiar transformations of the plane — translation, rotation, reflection, and dilatation — inversions form a group of transformations that map circles to circles and preserve angles. This is called the Möbius transformation group, named after August Ferdinand Möbius (1790-1868), a German mathematician and astronomer who used the group to study projective geometry. A colorful exploration of Möbius tranformations and symmetries can be found in [**?**].

Problem Set ??

1. Suppose x is the real axis of the complex plane, y the imaginary axis, and z an axis perpendicular to the complex plane. Then, the equation of the unit sphere tangent to the complex plane at $(0,0,0)$ is $x^2 + y^2 + (z-1)^2 = 1$. The north pole of the sphere is located at $(0,0,2)$. Let $(x, y, 1 + \sqrt{1 - x^2 - y^2})$ be the coordinates of a point P in the northern hemisphere, for $x^2 + y^2 \leq 1$. Compute the coordinates of the stereographic projection of P.

2. Find the coordinates of the stereographic projection of an arbitrary point in the southern hemisphere.

3. The equation of an arbitrary plane is $ax + by + cz + d = 0$, for constants a, b, c, d. Any circle on the sphere through N can be written as the intersection of the sphere $x^2 + y^2 + (z-1)^2 = 1$ and a plane described by the equation $ax + by + z - 2 = 0$. Prove, using coordinates, that stereographic projection takes a circle on the sphere through N to a line on the xy-plane.

4. Prove, using coordinates, that stereographic projection takes a circle on the sphere not through N to a circle on the xy-plane.