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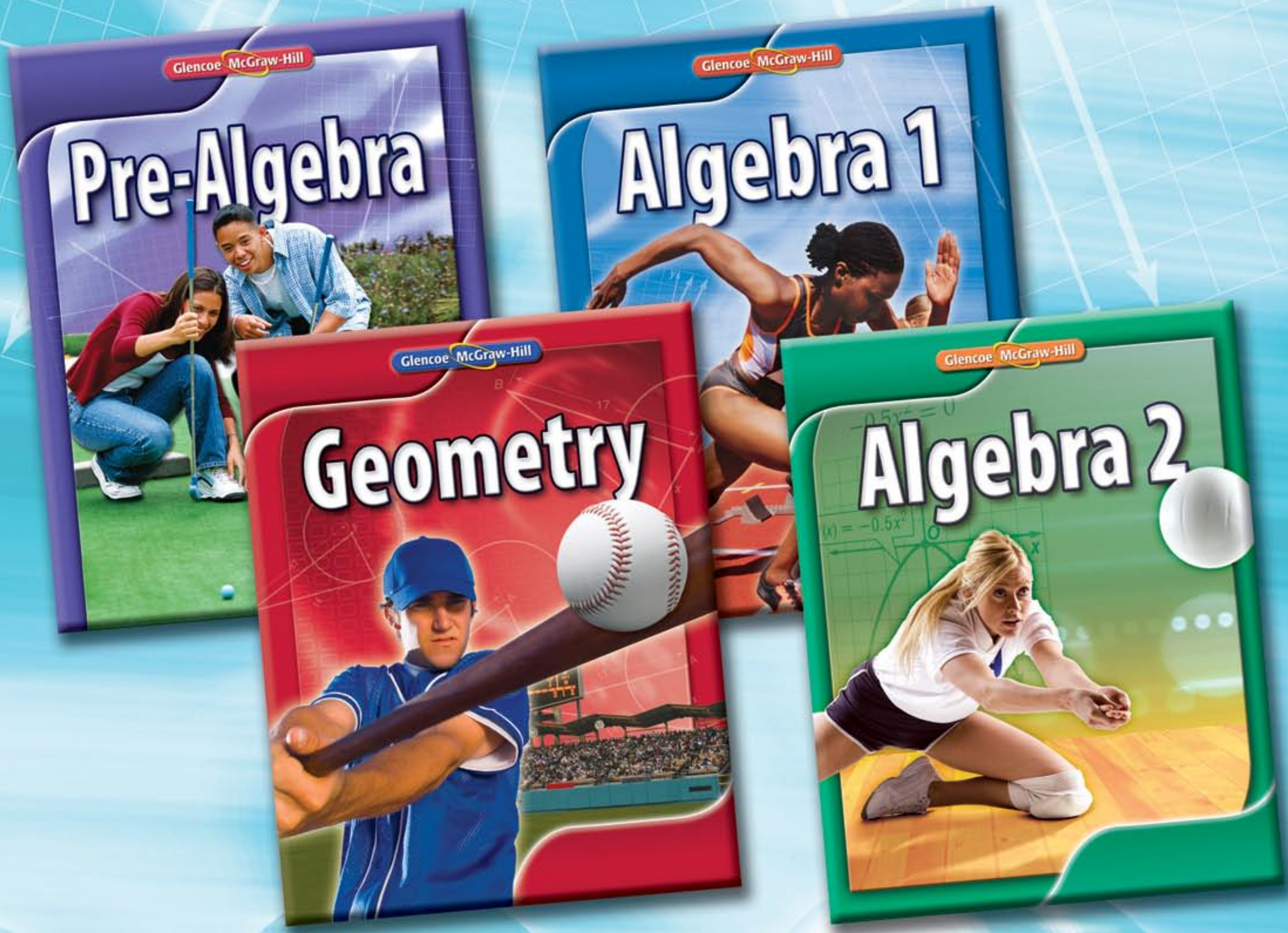

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Building Brighter Futures

Illuminating the Mathematics of Lamp Shades

Creating lamp shades to specific design parameters allows rich explorations in the mathematics of circles and triangles.

Michael E. Matthews and Greg Gross

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A mathematician is a blind man in a dark room looking for a black cat which isn't there.

—Charles Darwin

Darwin's adaptation of a well-known Chinese proverb to mathematics may parallel the feelings of futility that many students may have regarding mathematics. One response to these feelings is for mathematics teachers to show students the beauty of mathematics through realistic and intriguing projects. Using a lamp shade, we take several geometric concepts—similarity, right triangles, and arcs and sectors of circles—and integrate them in a context that encourages students to apply the mathematics they have learned.

This project explores how to create, from a rectangular piece of fabric, a lamp shade according to specific design parameters. We introduce this problem by having students imagine a specialty design company that creates lamps. Each customer specifies the height and the radii of the two circular bases of his or her lamp shade. In **figure 1**, the line segments BD and AF are the radii of the bases, and AB is the height of the shade. The challenge is to discover how to cut material into a lamp shade pattern given the lengths BD, AF, and AB.

The solution to this problem is complicated by the size and shape of the material from which the lamp shade is cut. We simplify the problem by restricting

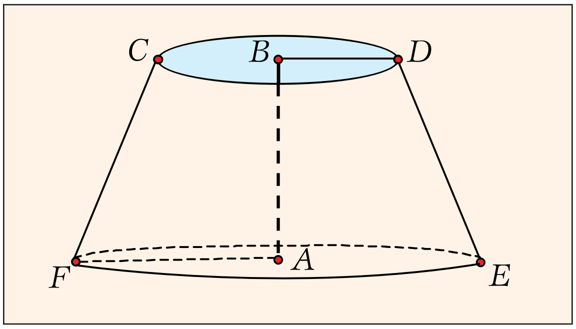


Fig. 1 Lamp shade with relevant quantities labeled

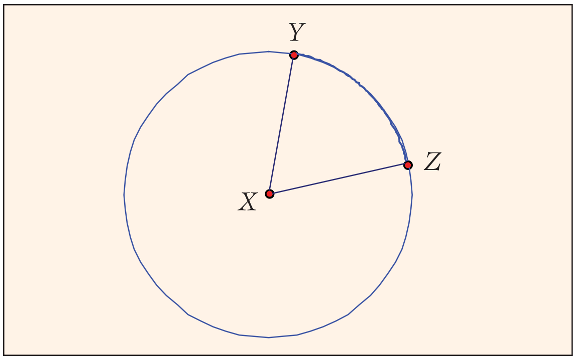


Fig. 2 The ratio of the measure of arc *YZ* to the circle's circumference is the same as the ratio of the measure of angle *YXZ* to 360°.

the size of the paper to 18-by-12-inch construction paper, thus narrowing the options of possible lamp shades but not excessively so. We also provide string-and-pencil compasses to help students create the larger circles needed during the project.

To proceed successfully, students need to be familiar with the following mathematical ideas: the Pythagorean theorem; the concept that corresponding angles formed when parallel lines are cut by a transversal are congruent; the concept of AA similarity, the concept that corresponding lengths of similar triangles are proportional; and the formula for the circumference of a circle. Further, we usually review the relationship between arc length and the corresponding central angle, as illustrated in **figure 2** and the equation below:

$$\frac{\text{measure of arc } YZ}{\text{circumference}} = \frac{\text{measure of angle } ZXY}{360°}$$

The first task we give students is to ask them to discover which basic net (two-dimensional shape) produces a lamp shade. We allow time for exploration. Rectangles and trapezoids (whose nets produce a shape similar to that of a party hat) are discovered and rejected. Eventually, the class determines that the appropriate net is a portion of a sector of a circle, something like that shown in **figure 3**. Once the students discover the net, we ask them to try to describe which quantities in the net would affect the shape of the lamp shade. As

Making a truncated cone to visualize point *G*'s location

students describe and label the quantities that affect the shape of the lamp shade (the measures of $\overline{GC}$, $\overline{GF}$, and $\angle CGC'$), we use a "prototype" net on the front board to facilitate the discussion.

Making the connection between what the customer gives the designer (*BD, AF,* and *AB*) and what is needed to draw the net (*GC, GF*, and $m\angle CGC'$) is a challenging spatial reasoning problem. At this point, we ask students to shed some light on where point *G* on the net (from **fig. 3**) would occur on the actual lamp shade (see **fig. 1**). Although some students are able to visualize point *G*'s location, others eventually make sample nets, wrap these nets up into a truncated cone (lamp shade), label the lamp shade as in **figure 1**, and finally unwrap the lamp shade to see where these labels go when the shade is flattened back into net form. Recognizing that point *G* is above the lamp shade is critical (see **fig. 4**).

Next we challenge students to locate points *A*–*F* on a lamp shade's net. These tasks challenge even those students who were able to find point *G* easily. Inevitably, all the students use labeled sample nets to solve these problems. Note that students do not draw **figure 3** with the labels *C* or *C′* and *F* or *F′* until after they have discovered the appropriate locations for points *C* and *F* (see **fig. 5**). Also note that points

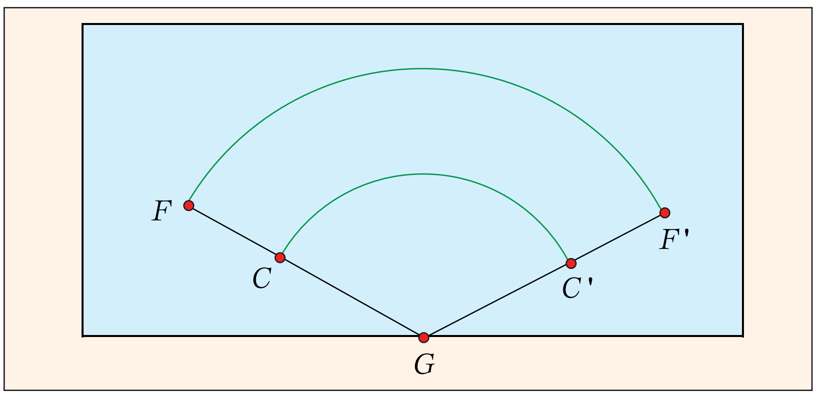


Fig. 3 The net of the lamp shade

A and *B* are not really located on the net. The next big "aha!" moment comes with the nonintuitive realization that the radii needed to make the net (GC and GF) are *not* the same as the radii of the top and bottom circles of the lamp shade (compare **figs. 3** and **4**). Thus, the problem becomes finding generalized formulas for GC, GF, and $m\angle CGC'$ from the given customer-specified lengths.

We generally encourage students to find GC first. They usually notice that if they could discover the measure of GB, they could then use the Pythagorean theorem to calculate the length of GC. One way to find GB comes from the similarity of triangles GBC and GAF. We eventually ask leading questions, such as "What can you say about the measures of angles GCB and GFA?" to guide students to see the AA similarity. Sometimes we need to collapse **figure 4** to help students see the relationship (see **fig. 6**). Using these similar triangles, students then come up with a general formula for finding GB from the known quantities. For convenience, we usually relabel some quantities. For instance, we might let r represent the lengths BD and BC, R the length AF, and h the length AB:

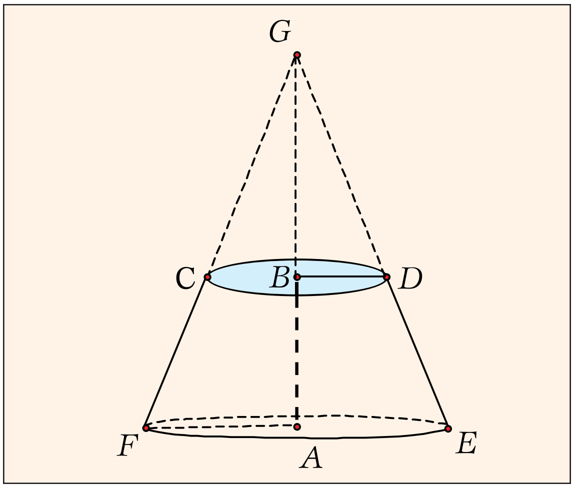


Fig. 4 Realizing that point *G* exists helps students visualize the problem.

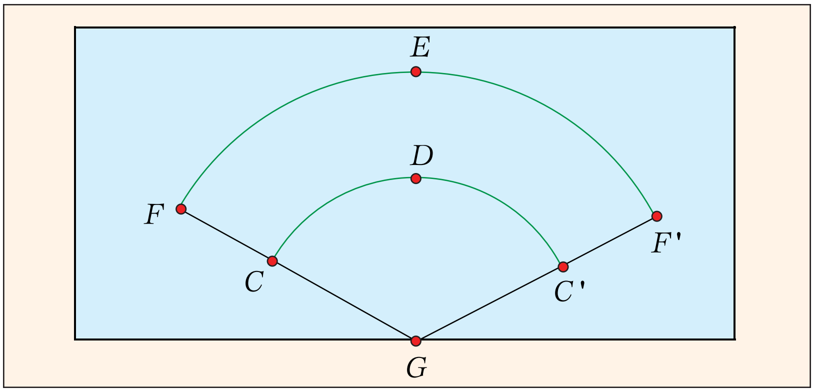


Fig. 5 Students are challenged to locate points on the net and relate them to the lamp shade.

$$\frac{GB}{r} = \frac{GB+h}{R} \to R \bullet (GB) = r \bullet (GB) + rh \to GB = \frac{rh}{R-r}$$

Then, using the Pythagorean theorem again, students get a general solution for GC, as shown here:

$$GC = \sqrt{\left(\frac{rh}{R-r}\right)^2 + r^2}$$

The students' next challenge is to find GF. Usually, students discover with little guidance that they can find a generalized solution for CF (and thus GF) once GC is known (see **fig. 7** and the following equation):

$$CF = \sqrt{h^2 + (R-r)^2} \to$$

$$GF = \sqrt{\left(\frac{rh}{R-r}\right)^2 + r^2} + \sqrt{h^2 + (R-r)^2}$$

The students' subsequent task is to come up with a general formula for the measure of $\angle CGC'$ on the basis of the known values. Students must apply the proportional relationship involving arc lengths, circumferences, and a circle's central angle to this project. Some students do not readily think of this relationship; at times we do a quick side activity to help them rediscover it. The students

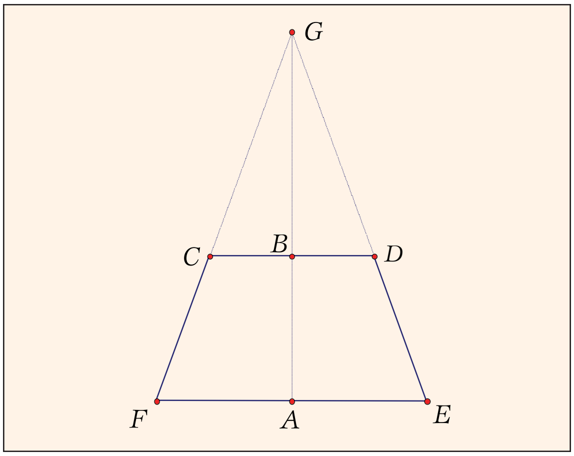


Fig. 6 A collapse of the image in **figure 4**

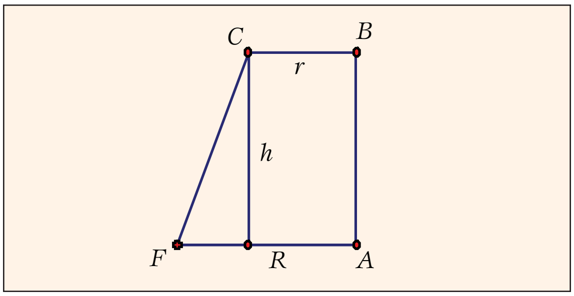


Fig. 7 A portion of the net may help in generalizing relationships.

Table 1

Dimensions (in inches) of lamp shades made from 18-by-12-inch construction paper

r	0.625	0.5	2.75	2.0	2.000	0.500	0.50	5.0
R	4.000	1.5	3.25	3.5	4.125	1.125	1.50	1.5
h	6.750	7.0	1.50	3.0	4.000	1.500	8.25	10.0

quickly see that they need the measure of arc CC'. Then, given time to think about it, they generally discover that CC' wraps around and becomes the top (or smaller) circle of the lamp shade. That is, they realize that the measure of arc CC' in the net equals the circumference of the smaller circle in the lamp shade (or $2\pi r$). With this realization, the students substitute appropriate known values into the proportion to come up with the following general formula for the measure of $\angle CGC'$:

$$\frac{\text{measure of arc } CC'}{\text{circum. of circle } GCC'} = \frac{m\angle CGC'}{360°} \rightarrow \frac{2\pi r}{2\pi\left(GC\right)}$$

$$= \frac{m\angle CGC'}{360°} \rightarrow m\angle CGC' = \frac{360\left(r\right)}{\sqrt{\left(\frac{rh}{R-r}\right)^2 + r^2}}$$

On the basis of these general formulas, the design company can now create any lamp shade to a customer's specifications. At this point in the project, students typically want to try a few examples to verify that the formulas really work. For example, if $r = 2$ inches, $R = 3.5$ inches, and $h = 3$ inches, then $GB = 4$ inches, $GC = \sqrt{20}$ inches, $GF = \sqrt{20}$ inches + $\sqrt{11.25}$ inches, and $m\angle CGC' \approx$ 161°. Using this example, a student could place his or her compass at the center of the 18-inch side of the construction paper; create circles with radii of approximately 7 3/4 inches and 4 1/2 inches; and then cut out a sector along an angle measuring 161°. This construction would create a net for a lamp shade that is 3 inches high with openings for circles with radii 3.5 and 2 inches. The dimensions (r, R, and h) given in **table 1** produce lamp shades that can be made from 18-by-12-inch sheets. These produce some distinct variations in the shapes of lamp shades; some flare out wide (like megaphones), while others are practically cylinders.

As with all rich problems, variations of the solution and the problem itself are plentiful. Studying the spectrum of variations of this problem brings out mathematical nuances that may be appropriate for different courses. For instance, sometimes students decrease the amount of wasted construction paper by placing point G off the construction paper. However, discovering point G's ideal placement requires trigonometry and so may be appropriate only for some

A variety of student results

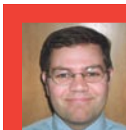
classes. Moreover, minimizing waste is a question better approached by using the tools of calculus, not yet in the repertoire of geometry students.

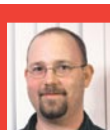
In the lamp shade project, students engage in some intricate mathematics as they develop their solutions. As their guides, we occasionally use leading questions and physical models to shine light on some of the corners too dark to see clearly. Informal comments from students—"Wow, I never knew that so many different topics could go together like this" and "Oh, yeah, this is just like the time when we cut those shades …"—show that they enjoy the hands-on, exploratory, and challenging nature of this project.

The lamp shade project has several benefits: Students see the blending of mathematical topics typically taught separately; disengaged students become enthusiastic about mathematics; and students improve their spatial reasoning, an important skill for successful nonstandard problem solving (Presmeg and Balderas-Cañas 2001). We believe that projects like this one can help students see mathematics as more than just a futile search in the dark—indeed, that they learn to see mathematical investigations as exciting, worthwhile, and rewarding.

REFERENCE

Presmeg, Norma C., and Patricia E. Balderas-Cañas. "Visualization and Affect in Nonroutine Problem Solving." *Mathematical Thinking and Learning* 3, no. 4 (2001): 289–313. ∞





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