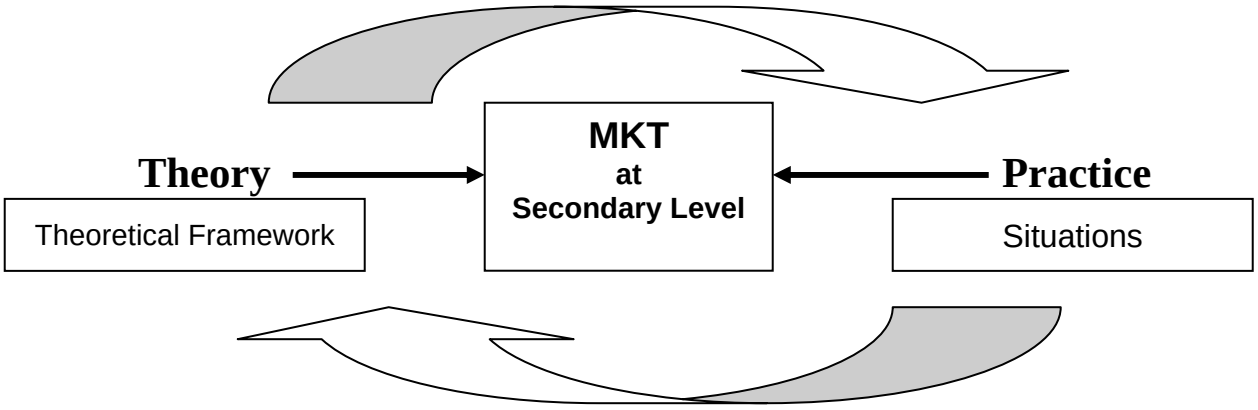
Theoretical Framework for Secondary Mathematical Knowledge for Teaching

In seeking to understand the construct of *mathematical knowledge for teaching* (MKT) as it might be applied to secondary school mathematics, we have developed a variety of sample situations. Each situation contains a prompt that portrays a scene about teaching secondary mathematics from a classroom or from outside class (e.g., departmental meeting or hallway conversation) in which some mathematical point is at issue. After the prompt, we offer a commentary that provides a rationale or explanation of the mathematical issues raised by the prompt, followed by three to six foci, each of which elaborates facets of those issues.

The purpose of these situations is to begin developing the construct of MKT from the bottom up. By identifying what we can agree are instances of the kind of special knowledge of secondary school mathematics that a teacher should possess so as to deal with the proposed situation—but that another user of mathematics does not necessarily need to know—we are attempting to identify some of the characteristics of MKT at the secondary school level.

At the same time, we are beginning to develop, from the top down, a framework within which to situate the mathematical knowledge that a teacher of secondary mathematics needs to know. This framework is much more tentative than the situations are, and it is currently less useful as a means of conveying what we mean by mathematical knowledge for teaching. We hope to refine the framework, iteratively working from the bottom up and the top down to achieve a clearer understanding of MKT. The diagram below suggests the interactive nature of our work.



The framework we propose has two dimensions: *mathematical content* and *mathematical proficiency*. The mathematical content dimension is taken from chapter 9 of *The Mathematical Education of Teachers* (Conference Board of the Mathematical Sciences, 2001). It contains the following major categories:

1. *Algebra and number theory*
2. *Geometry and trigonometry*
3. *Functions and analysis*
4. *Data analysis, statistics, and probability*
5. *Discrete mathematics and computer science*

Except for the first (which combines topics often kept separate) and the last categories, these are similar to the categories used in classifying mathematics achievement items for the 8th and 12th graders in the National Assessment of Education Progress (National Assessment Governing Board, 2004) and for the 8th graders in the Trends in Mathematics and Science Study (Mullis et al., 2005). Each category can easily be decomposed into subcategories, in which case the classification schemes are not so dissimilar.

The mathematical proficiency dimension is based on the strands of mathematical proficiency in chapter 4 of *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, Swafford, & Findell, 2001). These five strands are meant to be combined in instruction, and separating them entails some distortion. But they provide a useful scheme for capturing aspects of mathematical knowledge for teaching that may cut across the content dimension. The categories and subcategories of the mathematical proficiency dimension are as follows:

A. Conceptual understanding
- Separate big from small mathematical ideas
- Produce or interpret a representation
- Define a mathematical object
- Create an abstraction
- Give a concrete example of an abstraction

B. Procedural fluency
- Perform an operation, transformation, or other procedure
- Predict or estimate the result of an operation, transformation, or other procedure
- Explain or justify a procedure
- Describe the boundary conditions for a procedure

C. Strategic competence
- Extend a system while preserving a property
- Formulate a problem or a mathematical model
- Consider an extreme case
- Work backwards from an assumed result
- Solve a simpler problem

D. Adaptive reasoning
- Introduce an analogy
- Generalize from a set of instances
- Give an instance of a generalization
- Interpret, extend, or justify an argument
- Identify an error in an argument
- Analyze alternative arguments
- Identify assumptions in an argument

E. Productive disposition
- Identify an ambiguous formulation
- See a mathematical idea in nonmathematical phenomena
- Construct or recall a useful application of a mathematical idea

These categories do not capture every aspect of the mathematical knowledge needed for teaching secondary mathematics, but they do include many of the important aspects we have seen thus far in the situations we have analyzed.

References

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