

## ***Situation 10: Simultaneous Equations***

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### **Prompt**

A student teacher in a course titled “Advanced Algebra/Trigonometry” presented several examples of solving systems of three equations in three unknowns algebraically using the method of elimination (linear combinations). She started another example and had written the following

$$3x + 5y - 6z = -3$$

$$5x + y - 2z = 5$$

when a student asked, “What if you only have two equations?”

### **Commentary**

#### **Mathematical Foci**

##### ***Mathematical Focus 1***

Linear algebra.

Any linear combination of two equations in *two* variables produces a new equation. The solution of the original system, if one exists, is also a solution for the new equation. A linear combination of the two equations that eliminates one of the variables yields an equation in a single variable. The solution value for this univariate equation is necessarily the same as the solution value for that variable in the original system. Similarly, any linear combination of two equations in three variables produces a new equation. A solution of the original system is also a solution of the new equation. In the three-variable case, however, a linear combination of the equations of the system results in an equation in two variables, which has infinitely many solutions. Each of these solutions is also a solution of the two-equation system for an appropriate value of the third variable.

##### ***Mathematical Focus 2***

Graphical/geometric.

The graph of a linear equation in three unknowns is a plane. Any two nonparallel planes intersect in a line. Three pairwise nonparallel planes necessarily intersect in a single point. Thus a system of three linear equations in three unknowns has a unique solution if and only if the equations represent pairwise nonparallel planes. Given two nonequivalent equations in three unknowns, the intersection of their graphs is a line. There are infinitely many points on this line, each of which satisfies the pair of equations simultaneously.

### **Mathematical Focus 3**

Matrices.

Systems of linear equations are often solved by matrix methods, either by performing Gaussian elimination on the augmented matrix of coefficients and constants, or by multiplying the inverse of the coefficient matrix with the matrix of constants. Let  $\mathbf{A}$  be the matrix of coefficients of the  $2 \times 3$  system of equations and let  $\mathbf{B}$  be the matrix of constants.

Gaussian elimination involves the creation of an identity matrix within the augmented matrix  $[\mathbf{A}, \mathbf{B}]$ . In this case, this identity matrix would need to be of dimensions  $2 \times 3$ , but identity matrices are necessarily square. One possibility is to append a third row consisting of only zero entries onto  $[\mathbf{A}, \mathbf{B}]$ . This addition to the matrix creates a matrix of the proper dimensions, but one that still contains a row of zeroes when transformed into row echelon form, in this case indicating infinitely many solutions.

Similarly, any method based on matrix inverses cannot produce a unique solution. We would like to find a solution by the multiplication  $\mathbf{A}^{-1}\mathbf{B}$ , however,  $\mathbf{A}$  cannot have an inverse since it is not a square matrix. If one attempts a technological approach using these matrices, say with hand-held technology such as the TI 8x series of calculators, one gets a *dimension error* when trying to perform the multiplication  $\mathbf{A}^{-1}\mathbf{B}$ , since the dimensions of  $\mathbf{A}$  are not of the form  $n \times n$ . The only plausible option is, again, to append a row of zeroes to  $\mathbf{A}$ . But this appended matrix does not have an inverse. If one denotes by  $\mathbf{A}_+$  matrix  $\mathbf{A}$  appended with a row of zeroes and performs the calculator operations  $\mathbf{A}_+^{-1}\mathbf{B}$ , one gets a *division by zero error* when the calculator attempts to find the inverse of  $\mathbf{A}_+$ , since this matrix is singular.