MAC-CPTM Situations Project

Situation 10: Simultaneous Equations

Prepared at University of Georgia
13 April 2006 – Dennis Hembree and Erik Tillema

Edited at Penn State University
04 February 2007 – Evan McClintock
13 February 2007 – Rose Zbiek

Prompt

A student teacher in a course titled *Advanced Algebra/Trigonometry* presented several examples of solving systems of three equations in three unknowns algebraically using the method of elimination (linear combinations). She started another example and had written the following

$$3x + 5y - 6z = -3$$
$$5x + y - 2z = 5$$

when a student asked, "What if you only have two equations?"

Commentary

The problem seems centered on knowing necessary and sufficient conditions for unique solutions to systems of linear equations. Connections can be made to linear algebra through matrix representations. The foci build from systems of equations in two variables to systems of equations in three variables, and examine why three equations are necessary to produce a unique solution to a system of equations in three variables. In general, a system of n linear equations in m unknowns has solutions in a space of dimension n - m. The foci use physical models, symbolic representations, and graphical representations to examine systems of linear equations with unique solutions, infinite solutions, and no solutions.

Mathematical Foci

Mathematical Focus 1

In cases for which the solutions are non-negative values, a length model can illustrate necessary conditions for determining unique solutions to systems of linear equations in two and three variables.

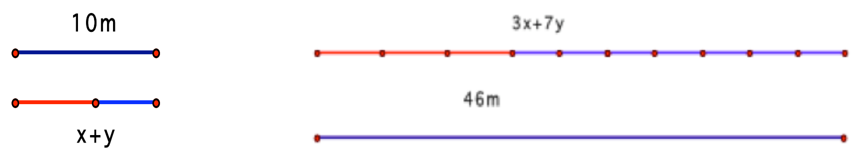
Consider the following example:

Two ropes have different lengths, and the sum of their lengths is 10 meters. To measure the length of a 46-meter bamboo rod requires 3 lengths of the first rope and 7 lengths of the second rope. Determine the length of each rope.

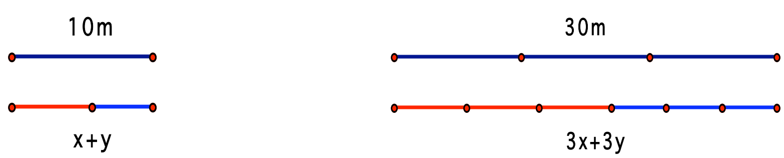
Letting x represent the length of the first rope and y the length of the second rope, the sum of the lengths of the rope can be expressed symbolically as $x + y = 10$. Many combinations of values for x and y satisfy this equation.

A second equation is needed to determine a unique value for the length of each rope. Since measuring a 46-meter bamboo rod requires 3 lengths of the first rope and 7 lengths of the second rope, a relationship between the lengths of the first and second rope can be expressed symbolically as $3x + 7y = 46$.

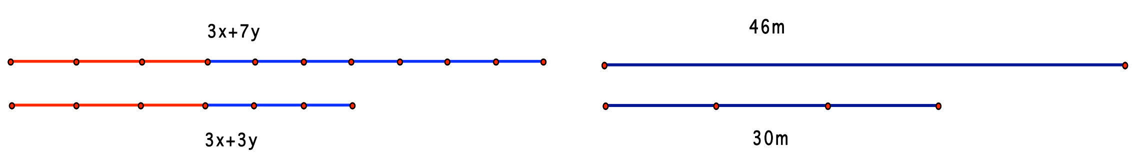
Using length models to represent the equations $x + y = 10$ and $3x + 7y = 46$:



Because the sum of one length of each rope is 10m, the sum of three lengths of each rope would be 30m. This relationship could be expressed symbolically with the equivalent equations $x + y = 10$ and $3x + 3y = 30$. Using length models to represent the equations $x + y = 10$ and $3x + 3y = 30$:



Since $3x + 3y = 30$ is equivalent to $x + y = 10$, we can express the lengths of each rope with the system of equations $3x + 3y = 30$ and $3x + 7y = 46$. Using length models to represent the equations $3x + 3y = 30$ and $3x + 7y = 46$:



By inspection, four lengths of the second rope must equal 16 meters, and therefore one length of the second rope is 4 meters. Expressed symbolically:

$$
\begin{aligned}
3x + 7y - 3x - 3y &= 46 - 30 \\
4y &= 16 \\
y &= 4
\end{aligned}
$$

Hence, a unique length for the first rope exists, namely 6 meters.

In the same way, a system of two linear equations in three variables will not have a unique solution.

Mathematical Focus 2

Two different methods for solving a type of problem (e.g., two different matrix methods for solving a systems of m *equations in* m *unknowns) might not be equally useful in concluding the absence of a solution for a related type of problem (e.g., solving a system of* m *equations in* n *unknowns where $m \neq n$).*

Systems of linear equations are often solved by matrix methods. One technique involves multiplying the inverse of the coefficient matrix and the matrix of constants, in that order. In the case of a system of two equations with three unknowns, the 2x3 coefficient matrix is not a square matrix. Thus the coefficient matrix does not have an inverse and a solution by this method is not possible. However, that does not necessarily mean that no solution exists.

Solving systems of linear equations may also be accomplished by performing Gaussian elimination on the augmented matrix of coefficients and constants. Consider the general case of a system of two equations in three variables:

$$
\begin{aligned}
a_1x + b_1y + c_1z &= k_1 \\
a_2x + b_2y + c_2z &= k_2
\end{aligned}
$$

Performing Gaussian elimination on the augmented matrix of coefficients and constants gives the augmented matrix:

$$
\begin{bmatrix} 1 & 0 & r_1z & s_1 \\ 0 & 1 & r_2z & s_2 \end{bmatrix}
$$, where $r_1, r_2, s_1, and\, s_2$ are constants.

The augmented matrix represents an equivalent system of two equations in three variables:

$$
\begin{aligned}
x + 0y + r_1z &= s_1 \\
0x + y + r_2z &= s_2
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
x &= s_1 - r_1z \\
y &= s_2 - r_2z
\end{aligned}
$$

These equations indicate that, although the values of *x* and *y* depend on the value of *z*, the value of *z* is arbitrary. Hence, a system of two equations in three variables has no unique solution.

Mathematical Focus 3

Existence or nonexistence of solutions to systems of linear equations arises in multiple representations of those systems.

A graphical representation of the points whose coordinates satisfy a linear equation in two variables is a line. A graphical representation of the points whose coordinates satisfy a linear equation in three variables is a plane. A graphical representation of the solution of a system of two or three linear equations in three variables would be the intersection of the two or three planes representing the solutions of each of the three equations.

Two planes will either be parallel or intersecting, as illustrated in figures 1 and 2.

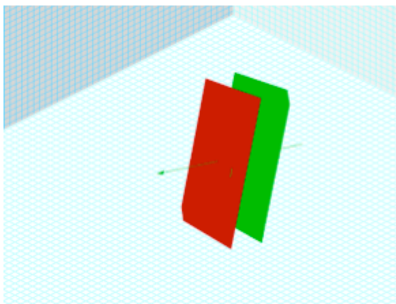


Figure 1: Two parallel planes

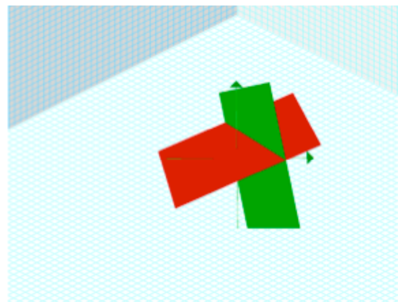


Figure 2: Two intersecting planes

Either two planes are parallel, as shown in Figure 1, and the system of two equations has no solution, or the planes intersect in a line, as shown in figure 2, and the system has infinitely many solutions, each of which lies on the line of intersection. Without a third plane to intersect that line of intersection, there is no unique point of intersection. Thus, a system of two equations in three unknowns cannot have a unique solution.

Three planes will either be parallel or can intersect in several ways, as illustrated in figures 3 through 7.

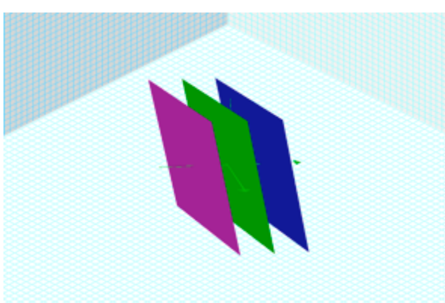


Figure 3: Three parallel planes

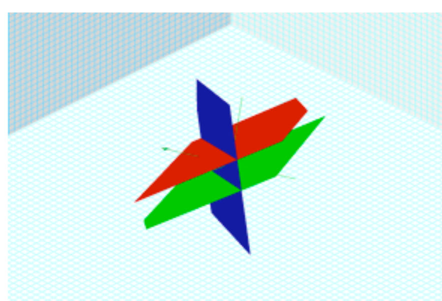


Figure 4: Three planes,

two of which are parallel

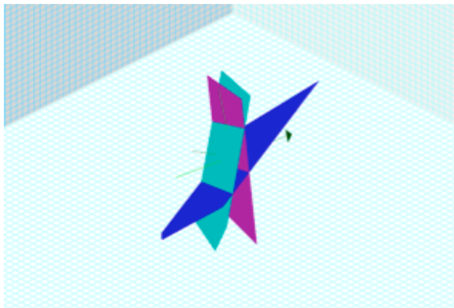


Figure 5: Three planes intersecting pairwise in three parallel lines

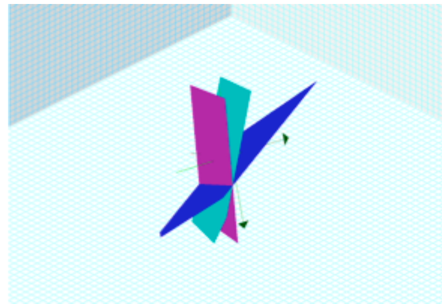


Figure 6: Three planes intersecting in a line

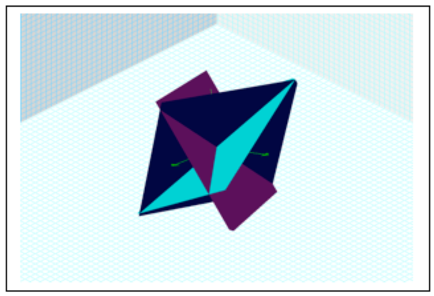


Figure 7: Three planes intersecting in a point

Thus a system of three linear equations in three variables may have no solutions, as represented in figures 3, 4, and 5. The system may have infinitely many solutions as represented in figure 6, or the system may have a unique solution as represented in figure 7.