Situation 10: Simultaneous Equations

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Prompt

A student teacher in a course titled "Advanced Algebra/Trigonometry" presented several examples of solving systems of three equations in three unknowns algebraically using the method of elimination (linear combinations). She started another example and had written the following

$$3x + 5y - 6z = -3$$
$$5x + y - 2z = 5$$

when a student asked, "What if you only have two equations?"

Commentary

From this prompt, we will explore three foci. The first and second focus grow out of thinking of x, y, and z as unknown quantities. The third focus relies on a functional approach in which x, y, and z are variables and the solution set is the set of all possible values that satisfy a particular equation. In this case, the solution set is represented as a point or line of intersection of the functions. The method of elimination works in both settings but the reasoning involved differs depending on whether one considers x, y, and z to be unknowns or variables. The approach that focuses on x, y, and z as unknown quantities might arise in situations that relate three as yet unmeasured quantities while the functional interpretation might arise in a situation in which x, y, and z vary over a specified domain. It is our belief that having algebraic notation grow out of mathematical examples in quantitative contexts helps students' develop ideas about quantitative equivalence, which are central to understanding the method of elimination.

Mathematical Foci

Mathematical Focus 1

Linear algebra.

Linear equations in two variables

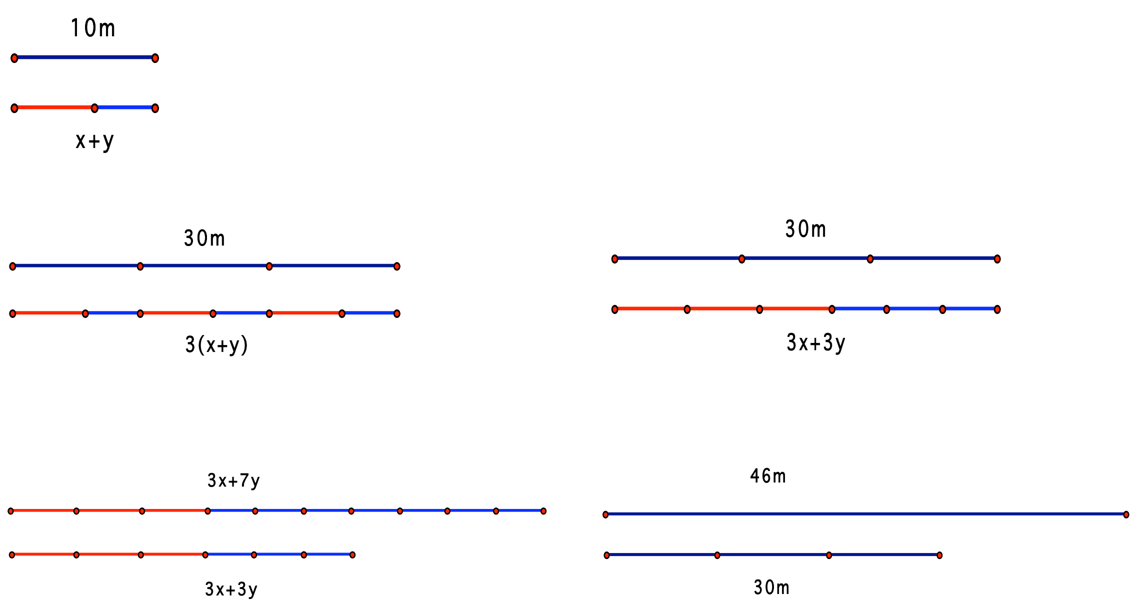
Any linear combination of two equations in *two* unknowns produces a new equation. The solution of the original system, if one exists, is also a solution for the new equation. A full algebraic proof of this statement is given in Appendix A. We start by giving an example of a situation in two unknowns and then look at a situation in three unknowns. The example is intended to give an intuitive sense for why solutions of equations are invariant under linear transformations. The sample situation is similar to situations found in the ancient Chinese text *Jiu Zhang Suanshu* (*JZSS*) or *The Nine Chapters of the Mathematical Arts* and many early Babylonian mathematical texts.

Suppose that I have two lengths of rope. I know that the sum of the two lengths of rope is 10 meters. Further, I know that to measure the length of a 46-meter bamboo rod requires 3 lengths of the first rope and 7 lengths of the second rope. Can you find the lengths of each rope?

This example in two unknowns suggests why having only one equation in two unknowns would be insufficient for a unique solution. For instance, letting x represent the length of the first rope and y the length of the second rope, we see that $x + y = 10$. However, the length of each rope remains indeterminate.

The second equation ($3x + 7y = 46$) is essential in order to determine a unique solution for the lengths of rope. In order to conceive of the method of elimination, it is necessary to move away from a literal interpretation of the situation and imagine a new situation that helps to resolve the problem. In order to do this, one can think of making an equivalent equation in a variety of ways. For example, one can multiply 3 times the sum of the length of the ropes. Doing so produces an equivalent equation, $3(x+y) = 3*10$ or an equation which we know must hold true if the sum of the lengths of the rope is 10 m. Notice that there are two different ways to refer to the measure of the same quantity, x+y and 10 m, as demonstrated in the diagram below. Therefore, multiplying the measure of the quantity by 3 implies the need to multiply both ways of referring to the quantity by 3 to preserve the equivalence.

Now it is possible to compare the new equation ($3x+3y = 10$) and the second equation ($3x + 7y = 46$). Having this insight may result from thinking about making an additive comparison of the two resulting lengths. Below is a diagram of the situation from which the algebraic notation can arise. If we arrange…



From the diagram, one might notice that the two "left over" lengths are 4y and 16m, giving $4y=16m$ and $y = 4m$.

As a final note, negative values for unknowns can arise in situations where the unknowns represent an amount of money someone owes, an unknown integer, or measurements of temperature.

Linear equations in three unknowns

In more advanced situations, like the one presented in the prompt, algebraic symbolism serves to alleviate some of the difficulty of representing more complicated quantitative situations. The method of elimination in three unknown quantities relies on reducing a situation to two equations in two unknowns and therefore is a generalization of the method shown above. When starting with *only* two equations in three unknowns, it is possible to only generate one new equation in two unknowns and therefore the solution remains indeterminate (as in the rope example when it was not possible to uniquely determine the length of each rope given only one equation). Beginning with three unknown length quantities and three conditions on their relatedness, it is possible to produce two equations in two unknowns with appropriate linear combinations of the three equations.

Below is an algebraic representation of this process that grows out of modeling the following quantitative situation: Suppose there are three lengths of rope and the sum of their lengths is 20 m. It takes 3 lengths of the first rope, 2 lengths of the second rope, and 2 lengths of the third rope to measure a 55 m bamboo pole. Further, we know that it takes 2 lengths of the first rope, 1 length of the second rope, and 3 lengths of the third rope to measure a 42 m bamboo pole. Find the lengths of each rope.

Letting x represent the length of the first rope, y represent the length of the second rope, and z represent the length of the third rope we get the following equations:

$x + y + z = 20$; $3x + 2y + 5z = 55$; $2x + y + 3z = 42$.

Selecting the first and second equation, creating a new equation, and additively comparing the two equations yields: $y + z = 5$ (which can be interpreted as the sum of the lengths of the second and third rope is 5 m).

Selecting the first and third equation, creating a new equation, and additively comparing the two equations yields: $-y + z = 2$ (which can be interpreted as the difference of the lengths of the second and third rope is 2 m). Solving this system as in the case of two equations and two unknowns we get that $x = 15$ m, $y = 3/2$ m, and $z = 7/2$ m.

Mathematical Focus 2

Matrices.

Systems of linear equations are often solved by matrix methods, either by performing Gaussian elimination on the augmented matrix of coefficients and constants, or by multiplying the inverse of the coefficient matrix with the matrix of constants. In the case of a two by three system of equations, the coefficient matrix is not a square matrix. Thus the coefficient matrix does not have an inverse and the latter method is not possible.

When Gaussian elimination is performed on the augmented matrix from a three by three system

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned} \quad \text{with unique solution } (x_1, y_1, z_1)$$

one expects the result to be

$$\begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \end{bmatrix}.$$

However, with a system of two equations and three unknowns, unless the equations represent parallel planes, Gaussian elimination gives

$\begin{bmatrix} 1 & 0 & r_1z & s_1 \\ 0 & 1 & r_2z & s_2 \end{bmatrix}$, where $r_1, r_2, s_1, and\, s_2$ are constants resulting from matrix manipulations.

Thus the original two by three system is equivalent to the system

$$\begin{aligned} x &= s_1 - r_1z \\ y &= s_2 - r_2z \end{aligned}$$

Clearly this system has infinitely many solutions, since any chosen value of *z* results in solutions values for *x* and *y*.

Mathematical Focus 3

Graphical/geometric.

The graph of a linear equation in three variables is a plane. The solution(s) for a system of linear equations in three variables are points in the intersections of their graphs. Either two planes are parallel, as shown in Figure 1, and the system of two equations has no solution, or the planes intersect in a line, as shown in figure 2, and the system has infinitely many solutions, each of which lies on this line of intersection.

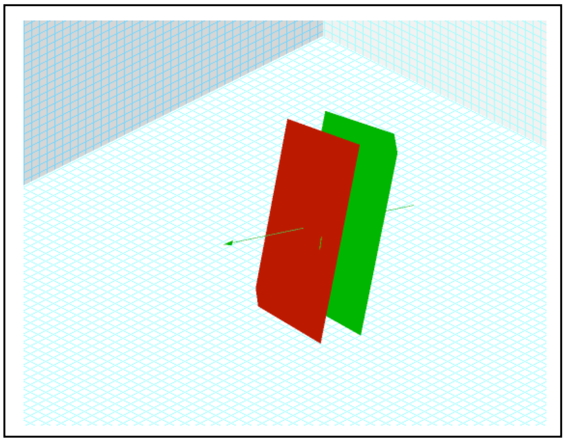


Figure 1: Two parallel planes

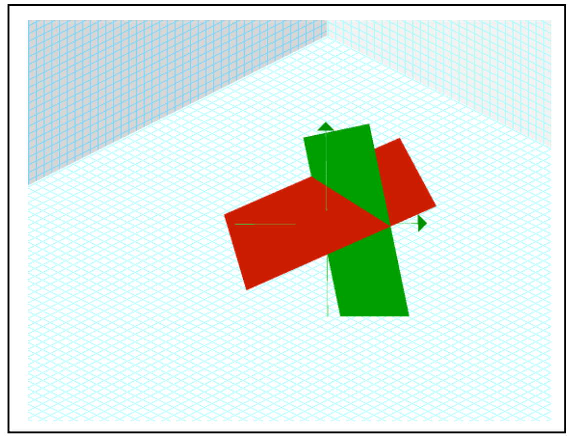


Figure 2: Two intersecting planes

The solution(s) of a system of *three* linear equations in three variables may be represented by the intersections of three planes. Three planes may intersect in a variety of ways, as illustrated in Figures 3 through 7.

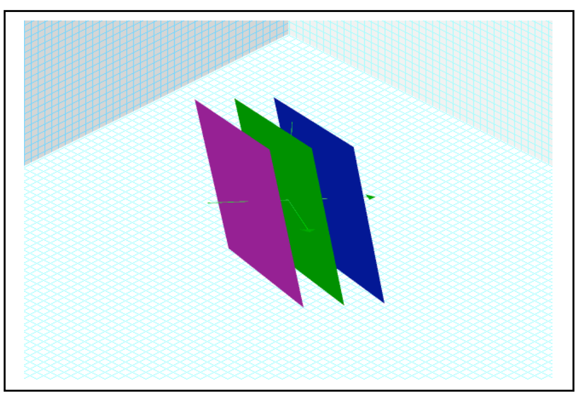


Figure 3: Three parallel planes

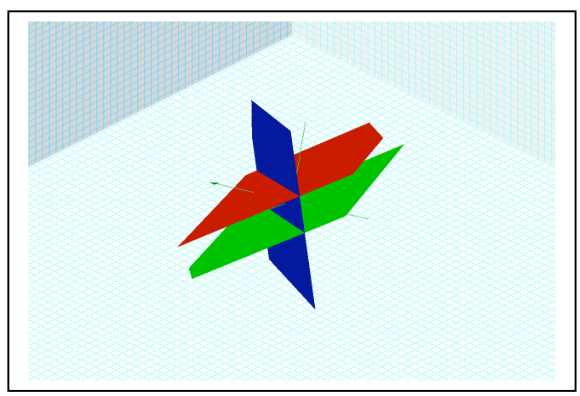


Figure 4: Three planes, two of which are parallel

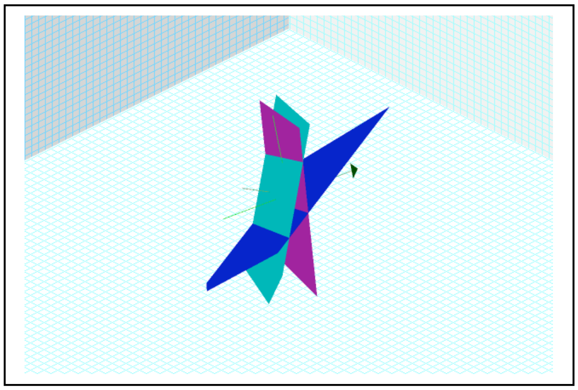


Figure 5: Three planes intersecting pair wise in three parallel lines

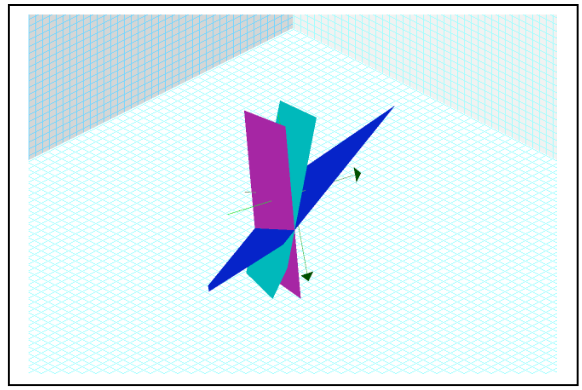


Figure 6: Three planes intersecting in a line

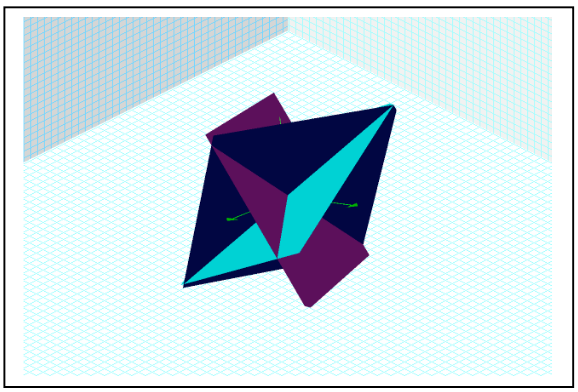


Figure 7: Three planes intersecting in a point

Thus a system of three linear equations in three variables may have no solutions, as in Figures 3, 4, and 5. The system may have infinitely many solutions as in Figure 6, or the system may have a unique solution as in Figure 7.

For the original prompt a geometric/graphical focus illustrates that, given only *two* equations in three unknowns, if the graphs intersect, that intersection is a line. There are infinitely many points on this line, each of which satisfies the pair of equations simultaneously. Thus, a system of two equations in three unknowns cannot have a unique solution. A third plane and thus a third linear equation are needed to produce a unique solution as in Figure 7.

Appendix A

Any linear combination of two equations in *two* unknowns produces a new equation. The solution of the original system, if one exists, is also a solution for the new equation.

Proof:

Given the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \text{with solution } (x_1, y_1),$$

consider the linear combination $k_1(a_1x + b_1y) + k_2(a_2x + b_2y) = k_1c_1 + k_2c_2$.

The ordered pair (x_1, y_1) is a solution for this new equation since

$k_1(a_1x_1 + b_1y_1) + k_2(a_2x_1 + b_2y_1)$ is equivalent to $k_1(c_1) + k_2(c_2)$ by substitution.

The method of linear combinations—sometimes called the addition method or the elimination method—"works" since appropriate choices of k_1 and k_2 will eliminate either *x* or *y*. That is, for appropriate choices of k_1 and k_2 the linear combination $k_1(a_1x + b_1y) + k_2(a_2x + b_2y) = k_1c_1 + k_2c_2$ results in either $(k_1a_1 + k_2a_2)x = k_1c_1 + k_2c_2$ or $(k_1b_1 + k_2b_2)y = k_1c_1 + k_2c_2$ after simplification. Based on the above proof, the unique solution to either one of these results must be the solution value of the corresponding variable in the original system.

In practice then, if we have a system such as

$$\begin{aligned} 2x - y &= -3 \\ x + 3y &= 16 \end{aligned}$$

and we use a linear combination of the two equations to eliminate y and arrive at the result $x = 1$, it must be true that the solution set for this new equation contains the solution set of the original system. In other words, for the original system, *x* is 1 and the value of *y* is as yet unknown, but easily found by substitution. If *x* and *y* are considered as variables, we might phrase these results in a different way. We might say that when we eliminate *y,* we have produced a new linear equation $x = 1$ that passes through, or contains, the point of intersection of the two original lines.

Linear equations in three variables

Any linear combination of two equations in three variables produces a new equation. A solution of the original system, if any exist, is also a solution of the new equation. A proof for this statement is simply an extension of the one given above. In the three-variable case, however, a linear combination of two of the equations of the system that eliminates one of the variables results in an equation in two variables. This resulting equation has infinitely many solutions. Each of these solutions is also a solution of the two-equation system for an appropriate value of the third variable.

In practice, for a system such as

$$\begin{aligned} x + y + z &= 6 \\ x + y - z &= 0 \\ 2x + y - z &= 1 \end{aligned}$$

we might use the first two equations to eliminate *z* and arrive at the equation $x + y = 3$. In variable terms, this new equation represents a plane perpendicular to the *x*-*y*-plane that contains the solution (passes through the intersection of) the three original equations.

Similarly using the first and third equations to arrive at $3x + 2y = 7$ again produces a new plane perpendicular to the *x*-*y*-plane that contains the solution (passes through the intersection of) the three original equations.

Using the two new equations $x + y = 3$ and $3x + 2y = 7$ and eliminating y results in the equation $x = 1$, which in variable terms represents a line perpendicular to the *x*-*y*-plane that contains the solution (passes through the intersection of) the three original equations.

Values of *y* and *z* may be found successively by substitution or by repeating a process similar to the one used to find *x.*