Situation 12: Quadratic Equations
PRIME at UGa
Eric Tillema

Commentary November 2005

Four ideas about giving comments for vignettes:

1) **I think it would help to know who has given comments because then it is possible to ask questions to the commenter if necessary and may allow for collaborative work.**
2) **In general, language seems like it will be important in the vignettes. So I think it is important to stay away from language such as the teacher *needs* to know x or a teacher *needs* to do x.**
3) **Comments that are focused on the vignette and the foci are of the most help in revising the vignette and foci. Comments related to the practices of the particular teacher in the vignette are more difficult to interpret back in the context of the vignette and foci.**
4) **Rhetorical questions are very difficult to interpret as comments. So a question like "Why the surprise?" directly after the vignette is difficult to interpret in terms of creating a more clear or coherent vignette or foci. If the comment after the vignette was: *In the vignette, the teacher seems only to be working on a procedure and not a concept. Therefore, the vignette seems somewhat limited. Perhaps framing the vignette with a different question and filling in the details of where this vignette took place would make it more powerful.* This type of comment is much more helpful in directing attention to how an author (me in this case ☺) might change the vignette or foci.**

Vignette 12:

Mr. Sing presents equations ~~problems~~ [Comment: As presented, this is an equation, not a problem.] like the following to his students.

$$(x+1)^2 = 9$$

He demonstrates to them that they need only take the square root of each side to get

x+1 = 3 or x +1 = -3.

Then we can solve for x = 2 or x = -4. He then turns his students loose to solve some problems like the ones he has presented and is surprised to find out that many of his students are multiplying the terms out to get

$$x^2 + 2x + 1 = 9$$

and then transforming the equation so that

$$x^2 + 2x - 8 = 0$$

-and factoring this equation. He notes, however, that many students still were not able to factor correctly.

He stops the class and reminds the students that they need only take the square root of both sides to solve these types of equations and then let's them continue working on the problems. A few days later, Mr. Sing grades the test covering this material and finds that many of his students are still not doing as he has suggested. At first he thinks that his students just didn't listen to him but then he reminds himself that during the class period the students seemed to be quite attentive.

What hypotheses do you have for why his students are acting in this way? What concepts are present in the material he is trying to communicate to his students? In what ways might Mr. Sing work with his students to develop the concepts he is trying to communicate? What knowledge might Mr. Sing need in order to develop these concepts with his students?

[Comment: -Why the surprise? Students should solve a quadratic equation in a way suitable to the problem. This is a suitable method since it is one of the few quadratics that can be factored. What if can't be factored? What about alternative (graphical) approaches to help build conceptual understanding?

What he is trying to communicate, as it is written, is not a concept. It is a procedure. Instead of the teacher racking his brain because he doesn't know why a student is not following his method instead of a method that they are already familiar with, maybe the question that could be asked by a student is, "Do we always have to have a zero on one side of the equal sign to solve a quadratic equation?" or "How do we solve a quadratic if it can't be factored?"or "Do we always get two answers when we solve a quadratic equation?"] Mr. Sing was a new teacher. Many of his teaching strategies were "procedural". He put a lot of effort in trying to think of ways to work with his students that might be less procedural but this required that he "rack his brain" because he did not have well articulated knowledge for teaching. I am not sure if your comments are indication that we need to be more explicit about who the teacher was and in what context the vignette occurred or perhaps that this vignette is not so good because it seems like the teacher is being procedural.

DISCUSSION
October 2005

Mathematical Foci

Mathematical Focus 1: Solving equations of the form $x^2=k$. [Comment: I like the visual, graphical, representation. However, I am not sure that a quantitative situation using area helps with a general knowledge about quadratics nor will it, I believe, help students (or teacher) understand anything outside of this isolated case. I think that the graph used is a nice representation of a unique example, but I think that other graphical methods can be used to aid in overall understanding of the concept that would be easier for students and the teacher to use. Please see my *Focus 1 addition*.]

Representations like this are possible for all quadratics including quadratics that have complex solutions, but as with any representation a teacher has to build up meaning for it with their students. I like this representation better than simply using a graphing calculator for several reasons. First, you see a dynamic relationship between elements of the domain and elements of the range. Second, you have to construct this relationship in the literal sense of the word, which requires engaging in a substantial amount of mathematical reasoning. Third, if one is oriented quantitatively, then x + 1 is "an amount, a number, an entity, an object" as you suggest in foci 5. It is a length, and when squared it produces an area. Fourth, it makes factoring a creative activity, which includes figuring out two ways of naming the area of a rectangle or square in order to factor a quadratic or solve an equation. Fifth, it leads to all sorts of problematic situations, some of which lead to a quantitative (geometric) interpretation of the quadratic formula. You may feel a graphing calculator representation is more to your liking for other reasons. It seems what is at issue here is that a teacher needs to feel comfortable with the representation they are using and have specific conceptual goals they hope to accomplish with its use. To interpret this issue in terms of the vignette project, it might beg the question when are two representations different and when are they the same. Thinking about this issue may help in figuring out when two foci are similar or different, an issue I think we as a group are trying to figure out.

Introduce a quantitative situation like the following: $(x+1)^2$ is of the form $x^2=k$ which can be thought of as $s^2=k$, where s is the length of a side of a square. ~~I have~~ Can you make a representation of ~~a~~ a square whose side is length *x + 1*. ~~. Can you make a representation~~ [Comment: What does this mean? Is your question about the word representation?] Can you ~~of this square and~~ figure out the length of *x* when the area of the square is 9 m2? In using this approach, a dynamic representation is helpful because it allows students to move the values of *x* (as seen in the GSP attachment). It leads to a number of possible levels of solution to the problem. At a basic level, it allows students to move the quantity (*x+1*)2 in GSP to find out when the area will be 9 m2. In constructing the representation, the student may also be able to coordinate some of the factoring issues that they seem to be experiencing in the vignette. For instance in creating the square whose sides are *x +1*, the student can name the square algebraically either as (*x +1)*2 or through the sum of its parts. This may allow the student to have a quantitative experience of why *(x +1)2 = x2 + 2x + 1* and further to coordinate what the area of each region of the function is when the total area is 9 m2. This approach can also lead to creating a linear representation of the area function and graphing it in the usual way in the plane which can lead to discussions about why this function is a translation of *x*^2. Note to get the second solution using this method the teacher would need to address the notion of negatively oriented area.

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Focus 1 Addition:

This teacher needs to be introduced to equations as two functions. For example, Mr. Sing needs to know that $(x+1)^2 = 9$ can be written as:

$$y = (x+1)^2 \quad \text{and} \quad y = 9$$

These two graphs can be graphed on the TI - __ and the intersection point can be found. Mr. Sing will be able to see (and pass it on to students if need be) that the intersections of:

$y = (x+1)^2$ and $y = 9$
$y = x^2 + 2x + 1$ and $y = 9$
$y = x^2 + 2x – 8$ and $y = 0$

will all yield the same points. I think that Mr. Sing needs to focus more on equivalent relations instead of the equal sign as an operator. Now the question needs to be asked if Mr. Sing is focusing too much on factoring and procedural methods of solving equations instead of on concepts. If so, what can we do to help him be a better teacher? Does he even know that some quadratics have no solution or only one solution. If he does, does he know why (outside of some algorithmic break down)? How and when does he convey this to his students? Maybe we should be including in this Focus what some graphs of what quadratics with two, one and no solutions would look like. I think that if he knew more about these, that he would move away from procedural teaching and ask different question than why aren't students following the procedure that he wants them to use. I am not clear about what you mean by quadratics with no solutions (Do you mean over the complex numbers? real numbers? or rational numbers?). To build up to thinking about the class of all quadratics (e.g. (x+1)^2 = -5), it seems like it would be best not to introduce the notion that some quadratics are un-factorable or non-solvable unless one also introduces the notion of fields. In my teaching experience, the introduction of fields in this context has not had much meaning for many of my students.

Mathematical Focus Number 2: General versus special case.
[Comment: I think that the part of Focus 2 that talks about teaching strategy (what comes first, general or specific methods) is unnecessary. The development of the quadratic equation is a necessary part of any teacher's and student's bank of knowledge. However, it needs to be developed for a reason. Do the students (and/or teacher) know that some quadratic equations can't be factored and why?- What do those equations look like? Do they already have a solid image of what all possible cases of quadratic equations would look like graphically (for example $(x+1)^2 = -5$)?

This type of equation is part of the reason I don't like using a TI-82 to solve quadratic equations because when you get to an equation where there is no intersection then many students conclude that there is no solution to this equation. I would be interested to know how you introduce imaginary and complex numbers using a graphical method.

Using an area model to introduce imaginary numbers can be done through thinking of i^2 = -1 as a square whose area has a negative orientation but the vectors that make up the square each have a positive orientation. Such a square can be found in the xz plane of the xyz coordinate system (if we take the x and z axis to represent imaginary numbers and the y axis to represent real numbers). If we take the conventional ways of orienting vectors and area, a square in the xz plane has two positive vectors and a negatively oriented area. This idea can be built up and eventually used in an area model that develops finding the difference of two squares and completing the square as a way for students to creatively produce the quadratic formula out of their factoring activities. This method for representing imaginary numbers connects to Hamilton's representation of the quaternion group, vector multiplication in multivariate calculus, and electrical engineers use of imaginary numbers in their work.) What is the reason for the quadratic equation (outside of it helps us solve "harder" ones) and how can we use it to build on understanding of what a solution to a quadratic could look like.]

This problem helps to demonstrate a common theme in mathematics—finding special cases whose solution strategy is an abbreviated form of a more general algorithm. Some of the mathematical content involved for teachers is deciding whether to develop the general solution strategy first and then look at special cases or to look at special cases that might build to the general solution strategy. Given the way I treated the problem in foci 1, it makes more sense to move from special cases to the more general solution strategy although this technique is usually reversed in the textbooks I have read on this topic. That is usually students learn to factor quadratic equations to find their zeros and then are presented with special cases like how to solve the equation $(x + 1)2 = 9$. In this case, I would probably start with specific cases and build to a generalized solution of quadratics, using the notion of the difference of two squares and completing the square to build up to the quadratic formula.

[Comment: If you build the quadratic formula from completing the square, does Mr. Sing need to know why completing the square works? Does he use it to develop what a quadratic equation is (or does he just use it as another procedure)?] The answer to these questions will depend on the teacher.

Mathematical Focus Number 3: Variable versus unknown
[Comment: If you are going to mention this as a focus, should there be more development of the idea of what a function is? I think that this may be unnecessary. If it isn't, then it should be developed more and not just mentioned as something that needs to be looked for.]

There is an issue in this problem also as to what x represents mathematically. In the statement of the problem x is an unknown. However, there may be significant confusion for what x stands for if a function approach is used to solve this problem. That is in a functional context x is a variable where x is any but no particular value whereas when we are solving for a particular area of square x becomes an unknown because we are looking for one particular value of x. Algebraically, however, there is no differentiation between the notation of these two ideas rather it is context dependent. Therefore, it may be important for teachers to be aware of the possible differences between thinking of x as a variable and x as an unknown and how and when they are switching between these two conceptions in given problem situations.

Mathematical Focus Number 4:

Important knowledge for addressing the issue in the vignette may be to have strategies for helping students see that general use of a process ~~formula~~ [Comment: which formula?]may not always lead to the most insightful solution of a problem. Such knowledge might be in part at the heart of helping students to build a habit of mind in which they search for efficient problem solutions. [I am not so sure that the habit we want to strive for is efficiency. I agree that students need to be flexible in their approaches to solving a problem, but I believe that they will use the methods most familiar (case in point, Mr. Sing's problem) so we need to familiarize them with all ways of thinking about a problem not just efficient strategies. If students (and Mr. Sing) have a complete concept of what quadratics are instead of only efficient methods of solving them, then they will have, in my opinion, a better "habit of mind."] Note foci 2 and 4 are mostly related to classroom culture or teacher pedagogy.

Mathematical Focus 5:

To be able to teach this idea I believe there is one more thing that students need to recognize. They need to be able to see the x+1 as an amount, as a number, as an entity, as an object so that they can treat this as something that can be doubled, tripled etc. In other words, I believe the unit is an issue. That is, what will the "unit" to find out the square of x+1 or cube, or quadruple of it.

Mathematical Focus 6

$f(x) = x^2$ $\qquad f^{-1}(x) = \sqrt{x}$ when $x \geq 0$

$\qquad\qquad\qquad = -\sqrt{|x|}$ when $x < 0$

$g(x) = x + 1$ $\qquad g^{-1}(x) = x - 1$

$$(f \circ g)(x) = 9$$
$$(f(g(x)) = 9$$
$$f^{-1}(f(g(x)) = f^{-1}(9)$$

(a) If $x \geq 0$ $\quad f^{-1}(f(g(x)) = f^{-1}(9) = 3 = (f^{-1} \circ f)(g(x) = g(x)$

So $g^{-1}(g(x)) = g^{-1}(3) = 2$

(b) If $x < 0$ $\quad f^{-1}(f(g(x)) = f^{-1}(9) = -3 = (f^{-1} \circ f)(g(x) = g(x)$

So $g^{-1}(g(x)) = g^{-1}(-3) = -4$

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