

MAC-CPTM Situations Project

Situation 15: Graphing Quadratic Functions

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Prompt

When preparing a lesson on graphing quadratic functions, a student teacher had many questions about teaching the lesson to a Concepts of Algebra class, an introductory algebra mathematics course. One of the concerns that the student teacher had was the graphing of the vertex of the parabola, which also means identifying the equation of the axis of symmetry. The textbook for this class claimed that $x = \frac{-b}{2a}$ was the equation of the line of symmetry. The student teacher wanted to know how to derive this equation.

Commentary

This prompt seems to be centered on the graphing of quadratic equations and specifically the derivation of the equation of the line of symmetry. Both foci in this situation deal with the general algebraic representation of any quadratic function, but they differ in the approaches used to obtain the equation in question. Focus 1 uses the first derivative to find the x-coordinate of the vertex of the parabola. Focus 2 utilizes transformations to the graph of $y = x^2$ to determine the coordinates of the vertex.

Mathematical Foci

Mathematical Focus 1

*The first derivative of a polynomial function can be used to obtain the coordinates of extreme points of the graph of the function (if any exist).
The graphs of some functions are symmetric in the line $x=k$ where k is the first*

coordinate of an extreme point of the graph.

A quadratic function can be written in the general form $y = ax^2 + bx + c$, where a , b , and c are real numbers and $a \neq 0$. The graph of any function of this form is a parabola that is symmetric about a vertical line through its vertex. The vertex corresponds to an absolute extreme value of the quadratic function and the line of symmetry is of the form $x=k$, where k is the value of x for which the extreme value of the function occurs. For this polynomial function, we can use the first derivative of the function to obtain the critical value, k , and then use the critical values to obtain the coordinates of the vertex of the parabola (the absolute minimum or maximum).

$$y = ax^2 + bx + c$$

$$y' = 2ax + b$$

To find the critical value, k , and thus the equation of the line of symmetry, set the derivative equal to 0 and solve for x :

$$2ax + b = 0$$

$$x = \frac{-b}{2a}$$

We can either use the second derivative of the function or investigate the signs of the first derivative of the function, we notice that the slope of the function changes from positive to negative (indicating an absolute maximum) or from negative to positive (indicating an absolute minimum).

Mathematical Focus 2

Using transformations, the graph of the function $y = x^2$ can be mapped to the graph of any quadratic function of the form $y = ax^2 + bx + c$.

The graph of the function given by $g(x) = af(b(x - c)) + d$ is the image of the graph of f under the composition of a horizontal translation, a horizontal stretch, a vertical stretch, and a vertical translation related to the values of c , b , a , and d , respectively.

The point $(0,0)$ is the vertex of the graph of the function given by $y = x^2$. To find the coordinates of the vertex of the general parabola, we start with the general quadratic equation and apply the completing the square method.

$$\begin{aligned}
y &= ax^2 + bx + c \\
y - c &= ax^2 + bx \\
\frac{1}{a}(y - c) &= x^2 + \frac{b}{a}x \\
\frac{1}{a}(y - c) + \left(\frac{b}{2a}\right)^2 &= x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 \\
\frac{1}{a}\left(y - c + \frac{b^2}{4a}\right) &= \left(x + \frac{b}{2a}\right)^2
\end{aligned}$$

Now solving for y , we get a symbolic form of the equation that we can compare to $y = x^2$, the equation of the particular case, to get information about the needed transformation:

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

The presence of $\left(x + \frac{b}{2a}\right)^2$ rather than x^2 represents a horizontal translation through $\frac{b}{2a}$ units in the negative direction. This horizontal translation of the graph maps the vertex of the parabola associated from $(0,0)$ to $\left(\frac{-b}{2a}, 0\right)$. The rest of the equation suggests a stretch (by a factor of a) and a vertical translation (through $\frac{b^2}{4a}$ units in the positive direction). The x -coordinate of the vertex of the image parabola under each of the last two transformations would be $\frac{-b}{2a}$. Thus the line of symmetry that passes through the vertex of the graph of the general parabola has the equation $x = \frac{-b}{2a}$.

Post Commentary

In both foci, the graph is treated as a geometric object with properties that can be captured in symbolic forms. Both foci implicitly use equivalent forms of an equation to obtain different information about what the relationships embodied in the symbolic forms.