Situation 15: Graphing Quadratic Functions

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Prompt

When preparing a lesson on graphing quadratic functions, a student teacher had many questions about teaching the lesson to a Concepts of Algebra class, a remedial mathematics course. One of the concerns that the student teacher had was the graphing of the vertex of the parabola, which also means identifying the equation of the axis of symmetry. The textbook for this class described that $x = -\frac{b}{2a}$ was the equation of the line of symmetry. The student teacher did not feel confident in his own understanding of how to derive this equation.

Commentary

Focus 1 is a discussion on how first derivatives can be implemented to illustrate the formula for the x coordinate of the vertex. In Focus 2, applying the completing the square method will result in a formula that is a composition of transformations being applied to $y = x^2$, one of which is a horizontal translation. Another important topic covered in secondary mathematics, particularly algebra classes, is the quadratic formula as determining solutions of a quadratic equation. The fact that the x coordinate of the vertex is contained within the quadratic formula is discussed in Focus 3.

Mathematical Foci

Mathematical Focus 1

The vertex of a parabola is the absolute minimum (or maximum) functional value of a quadratic function. We can find the absolute minimum (or maximum) by taking the first derivative of the quadratic function.

$$y = ax^2 + bx + c$$
$$y' = 2ax + b$$

To find the extrema, set the derivative equal to 0 and solve for x:

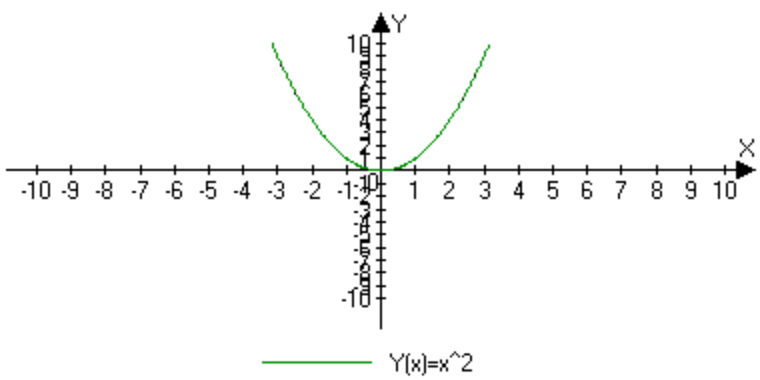
$$0 = 2ax + b$$
$$-b = 2ax$$
$$\frac{-b}{2a} = x$$

If we investigate the signs of the first derivative of the quadratic, the function either changes signs from decreasing to increasing (or from increasing to decreasing), implying that this x value is an extremum. Since there is no other extrema, then this point is the absolute minimum (or maximum).

Calculus can be a quick approach to find the minimum or maximum value of the function. Taking the derivative of the quadratic equation would yield a linear equation, and solving for a linear equation can be done with little effort on the teacher's part.

Mathematical Focus 2

It can be shown that the coordinate (0, 0) is the absolute minimum for graph of the parabola $y = x^2$.



We start with the general quadratic formula and then apply the completing the square method to find the coordinates of the vertex of any quadratic function.

$$y = ax^2 + bx + c$$
$$y - c = ax^2 + bx$$
$$\frac{1}{a}(y-c) = x^2 + \frac{b}{a}x$$
$$\frac{1}{a}(y-c) + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$
$$\frac{1}{a}\left(y - c + \frac{b^2}{4a}\right) = \left(x + \frac{b}{2a}\right)^2$$
$$y + \frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$
$$y = a \times f\left(x + \frac{b}{2a}\right) + \left(c - \frac{b^2}{4a}\right)$$

By converting the general quadratic to this form, we can show all of the transformations from the graph of the parabola $f(x) = x^2$ to the graph of this function. Among the transformations of the graphs from y = x^2 to this function, $f\left(x + \frac{b}{2a}\right)$ represents a horizontal translation of the graph of the parent graph. We can show that this transformation will affect the coordinates of the vertex. In $f(x) = x^2$ the vertex is at *x* = 0. In the function $y = a \times f\left(x + \frac{b}{2a}\right) + \left(c - \frac{b^2}{4a}\right)$, the x coordinate of the vertex will be at $x + \frac{b}{2a} = 0$. This means that the transformed graph has vertex with an x coordinate of x = -b/2a.

Mathematical Focus 3

Assume the graph of the parabola has two real roots. The quadratic formula gives the roots of the function to be $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The midpoint of those roots is (-b/2a, 0). If the graph has two real roots, then the discriminant, $b^2 - 4ac$, is some positive number. As the discriminant approaches zero, the distance between the roots decreases, yet the midpoint remains the same. When the distance between points becomes zero, the discriminant is zero, and the point that was once the midpoint of the two roots now becomes the vertex of the graph. The x coordinate of the vertex is $-b/2a$.