

Situation 15: Graphing Quadratic Functions

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Prompt

When preparing a lesson on graphing quadratic functions, a student teacher had many questions about teaching the lesson to a Concepts of Algebra class, a remedial mathematics course. One of the concerns that the student teacher had was the graphing of the vertex of the parabola, which also means identifying the equation of the axis of symmetry. The textbook for this class described that

$x = -\frac{b}{2a}$ was the equation of the line of symmetry. The student teacher did not feel confident in his understanding of how to derive this equation.

Commentary

Graphing parabolas can be done in one of two different algorithms. One is to simply determine a collection of points and, assuming that the function is continuous across its domain, connect the dots in a fashion that resembles the desired shape. However in this approach there is no guarantee that sketch of the parabola would match the shape of the parabola given more precisely in graphing software packages, whether they be from a handheld graphing calculator or other computer software programs. What is desired is a different approach: a systematic way to draw essential components of the parabola and be able to sketch a more precise shape than from just a free form approach. Out of this we need ways to always find those essential components. One such component is the axis of symmetry. By having this line of reflection, we can assume that what we draw or sketch on side of this line is going to be the mirror reflection that of that same shape on the other side of the line.

Focus 1 is a discussion on how first derivatives can be implemented to illustrate the formula for the x coordinate of the vertex. In Focus 2, applying the completing the square method will result in a formula that is a composition of transformations being applied to $y = x^2$, one of which is a horizontal translation.

Mathematical Foci

Mathematical Focus 1

The vertex of a parabola is the absolute minimum (or maximum) functional value of a quadratic function. We can find the absolute minimum (or maximum) by taking the first derivative of the quadratic function.

$$y = ax^2 + bx + c$$
$$y' = 2ax + b$$

To find the extrema, set the derivative equal to 0 and solve for x:

$$0 = 2ax + b$$
$$-b = 2ax$$
$$\frac{-b}{2a} = x$$

If we investigate the signs of the first derivative of the quadratic, the function either changes direction from decreasing to increasing (or from increasing to decreasing), implying that this x value is indeed an extremum.

Now we might want to explain that the extreme value of the parabola helps us to determine the equation of the line for the axis of symmetry. In our discussion above, we see that there is only one value for an extremum. If we place our line of symmetry through the parabola that does not contain the extremum, then we will end up with a reflected extremum on the opposite side of the line of symmetry. This would then give two extrema, which would be a contradiction.

Mathematical Focus 2

It can be shown that (0,0) is the absolute minimum value for the graph of the function $y = x^2$. In the real numbers it is not possible to multiply a number by itself and receive a negative number product. There is only possible way for a real number to be multiplied by itself to get 0: multiply 0 by 0. For a Graphing Calculator file further illustrating this point see [the attached document](#).

Now what we can do is to explain that we can find the vertex of any parabola only knowing that the smallest value the graph of the parabola $y = x^2$ occurs at the coordinate (0,0). We start with the general quadratic formula and then apply the completing the square method to find the coordinates of the vertex

of any quadratic function.

$$y = ax^2 + bx + c$$

$$y - c = ax^2 + bx$$

$$\frac{1}{a}(y - c) = x^2 + \frac{b}{a}x$$

$$\frac{1}{a}(y - c) + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$

$$\frac{1}{a}\left(y - c + \frac{b^2}{4a}\right) = \left(x + \frac{b}{2a}\right)^2$$

$$y + \frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

$$y = a \times f\left(x + \frac{b}{2a}\right) + \left(c - \frac{b^2}{4a}\right)$$

By converting the general quadratic to this form, we can show all of the transformations from the graph of the parabola to the graph of this function. Among the transformations of the graph from $y = x^2$ to this function, $f\left(x + \frac{b}{2a}\right)$ represents a horizontal translation of the graph of the parent graph. We can show that this transformation will affect the coordinates of the vertex. In $y = x^2$ the vertex is at $x = 0$. In the function $y = a \times f\left(x + \frac{b}{2a}\right) + \left(c - \frac{b^2}{4a}\right)$, the x coordinate of the vertex will be at $x + \frac{b}{2a} = 0$. This means that the transformed graph has vertex with an x coordinate of $x = -b/2a$.

Mathematical Focus 3

Assume the graph of the parabola has two real roots. The quadratic formula gives the roots of the function to be $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The midpoint of those roots is $(-b/2a, 0)$. If the graph has two real roots, then the discriminant, $b^2 - 4ac$, is some positive number. As the discriminant approaches zero, the distance

between the roots decreases, yet the midpoint remains the same. When the distance between points becomes zero, the discriminant is zero, and the point that was once the midpoint of the two roots now becomes the vertex of the graph. The x coordinate of the vertex is $-b/2a$.