Situation 15: Graphing Quadratic Functions

Prepared at University of Georgia/Penn State
Center for Proficiency in Teaching Mathematics OR
Ginger Rhodes
Revised by Ryan Fox

Prompt

Matt and I worked together to plan a lesson on graphing quadratic functions. I asked Matt what he remembered about quadratic functions, and he said he remembered very little: he said it was a parabola and that was about all he knew. I asked him what he wanted students to be to do and he said he wanted them to be able to identify the domain/range, max/min point, and vertex, as well as graph the function. The book had

$$x = -\frac{2a}{b}$$

as the equation of the line of symmetry, which was also used to find the x-coordinate of the vertex. The book did not show how to derive this equation, so I asked Matt if he knew how - he said no. So then we talked about how to derive this equation using the quadratic formula, and whether this was something that his students should understand. He thought this was too difficult for his concepts of algebra students.

So then we talked about what the students had been doing recently, which was domain/range of functions and linear functions. We discussed how this connected to the lesson he was planning for, and Matt really struggled. I was thinking about the connection between linear functions and quadratic functions: such as what students can find from the graphs of functions, or from the equations. We looked at the linear function

$$f(x) = 2x + 7$$

and the quadratic function

$$f(x) = x^2 + 10x + 25$$

I asked him what he thought his students could do with the linear function, and he said they could find the slope, the y-intercept, and probably graph the line. I asked him if he thought his students knew what the domain and range was for a linear function, and he did not think so. By the end of our planning session we had decided

to begin with a warm-up reviewing linear functions (how to graph, domain/range, slope, and y-intercept), and then introduce a quadratic function by asking how it looks different, what they know about it, and connecting to the linear function. Matt was not confident with the mathematical connections, and therefore this conversation did not go the way I envisioned it. The conversation was very procedural and the connections were not made.

Matt is a student teacher. What else should I do to help him?

Commentary

[My idea for this prompt is to direct attention on where the formula for the x coordinate of the vertex of the parabola x = -b/2a comes from. RF]

The graphing of the parabola $y = x^2$ is a graph teachers expect their students to know and apply parabolas to model real world phenomena. The graphing of the vertex is useful to graph a quadratic function because the functional value of the vertex gives the absolute minimum or maximum functional value. Focus 1 is a discussion on how first derivatives can be implemented to illustrate the formula for the x coordinate of the vertex. In Focus 2, applying the completing the square method will result in a formula that is a composition of transformations being applied to $y = x^2$, one of which is a horizontal translation. Another important topic covered in secondary mathematics, particularly algebra classes, is the quadratic formula as determining solutions of a quadratic equation. The fact that the x coordinate of the vertex is contained within the quadratic formula is discussed in Focus 3.

Mathematical Foci

Mathematical Focus 1

The vertex of a parabola is the absolute minimum (or maximum) functional value of a quadratic function. We can find the absolute minimum (or maximum) by taking the first derivative of the quadratic function.

$$y = ax^2 + bx + c$$
$$y' = 2ax + b$$

To find the extrema, set the derivative equal to 0 and solve for x:

$$0 = 2ax + b$$
$$-b = 2ax$$
$$\frac{-b}{2a} = x$$

If we investigate the signs of the first derivative of the quadratic, the function either changes signs from decreasing to increasing (or from increasing to decreasing), implying that this x value is an extremum. Since there is no other extrema, then this point is the absolute minimum (or maximum).

Mathematical Focus 2

It can be shown that the coordinate (0, 0) is the lowest point in the graph of the parabola $y = x^2$. We start with the general quadratic formula and then apply the completing the square method.

$$y = ax^2 + bx + c$$
$$y - c = ax^2 + bx$$
$$\frac{1}{a}(y - c) = x^2 + \frac{b}{a}x$$
$$\frac{1}{a}(y - c) + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$
$$\frac{1}{a}\left(y - c + \frac{b^2}{4a}\right) = \left(x + \frac{b}{2a}\right)^2$$

Among the transformations from $y = x^2$ to the original function, the graph of the new function has been translated b/2a units to the left. This means that the transformed graph has vertex with an x coordinate of x = -b/2a.

Mathematical Focus 3

Assume the graph of the parabola has two real roots. The quadratic formula gives the roots of the function to be $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The midpoint of those roots is (-b/2a, 0). If the graph has two real roots, then the discriminant, $b^2 - 4ac$, is some positive number. As the discriminant approaches zero, the distance between the roots decreases, yet the midpoint remains the same. When the distance between points becomes zero, the discriminant is zero, and the point that was once the midpoint of the two roots now becomes the vertex of the graph. The x coordinate of the vertex is –b/2a.