Situation #22: Operations with Matrices

Prepared at University of Georgia
Center for Proficiency in Teaching Mathematics
28 June 2005 – Pat Wilson

Edited at University of Georgia
Center for Proficiency in Teaching Mathematics
21 September 2006 – Ryan Fox

Prompt

Students in an Algebra II class had been discussing the addition of matrices and had worked on several examples of n x n matrices. Most were proficient in finding the sum of two matrices. Toward the end of the class period, the teacher announced that they were going to being working on the multiplication of matrices, and challenged the students to find the product of two 3 x 3 matrices:

$$\begin{bmatrix} 2 & 4 & 5 \\ 5 & 2 & 3 \\ 1 & 4 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \\ 5 & 2 & 3 \end{bmatrix}$$

Students began to work on the problem by multiplying each corresponding term in a way similar to how they had added terms. One student shared his work on the board getting a product of

$$\begin{bmatrix} 2 & 12 & 10 \\ 10 & 12 & 15 \\ 5 & 8 & 12 \end{bmatrix}$$

As the period ended, the teacher asked students to return to the next period with comments about the proposed method of multiplying and alternative proposals.

Commentary

This prompt illustrates the notion that how one binary operation acts on two elements of one set can be very different in comparison to how it acts on two elements from a second set. Too often the temptation is to apply the algorithm to multiply and add elements of one set and apply those rules and algorithms to other sets. We can find very quickly this method does not always work. Multiplication of rational numbers seemingly works like multiplication of integers. However, addition of rational numbers becomes quite complicated; no longer are we allowed to add numerators or denominators individually, but rather a more elaborate algorithm is devised, one involving a secondary operation: multiplication of integers.

The matter at hand for this prompt is the concept of a matrix. The definition of matrix seems simple enough: a collection or an array of elements. At the secondary school level, we pay attention to matrices whose elements are real numbers, or maybe even rational numbers to be more specific. While matrices are composed of real numbers, not all properties that work for real numbers will necessarily work for matrices. For example, in multiplication of all non-zero real numbers, each number has a multiplicative inverse, where only a select set of matrices have an inverse under matrix multiplication. Also, multiplication of real numbers is a closed operation: one can multiply any two real numbers and get a real number product. In this operation called matrix multiplication, the operation is not considered closed: one cannot always two matrices and obtain a matrix product.

The foci presented try to offer both an explanation as to why the corresponding entries approach, a name given to for the method used in this prompt, gives us results that are not desirable for our study of matrix multiplication. The final three foci explain the inclusion of addition to matrix multiplication: the second focus shows a graphical understanding of multiplying matrices and the third focus attempts to justify the formula through a modeling approach. Lastly, we examine the inclusion of the transformation of coordinates through linear combinations. Linear combinations, as a type of linear transformation, will help us to show the interaction of the combination of multiplication and addition.

Mathematical Foci

Mathematical Focus 1

We want to show that if we did the corresponding entries approach, we would encounter results we do not want.

First, this approach does not seem to work with two matrices of different dimensions. For example, using the corresponding entries approach, we would not be able to multiply the following matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

However, especially in the context of solving a system of three equations with three unknowns, we want this product to exist. The first matrix in the above problem represents the coefficients of three equations with three unknowns. The second matrix represents those three unknown quantities. The desired product of these two matrices is the linear combination of these three unknowns needed to solve the system of equations.

Mathematical Focus 2

We can think of matrices as representing vectors, an idea that will be developed in a later focus. However, an idea that can be taken from the study of vectors is the concept of the dot product. The concept of the dot product can be thought of as the scalar answer from multiplying two vectors:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots a_n b_n$$

We can think of the two matrices as the collection of row and column vectors respectively. We can look at each entry in the product a_{ij} as the dot product of the i^{th} row vector of the first matrix and the j^{th} column vector of the second matrix.

We can use the concepts of transformations and dot products to explain matrix multiplication. Starting with the dilation, which is the stretching (or shrinking) of the x and y coordinates by the same value. In order to dilate the x coordinate we want to multiply our x coordinate by our factor and do not want to manipulate the y coordinate. What we have so far is $x \cdot c + y \cdot 0$. We can do the same approach for the y coordinate to give us $x \cdot 0 + y \cdot c$. Combining what we have so far and using our dot product approach to multiplication, we have shown that $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} x \cdot c + y \cdot 0 & x \cdot 0 + y \cdot c \end{bmatrix}$.

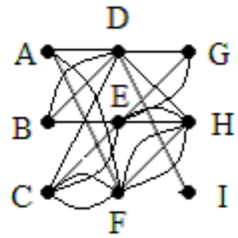
A reflection about one of the coordinate axes will change the sign of one of the coordinates will leaving the other coordinate unchanged. For example, a reflection about the x-axis will change the sign of the y coordinate, but leave the x coordinate unchanged. We can represent this as $x \cdot 0 + y \cdot -1$ and $x \cdot 1 + y \cdot 0$. If we convert these to matrices and use our dot product idea we can create

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} x \cdot 1 + y \cdot 0 & x \cdot 0 + y \cdot -1 \end{bmatrix}$$

Likewise, a reflection about the y-axis will change the sign of the x coordinate but leave the y coordinate unchanged. We can represent this as $x \cdot -1 + y \cdot 0$ and $x \cdot 0 + y \cdot 1$. If we use the dot product formulas we can convert this to the matrix multiplication problem $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -x \cdot 1 + y \cdot 0 & x \cdot 0 + y \cdot 1 \end{bmatrix}$

Mathematical Focus 3

We can use graph theory to develop the process of matrix multiplication. Let us come up with any graph of vertices and edges. For this problem, we will create a graph of nine vertices, labeled A through I. We will divide these nine vertices into three sets: {A, B, C}, {D, E, F}, and {G, H, I}. We will connect the vertices from {A, B, C} to {D, E, F} and then connect {D, E, F} to {G, H, I}. We will connect these vertices using as many edges as we want. These properties can be seen in the graph below:



Our goal is to determine the number of paths from a vertex in {A, B, C} to a vertex in {G, H, I} going through {D, E, F}. Now we can represent the connections between sets of vertices in matrix, using an adjacency matrix:

$$\begin{array}{cc} \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \\ & \begin{matrix} D & E & F \end{matrix} \end{array}$$

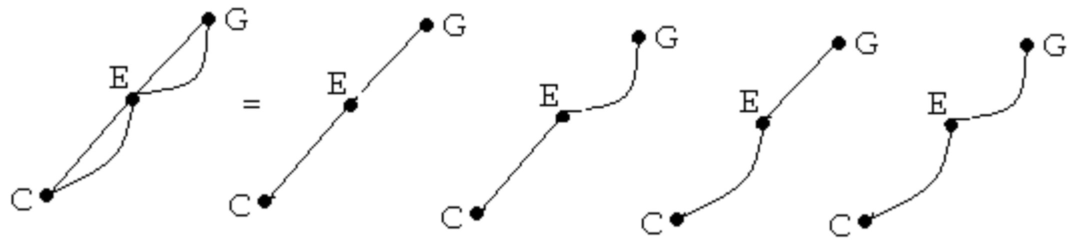
Each entry in the matrix represents the number of ways the two vertices are connected. For example, there are two ways to connect the vertices A and F, yet there is no direct connection between the vertices A and E. In the same manner, we can create an adjacency matrix between {D, E, F} and {G, H, I}.

$$\begin{array}{cc} \begin{matrix} D \\ E \\ F \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix} \\ & \begin{matrix} G & H & I \end{matrix} \end{array}$$

Now we want to find the number of ways between {A, B, C} and {G, H, I}. The only connection between these two sets is through the vertices {D, E, F}. To find the ways between {A, B, C} and {G, H, I}, find the number of connections to the individual vertices and then the total number of connections.

$$\begin{array}{cc} \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 1 & 7 & 1 \\ 4 & 4 & 3 \\ 5 & 11 & 3 \end{bmatrix} \\ & \begin{matrix} G & H & I \end{matrix} \end{array}$$

For example, in this product, there are five ways from vertex C to vertex G. There is the one path from C to D to G; there are four ways from C to E to G, as illustrated below:



We could use the Fundamental Counting Principle to assert that four is indeed the product of the two ways from C to E and the two ways from E to G. Similarly, there are 11 total ways from C to H, using the Fundamental Counting Principle to arrive at this sum: one path from C to D to H, four paths from C to E to H (the product of two from C to E and two ways from E to H), and six paths from C to F to H (two ways from C to F and three ways from F to H).

It should be noted that both of this approach would be able to demonstrate multiplying two matrices that are both not square. For an explanation, please see this document.

Mathematical Focus 4

One of the ways students can visualize matrix multiplication is through a problem solving approach. Since matrices represent arrays of data, the process of matrix multiplication can be arrived at intuitively. This problem solving approach can be illustrated by the School Mathematics Study Group (1961a, pgs. 24-26). In this approach, their textbook describes a manufacturer needing multiple parts to make different televisions, which can be written in matrix form:

$$\begin{array}{ll} \text{Tubes} & \\ \text{Speakers} & \end{array} \begin{bmatrix} 13 & 18 \\ 2 & 3 \end{bmatrix} \atop {\scriptsize \text{Model A} \quad \text{Model B}}$$

Then the numbers of televisions sold in two different months are:

$$\begin{array}{ll} \text{Model A} & \\ \text{Model B} & \end{array} \begin{bmatrix} 12 & 6 \\ 24 & 12 \end{bmatrix} \atop {\scriptsize \text{January} \quad \text{February}}$$

To find the total number of tubes and speakers sold during the two month period, it would make sense to determine the number of each part sold based on the number of models sold in each month. For example, the total number of tubes sold in January would equal the number of tubes for Model A times the number of televisions sold in January and the number of tubes sold for model B times the number of televisions sold in January: $13(12)+18(24)=156+432=588$. To figure out the remaining parts for each month, we would continue the same process:

$$
\begin{array}{r}
\text{Tubes} \\
\text{Speakers}
\end{array}
\begin{bmatrix} 13(12)+18(24) & 13(6)+18(12) \\ 2(12)+3(24) & 2(6)+3(12) \end{bmatrix}
$$
$$\begin{array}{cc} \text{January} & \text{February} \end{array}$$

$$
\begin{array}{r}
\text{Tubes} \\
\text{Speakers}
\end{array}
\begin{bmatrix} 588 & 294 \\ 96 & 48 \end{bmatrix}
$$
$$\begin{array}{cc} \text{January} & \text{February} \end{array}$$

This method too can be used to determine the product of two non-square matrices.

Mathematical Focus 5

Let us create a transformation for the coordinate (x, y) to a new coordinate (x_1, y_1) by performing a linear combination of both original coordinates to apply to the new coordinate: $\begin{cases} x_1 = ax + by \\ y_1 = cx + dy \end{cases}$ for real numbers a, b, c, and d. We will now take these four numbers and put the in matrix form: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This matrix is simply taking the four real numbers from the linear combination and placing them in the array exactly as we see them in the transformation. Now we will create a transformation of the transformation. We will transform the coordinate (x_1, y_1) to a new coordinate (x_2, y_2) by making each new coordinate a linear combination of the previously transformed coordinates: $\begin{cases} x_2 = ex_1 + fy_1 \\ y_2 = gx_1 + hy_1 \end{cases}$. Once again, a matrix can represent this transformation as well: $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$. We can represent the transformation from the first coordinate to the last coordinate as a composition of transformations.

$$
\begin{cases} x_2 = ex_1 + fy_1 = e(ax + by) + f(cx + dy) \\ y_2 = gx_1 + hy_1 = g(ax + by) + h(cx + dy) \end{cases}
$$
$$
\begin{cases} x_2 = eax + eby + fcx + fdy = (ea + fc)x + (eb + fd)y \\ y_2 = gax + gby + hcx + hdy = (ga + hc)x + (gb + hd)y \end{cases}
$$

. When describing a composition of transformations, we write the transformations in reverse order.

$\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{bmatrix}$, which gives us our commonly accepted formula for the multiplication of two matrices.

References

Barnett, Stephen. (1990). *Matrices*: *Methods and Approximations*. Oxford, England: Clarendon Press.

School Mathematics Study Group. (1961). *Mathematics for High School: Introduction to Matrix Algebra*. New Haven, CT: Yale University Press.

School Mathematics Study Group. (1961). *Introduction to Matrix Algebra: Teacher's Commentary*. Stanford, CA: School Mathematics Study Group.

END OF SITUATION – END OF SITUATION