# MAC-CPTM Situations Project

## Situations 24:

## *Absolute Value Equations and Inequalities*

### (includes material from Situation 26)

Situation 24 Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
29 June 2005 – Shari & Anna

Situation 24 Edited at University of Georgia
Center for Proficiency in Teaching Mathematics
15 February 2007 - Sarah Donaldson
20 February 2007 - Sarah Donaldson

Situation 26 Prepared at University of Georgia
Center for Proficiency in Teaching Mathematics
16 June 2006 – Kanita DuCloux and Sarah Donaldson

Edited and Combined at University of Georgia
Center for Proficiency in Mathematics Teaching
23 October 2008 – Kelly Edenfield and Erik Jacobson

## Prompt

A student teacher begins a tenth-grade geometry lesson on solving absolute value equations by reviewing the meaning of absolute value with the class. They discuss that the absolute value represents a distance from zero on the number line and that the distance cannot be negative. He then asks the class what the absolute value tells you about the equation $|x| = 2$. A student responded, "anything coming out of it must be 2." The student teacher states, "*x* is the distance of 2 from 0 on the number line." Then on the board, the student teacher writes

$$|x+3| = 5$$
$$x+3=5 \quad and \quad x+3=-5$$
$$x=2 \qquad\qquad x=-8$$

and graphs the solution on a number line. A puzzled student asks, "Why is it 5 and –5? How can you have –8? You said that you couldn't have a negative distance?"

## Commentary

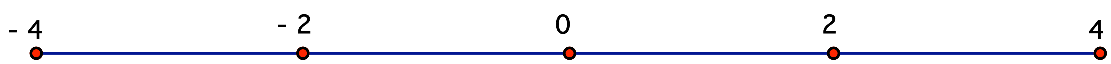
The primary issue is to understand the nature of absolute value in the real numbers. (Another Situation discusses absolute value of complex numbers.) In working with absolute value, the goal is not simply to know the steps and methods for solving absolute value equations and inequalities, but to have a deeper knowledge about the reasons why certain solutions are valid and others are not. The absolute value of a number is defined as the number's distance from zero. There are several interpretations of absolute value, some of which will be addressed in the following foci. Each view adds a new way to think about absolute value and to solve absolute value equations or inequalities. Foci 1 and 2 consider the absolute value as distance from zero on a number line and discuss how this characterization allows equations and inequalities to be solved graphically and algebraically. Foci 3 and 4 center on two-dimensional view of absolute value as a piecewise function, and Focus 5 addresses how describing absolute value of *x* as the square root of $x^2$ can be employed to solve absolute value equations and inequalities.

## Mathematical Foci

### Mathematical Focus 1

*Describing absolute value as distance from zero on a number line provides a clear representation of the meaning of* $|x + 3| = 5$ *and related inequalities.*

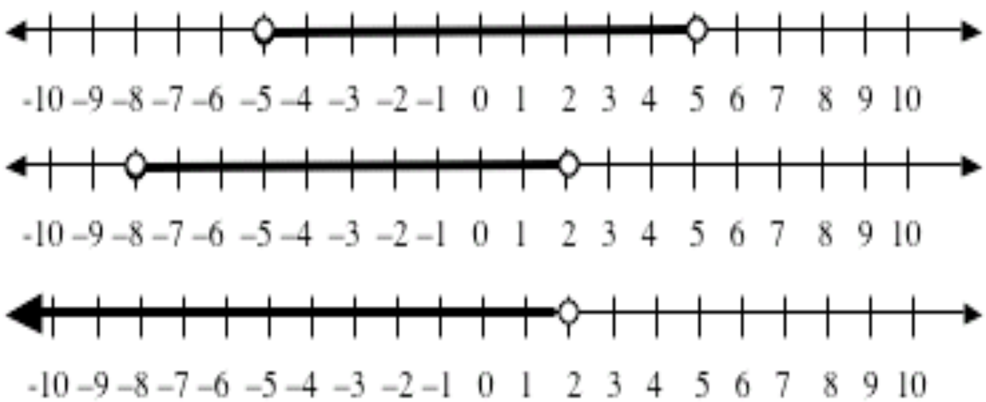
The absolute value of real numbers is often defined as the number's distance from zero on the number line. This can be seen on the number line below. For example, $|2| = 2$ because 2 is 2 units away from zero, and $|-2| = 2$ because -2 is also 2 units from zero.




Absolute value equations can also be solved using number lines. These graphs may be expressed in terms of the value of the variable (e.g., *x*) that produce the solution, or in terms of the values of a variable expression (e.g., *x* + a) that convey the solution. In these cases, graphs for *x* + a are translations of the graphs for *x*.

In $|x + 3| = 5$, *x* represents a number such that, when 3 is added to it, the result will be 5 units away from zero on a number line. Consider $|x| = 5$. The solutions to $|x| = 5$ would be 5 and -5. However, translating the solutions three units to the left to reflect the problem in the prompt yields the solutions of 2 and -8. Therefore $x = -8$ and $x = 2$ are solutions to $|x + 3| = 5$.

Likewise, real-number solutions to inequalities can be represented using number lines. Consider the related inequality, $|x+3| < 5$. One could interpret this problem as asking for all numbers (written *x* + 3) that are less than five units from zero. Graphing the set of numbers, $x$, that are less than five units from zero yields the first graph shown in the figure. To compensate for the "+3" and thus have a graph of the values of $x$ that satisfy the inequality $|x + 3| < 5$ requires translating the first graph three units to the left, as shown in the second graph. A graph of $x < 2$ (the solution set of $x + 3 < 5$) is shown as the third graph in the figure. Comparing the second and third graphs illustrates that some solutions of $x + 3 < 5$ are not solutions of $|x + 3| < 5$.




**Mathematical Focus 2**

*Describing absolute value as distance from zero on the number line also allows one to solve absolute value equations and inequalities algebraically.*

When solving for an unknown (such as "$x$") in an equation, one must list all the possible real solutions (values for $x$) that make the equation true. Some equations yield only one real solution. For example, in $x + 3 = 5$, the only real number that $x$ could be to make the equation true is 2. Other equations yield more than one solution. For example, if $x^2 = 9$, then $x$ could be either 3 or -3 because both $(3)^2$ and $(-3)^2$ equal 9.

Absolute value equations often yield more than one solution. In $|x| = 5$, for example, there are two values for $x$ that make the equation true, 5 and -5, because both $|5|$ and $|-5|$ are 5.

Expanding this notion to other absolute value equations, such as $|x + 3| = 5$, there are two possibilities for the value of (x + 3). The expression $(x + 3)$ could be 5 or -5 because both $|5|$ and $|-5|$ equal 5. Listing each of these possibilities as equations, then,

| | | | |
|---|---|---|---|
| | $x + 3 = 5$ | and | $x + 3 = -5$ |
| So | $x = 2$ | and | $x = -8$ |

Each solution can be checked in the original equation to see that are, indeed, in the solution set of the absolute value equation.

When an inequality involves a numerical constraint on the absolute value of an algebraic expression in one variable, distance from zero can be used to write an extended inequality that can then be solved algebraically. One could interpret the question as asking for all numbers (written $x + 3$) that are less than five units from zero, thus generating the inequality $-5 < x + 3 < 5$ which can be solved algebraically to find the solution set $-8 < x < 2$.

Similarly, if the inequality was reversed, $|x + 3| < 5$, a disjoint statement could be written. One could interpret this question as asking for all numbers (written $x + 3$) that are more than five units (in either direction) from zero. This yields the following:

$x + 3 < -5$ or $x + 3 > 5$

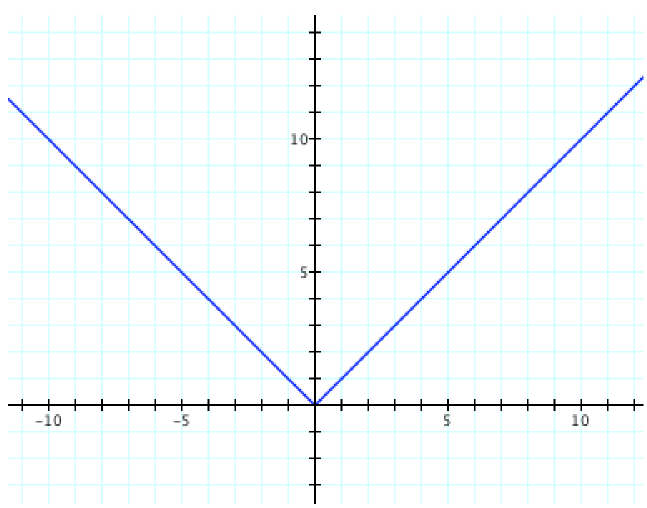
$x < -8$ or $x > 2$

This solution is seen quite clearly when examining a number line such as number line 2 in the above figure.

**Mathematical Focus 3**

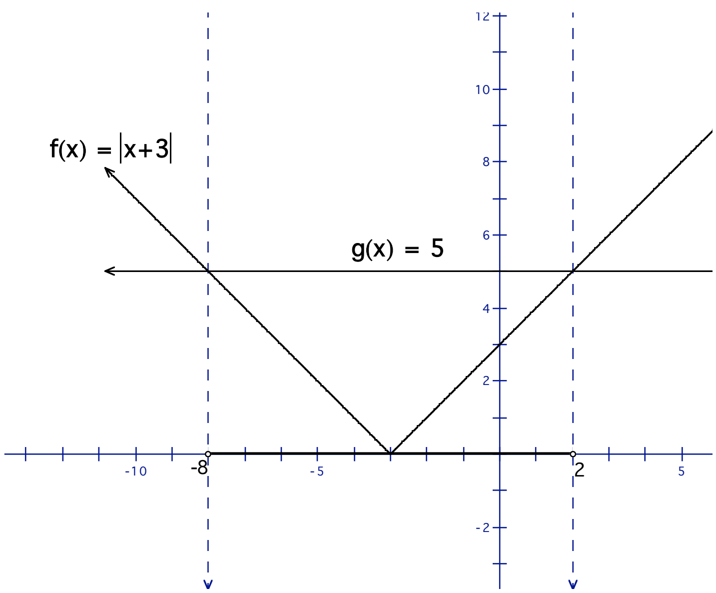
*Absolute value equations can be expressed as functions and can be represented graphically. This provides visual representations of domain and range and allows for graphical solutions to absolute value equations and inequalities.*

Expressing $y = |x|$ as a function yields $f(x) = |x|$. The domain of the function $f(x) = |x|$ is $\{x : x \in \Re\}$, and the range is $\{y : y \geq 0\}$. The $y$-value (range) is never negative (the function does not exist below the $x$-axis), but the $x$-value (domain) could be any real number. Using absolute value notation $y = |x|$, this idea of domain and range means that what is inside the absolute value symbols ($x$) **can** be negative while the absolute value itself ($y$) **cannot** be negative.



The solution of an equation can be found by graphing the function related to the left member of the equation and the function related to the right member of the equation and finding the $x$-value of intersection point(s) of the graphs. Similarly, graphs of related functions can be used to determine the solution of an inequality. To solve the inequality $f(x) < g(x)$, find the values of $x$ for which the graph of $f$ indicates smaller output values than the graph of $g$ (one might think of this as the graph of $f$ is "below" the graph of $g$).

Consider the functions $f(x) = |x + 3|$ and $g(x) = 5$. The solution to $|x+3| = 5$ will be all values in the domain for which $f(x)$ is equal to $g(x)$--the intersection points of the graphs of $f(x)$ and $g(x)$. Similarly, the solution to $|x+3| < 5$ will be all values in the domain for which $f(x)$ is less than $g(x)$. The two functions are graphed in the figure below. The solution to $|x + 3| < 5$ can be seen by determining for which $x$-values the graph of $f(x) = |x + 3|$ is below the graph of $g(x) = 5$. As shown below, $f(x)$ is less than $g(x)$ exactly when $-8 < x < 2$.



**Mathematical Focus 4**

*Absolute value equations can be expressed as piecewise functions. This representation allows absolute value equations and inequalities to be solved algebraically.*

The representation of absolute value as a piecewise function is $|x| = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$.

Considering the problem in this prompt, this means $|x+3| = \begin{cases} x+3, \text{if } x+3 \geq 0 \\ -x-3, \text{if } x+3 < 0 \end{cases}$.

To solve the equation $|x+3| = 5$, set each element of piecewise function equal to 5 and solve.

$$\begin{array}{lcl} x+3=5 & \text{and} & -x-3=5 \\ x=2 & & -x=8 \\ & & x=-8 \end{array}$$

So, again, the solution to the absolute value equation is $x = 2$ and $x = -8$.

Similarly, the solution set of the inequality $|x+3| < 5$ is the union of the solution sets of the inequalities generated by these two pieces. Solving the system of inequalities suggested by the first piece, $x + 3 < 5$ and $x + 3 \geq 0$, gives us $-3 \leq x < 2$. Solving the system of inequalities suggested by the second piece, $-x - 3 < 5$ and $x + 3 < 0$, yields $-8 < x < -3$. The union of these two solution sets yields $-8 < x < 2$.

**Mathematical Focus 5**

*Absolute value of a real number viewed as the positive square root of the square of the number provides another way of understanding absolute value, and another tool for solving absolute value equations and inequalities.*

Algebraically, absolute value can also be described as the positive square root of the square of that number. That is, $|x| = +\sqrt{x^2}$. Use of this characterization to solve $|x + 3| = 5$ is shown here:

$$|x + 3| = 5$$
$$+\sqrt{(x+3)^2} = 5$$
$$(x + 3)^2 = 25$$
$$x^2 + 6x + 9 = 25$$
$$x^2 + 6x - 16 = 0$$
$$(x + 8)(x - 2) = 0$$
$$(x + 8) = 0 \text{ and } (x - 2) = 0$$
$$x = -8 \text{ and } x = 2$$

Similarly, to solve $|x + 3| < 5$, most of the same computations can be used. However, after factoring, number sense comes into consideration.

If the product of 2 factors is less than zero, *one and only one* factor is less than 0. Consider $(x + 8)(x - 2) < 0$.
Now let $(x+8) < 0$, then $x < -8$. If $x < -8$, then $(x-2) < 0$. This is a contradiction, so $(x + 8)$ is not less than zero; it must be greater than zero. Therefore, $(x-2) < 0$. So $x > -8$ and $x < 2$. Then, as before, $-8 < x < 2$.