Situation 24: Isn't Absolute Value Always Positive?

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Prompt

The course was a first-year algebra course for lower-level students.

The discussion centered on solving absolute-value inequalities. One student said that since everything in $|x + 3| < 5$ is positive, it could be solved by solving $x + 3 < 5$, so the solution was $x < 2$.

What is the mathematical relationship between $|x + 3| < 5$ and $x + 3 < 5$?

Commentary

Mathematical Foci

Mathematical Focus 1

If solving one inequality can be used to obtain the solution to another inequality, the solution sets for both inequalities must be equal. In fact, the two inequalities must be equivalent. The goal is to identify *all* solutions of the first inequality. This requires one to ask if there are any solutions to the first inequality that are not solutions to the second inequality as well as to ask if there are any solutions to the second inequality that are not solutions to the first inequality.

To determine the solution set of one inequality, one can first find the solution of a related equation to determine boundaries for the solution set of the inequality. One might use any one of a variety of methods, including successive approximations, to determine the solutions of the equation. In this case, the solutions of the equation would not be solutions of the strictly-less-than inequality.

To determine the specific solution set for the inequality requires further reasoning about the relationship of the solutions of the equation to the solutions of the inequality. In this case, because the expressions for either of the inequalities involved relate to functions that are continuous and monotonic over the interval of real numbers that forms the solution of the original inequality, this approach works and all values between the least and greatest solution of the equation will be appropriate solutions of the inequality.

The student's solution set contains numbers that are solutions to the given inequality and numbers that are not solutions to the given inequality. We could consider particular numbers that are solutions to the second but not to the first. For example, considering -10 as a solution could lead to recognition of differences in the solution sets. Exploring other values could ultimately lead to a determination of the smallest and largest solutions to $|x + 3| = 5$. Solutions to this equation are the boundaries on the solutions to $|x + 3| < 5$. All numbers between -8 and 2 are solutions to this inequality. Comparing $-8 < x < 2$ with $x < 2$ demonstrates that the solution sets of the two inequalities are not equal.

NOTE: THE DISCUSSION HERE MAY BE A WAY TO HAVE THE STUDENT SEE THAT THE STUDENT'S PROPOSED SOLUTION DOES NOT YIELD A CORRECT ANSWER AND THEN GET AN ANSWER BUT IT DOES NOT UNPACK THE MATHEMATICS IN THE STUDENT'S COMMENTS.

Mathematical Focus 2

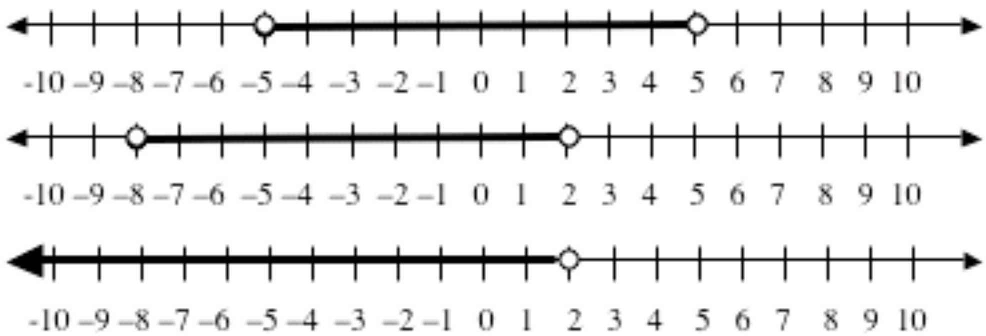
One interpretation of absolute value of real numbers involves distance from 0 on the number line. When an inequality involves a numerical constraint on the absolute value of an algebraic expression in one variable, distance from 0 can be used to write an extended inequality that can then be solved algebraically.

One could interpret the question as asking for all numbers (written $x + 3$) that were at most five units from zero, thus generating the inequality $-5 < x + 3 < 5$ which can be solved algebraically to find the solution set $-8 < x < 2$. This solution set is more limited than the solution set to $x + 3 < 5$ ($x < 2$), that is, all x such that $x \le -8$ are solutions to $x + 3 < 5$, but not to $|x + 3| < 5$.

Mathematical Focus 3

Absolute value as distance from 0 can be represented using number lines. Real-number solutions to inequalities can be expressed as graphs using number lines. These graphs may be expressed in terms of the value of the variable (e.g., x) that produce the solution or in terms of the values of a variable expression (e.g., x + a) that conveys the solution. In these cases, graphs for x + a are translations of the graphs for x.

One could interpret the question as asking for all numbers (written $x + 3$) that were at most five units from zero. Graphing the set of numbers, $x+3$, that are at most five units from zero yields the first graph shown in the figure. To compensate for the "+3" and thus have a graph of the values of x that satisfy the second inequality requires translating the first graph three units to the left, as shown in the second graph in the figure. A graph of $x < 2$ (the solution set of $x + 3 < 5$) is shown as the third graph in the figure. Comparing the second and third graphs illustrates the solutions to $x + 3 < 5$ that are not solutions to $|x + 3| < 5$.



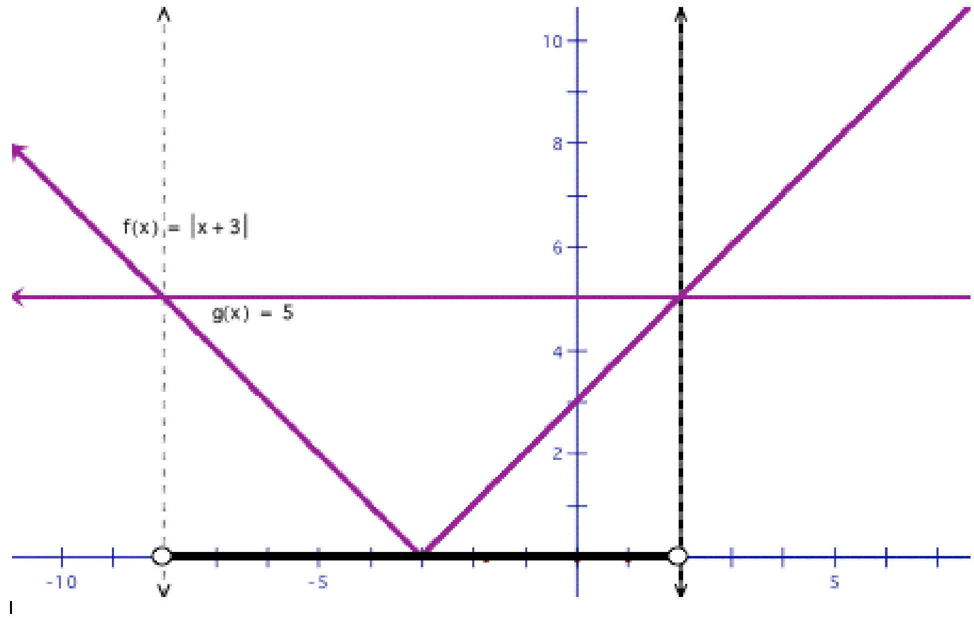
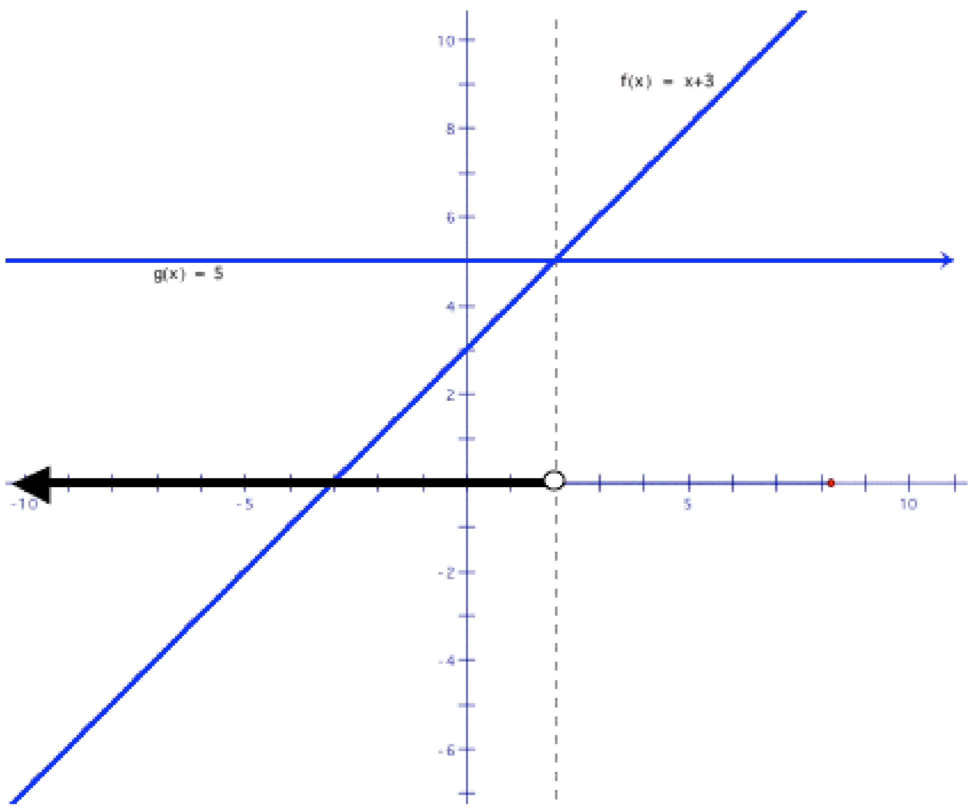
Mathematical Focus 4

The solution of an equation can be found by graphing the function related to the left member of the equation and the function related to the right member of the equation and finding abscissa of intersection point(s) of the graphs. Similarly, graphs of related functions can be used to determine the solution of an inequality. To solve the inequality $f(x) < g(x)$, find the values of x for which the graph of f indicates greater output values than the graph of g (one might think of this as the graph of f is "above" the graph of g).

Consider the functions $f(x) = |x + 3|$ and $g(x) = 5$. The solution to $|x + 3| < 5$ will be all values in the domain for which f(x) is less than g(x). The two functions are graphed in the first figure below. The solution to $|x + 3| < 5$ can be seen by determining for which x-values the graph of $f(x) = |x + 3|$ is below the graph of $g(x) = 5$. As shown below, f(x) is less than g(x) exactly when $–8 < x < 2$.

Now consider the inequality $x + 3 < 5$. Its solution will be all values in the domain for which the graph of $f(x) = x + 3$ is below the graph of $g(x) = 5$. As can be seen from the graph in the second figure below, f(x) is less than g(x) exactly when $x < 2$.

Comparing the two graphs, it is clear that the two solution sets are not equivalent. In the second figure below, we can see that the graph of $f(x) = x + 3$ is below the graph of $g(x) = 5$ for all x-values less than 2. However, in the first figure, it is clear that f(x) is greater than g(x) for all $x \le -8$.


f(x) = |x + 3|
g(x) = 5
10
8
6
4
2
-10
-5
5

f(x) = x+3
g(x) = 5
10
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-6
-10
-5
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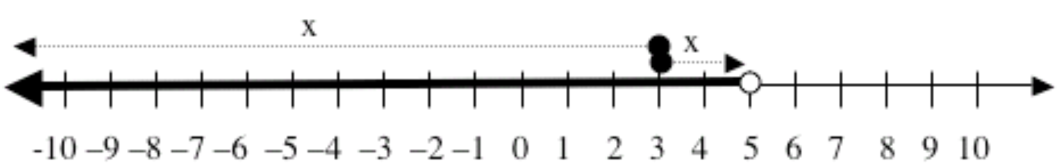
10

Mathematical Focus 5

We can consider the absolute value inequality numerically on a number line. Our solution would be all values of x for which $|x + 3| < 5$. We can think of x + 3 as "add 3 to x" and know we get the same result as "add x to 3". We graph all numbers x + 3 that are less than five units from zero. We have the interval (-5,5). We can think of the solutions as answering the following question: If we start at 3, how far can we go and still stay in the interval (-5,5)? We then consider what numbers x we could add to 3 in order to stay within the interval (-5, 5). The lengths and directions of the arrows in the first figure below illustrate that x can be any negative number greater than –8 and any non-negative number less than 2. Since we know our solution set is an interval, our solution set is (-8, 2).

Similarly, the second figure illustrates the solution of x + 3 < 5. On the number line is graphed all numbers x + 3 that are less than five. The lengths and directions of the arrows in the second figure illustrate that x can be any negative number and any non-negative number less than 2. The solution set is (-∞, 2).

Since the arrow to the left in the second figure extends indefinitely to the left, it is clear this solution set includes values that are not in the solution illustrated in the first figure.



Mathematical Focus 6

The definition of absolute value as a piecewise function is: $|x| = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$. We can then define the left-hand member of the inequality as $|x+3| = \begin{cases} x+3, \text{if } x+3 \geq 0 \\ -x-3, \text{if } x+3 < 0 \end{cases}$. The solution set of the given inequality is the union of the solution sets of the inequalities generated by these two pieces. Solving the system of inequalities suggested by the first piece, $x + 3 < 5$ and $x + 3 \geq 0$, gives us $-3 \leq x < 2$.Solving the system of inequalities suggested by the second piece, -

x – 3 < 5 and x + 3 < 0,yields –8 < x < -3. The union of these two solution sets yields –8 < x < 2.

Solving x + 3 < 5 in the same way immediately highlights the differences between the two inequalities. To solve this inequality, the definition of absolute value is not involved. We can see that |x + 3| is equivalent to x + 3 only when their domains are limited by x + 3 ≥ 0.

References

Include books, articles, curriculum materials, URLs, etc. from which any ideas or quotes are drawn. (There may be none.)

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