MAC-CPTM Situations Project

Situation 28: Adding Square Roots

(Includes material from Situation 32 Radicals)

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Prompt

Mr. Fernandez is bothered by his ninth-grade algebra students' responses to a recent quiz on radicals, specifically those in response to a question about square roots in which students added $\sqrt{2}$ and $\sqrt{3}$ and obtained $\sqrt{5}$ as a result.

Commentary

The mathematical basis for determining the appropriateness of the students' work is that the sum of the square roots of two numbers is not, in general, equal to the square root of the sum of the two numbers. Establishing that a statement is not true can be accomplished in different ways, including finding a counterexample, indirect proof, and lack of correspondence of function graphs. The students' statement, $\sqrt{2} + \sqrt{3} = \sqrt{5}$, can be disproved using numeric, geometric, symbolic, and graphical representations. Connections are made to linear transformations from linear algebra and to multivariate functions and function composition.

Mathematical Focus 1

Geometric constructions can be used to disprove statements about specific real numbers.

Construct an irrational spiral by constructing an isosceles right triangle with legs of length 1. This will have a hypotenuse with length $\sqrt{2}$. Then construct another right triangle with legs of length 1 and $\sqrt{2}$. This will have hypotenuse with length $\sqrt{3}$. The figure below demonstrates the continuation of this pattern through the construction of a right triangle with hypotenuse $\sqrt{5}$.

If $\sqrt{2} + \sqrt{3} = \sqrt{5}$ is a true statement, then the sum of the measures of the segments with lengths $\sqrt{2}$ and $\sqrt{3}$ must equal the measure of the segment with length $\sqrt{5}$. As the figure below demonstrates, the sum of the measures of the segments with lengths $\sqrt{2}$ and $\sqrt{3}$ is greater than the measure of the segment with length $\sqrt{5}$. Hence $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$.
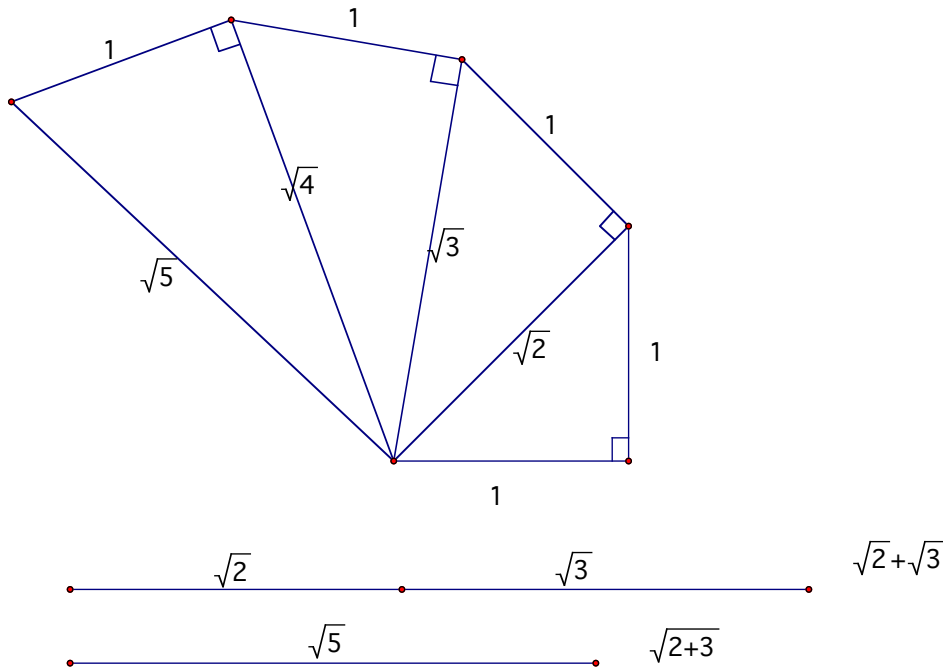
Mathematical Focus 2

A given statement can be shown to be false by supposing that it is true and showing that the supposition leads to a contradiction. This indirect proof uses the property "If $a = b$, then $a^2 = b^2$" and the fact that squares of numbers can be represented geometrically as areas of squares.

If $\sqrt{2} + \sqrt{3} = \sqrt{5}$ is a true statement, then $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$ and $\left(\sqrt{2} + \sqrt{3}\right)^2 = \left(\sqrt{2+3}\right)^2$ would also be true statements. Consider a square with side length $\sqrt{2} + \sqrt{3}$. The area of that square would be $\left(\sqrt{2} + \sqrt{3}\right)^2$. The square can be subdivided into four regions, having areas 2, $\sqrt{6}$, $\sqrt{6}$, and 3, as shown subsequently. Therefore, $\left(\sqrt{2} + \sqrt{3}\right)^2 = 2 + \sqrt{6} + \sqrt{6} + 3 = 5 + 2\sqrt{6}$. Now consider a square with side length $\sqrt{2+3}$. The area of that square would be $\left(\sqrt{2+3}\right)^2$, and $\left(\sqrt{2+3}\right)^2 = \left(\sqrt{5}\right)^2 = 5$. Since $5 + 2\sqrt{6} \neq 5$, $\left(\sqrt{2} + \sqrt{3}\right)^2 \neq \left(\sqrt{2+3}\right)^2$. Therefore, $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$.
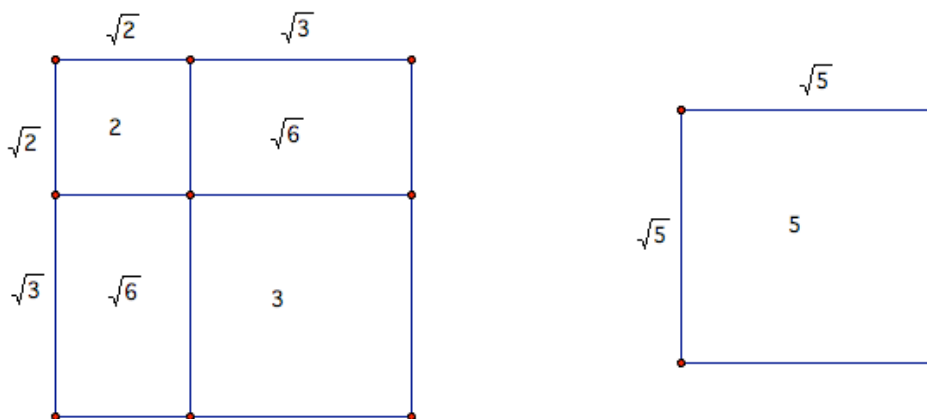
Mathematical Focus 3

A larger question than whether $\sqrt{2} + \sqrt{3} = \sqrt{5}$ is true is whether $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ is true for all $a,b \in \Re^+ \cup \{0\}$. A statement such as $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ can be disproved by identifying values of a and b that provide a counterexample.

If $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ were true for all $a,b \in \Re^+ \cup \{0\}$, then $\sqrt{9} + \sqrt{16} = \sqrt{25}$ would be true. But $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$ and $\sqrt{9+16} = \sqrt{25} = 5$, so $\sqrt{9} + \sqrt{16} \neq \sqrt{25}$ and therefore $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$.

The counterexample establishes that, as a general rule $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$, is not true. However, it does not establish that it is never true. To determine values of a and b that make the statement true, we need only solve the equation: $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$. Squaring both sides yields $a + 2\sqrt{a}\sqrt{b} + b = a + b$ which is true if and only if $2\sqrt{a}\sqrt{b} = 0$, that is, when $\sqrt{a} = 0$ or $\sqrt{b} = 0$. So, $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ is true only when $a = 0$ or $b = 0$.

Mathematical Focus 4

An expression such as $\sqrt{2} + \sqrt{3}$ can be thought of as a sum of function values, $f(2) + f(3)$, values of the square root function, $f(x) = \sqrt{x}$, $x \in \Re^{+} \cup \{0\}$. Some functions satisfy one of the conditions of the linearity property, namely, $f(a+b) = f(a) + f(b)$, and others do not. If a function f does not, in general, satisfy some property, it does not mean that there do not exist <u>some</u> values for which that property might hold.

If $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$, then if $f(x) = \sqrt{x}$, $x \in \Re^{+} \cup \{0\}$, $f(2+3)$ must be equal to $f(2) + f(3)$. In general, when $f(x) = \sqrt{x}$ and $a,b \in \Re^{+} \cup \{0\}$, does $f(a+b) = f(a) + f(b)$?

In general, if a function defined over nonnegative reals has the properties $f(a+b) = f(a) + f(b)$ and $f(ca) = c * f(a)$, then it is said be a linear transformation. Linear functions of the form $f(x) = kx$ that represent a direct variation relationship between variables satisfy the linearity property. Thus $f(a+b) = f(a) + f(b)$ for such functions. This is true because for the linear function $f(x) = kx$, $x \in \Re$, for all $a,b \in \Re$, $f(a+b) = k(a+b) = ka + kb = f(a) + f(b)$.

To determine whether $f(a+b) = f(a) + f(b)$ holds for $f(x) = \sqrt{x}$, $x \in \Re^{+} \cup \{0\}$, one can attempt to find a counterexample. When $a = 4$, $b = 1$, $f(4+1) = \sqrt{4+1} = \sqrt{5}$ and $f(4) + f(1) = \sqrt{4} + \sqrt{1} = 3$. Since $\sqrt{5} \neq 3$, we can conclude that $f(a+b) \neq f(a) + f(b)$ for $f(x) = \sqrt{x}$, and for all $x \in \Re^{+} \cup \{0\}$. However, this only establishes that *f* does not (in general) satisfy the linearity property, it does not establish that for every two values of *a* and *b*, $f(a+b) \neq f(a) + f(b)$. For example, $\sqrt{0} + \sqrt{0} = \sqrt{0+0}$. So this linearity property argument does not necessarily establish that $\sqrt{2} + \sqrt{3} \neq \sqrt{2+3}$, it only establishes that the statement in question cannot be deemed true by using an argument that all such statements are true for this particular function. Hence, one can evaluate $\sqrt{2} + \sqrt{3}$ and $\sqrt{2+3}$, as done previously, to disprove $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$.

Mathematical Focus 5

Expressions in two variables can be thought of as function values for multivariate functions. Function composition is not commutative.

Consider $f(x,y) = \sqrt{x+y}$ with domain $x + y \in \mathfrak{R}^+ \cup \{0\}$ such that $x,y \in \mathfrak{R}^+ \cup \{0\}$. Another way to express $f(x,y) = \sqrt{x+y}$ is as the composition of two other functions, $g(x,y) = x + y$, with domain $x,y \in \mathfrak{R}$ and $h(x) = \sqrt{x}$ with domain $x \in \mathfrak{R}^+ \cup \{0\}$. Since $f(x,y)$ is the composition of the functions represented by $g(x,y) = x + y$ and $h(x,y) = \sqrt{x}$, we can express $f(x,y)$ as $f(x,y) = h(g(x,y)) = h(x+y) = \sqrt{x+y}$.

Now consider the function $s(x,y) = \sqrt{x} + \sqrt{y}$ with domain $x,y \in \mathfrak{R}^+ \cup \{0\}$. We can express $s(x,y)$ as the composition of the functions represented by $g(x,y) = x + y$ and $h(x,y) = \sqrt{x}$, where $s(x,y) = g(h(x,y)) = g(\sqrt{x},\sqrt{y}) = \sqrt{x} + \sqrt{y}$.

The composition $s(x,y) = g \circ h(x,y) = g(h(x,y))$ applies the square root first, then the sum; whereas the composition $f(x,y) = h \circ g(x,y) = h(g(x,y))$ applies the sum first, then the square root. If the composition operation on functions were commutative then the functions $f(x,y) = \sqrt{x+y}$ and $s(x,y) = \sqrt{x} + \sqrt{y}$ would be equal.

However, the graphs of $f(x,y) = \sqrt{x+y}$ and $s(x,y) = \sqrt{x} + \sqrt{y}$ are not identical. Therefore, the functions $f(x,y) = \sqrt{x+y}$ and $s(x,y) = \sqrt{x} + \sqrt{y}$ are not equal, and composition is not commutative.

$f(x,y) = \sqrt{x+y}$ $s(x,y) = \sqrt{x} + \sqrt{y}$
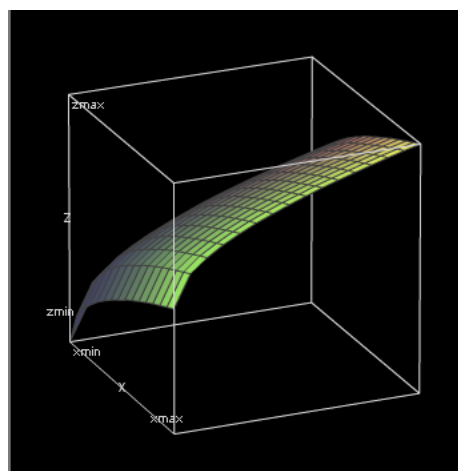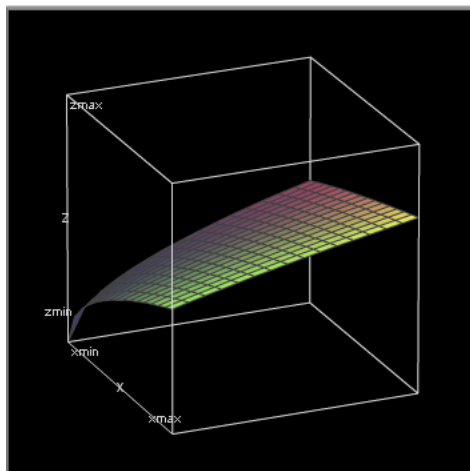
Recall that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ only when x=0 or y=0 (See Mathematical Focus 3 for a more detailed discussion.) In the graphs shown above, the graph of $f(x,y) = \sqrt{x+y}$ is identical to the graph of $s(x,y) = \sqrt{x} + \sqrt{y}$ on the xz plane (when y=0) and on the yz plane (when x=0) and at the origin (when x=0 and y=0).