MAC-CPTM Situations Project

Situation 31: Expanding Binomials

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Prompt

In a high school Algebra I class, students were given the task of expanding $(x + 5)^2$. A student responds, "That's easy! Doesn't $(x+5)^2 = x^2 + 25$?"

Commentary

The five foci for this situation provide different approaches to reasoning about the prompt. The first three provide ways to show that the expression $(x + 5)^2$ is different from $x^2 + 25$: (1) using field properties, (2) interpreting multiplication as a linear operator, and (3) using an area model of multiplication. Foci 4 and 5 suggest ways to demonstrate, using counterexamples and graphs, respectively, that $(x + 5)^2$ and $x^2 + 25$ are not equivalent expressions.

Mathematical Foci

Mathematical Focus 1

The distributive and commutative properties can be used to show that the two expressions are not equivalent for all real numbers x.

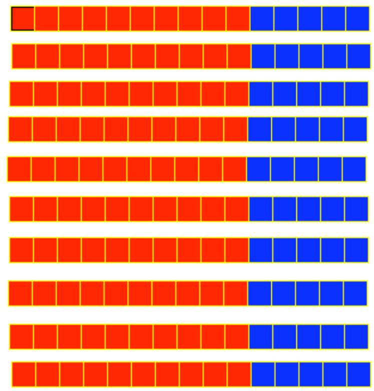
In any field, multiplication distributes over addition, and multiplication is commutative. Let $a = (x + 5)$, then $(x + 5)^2 = (x + 5)(x + 5) = a(x + 5) = ax + a5 = xa + 5a = x(x + 5) + 5(x + 5) = x^2 + 5x +5x + 25 = x^2 + 10x + 25$. Unless $10x = 0$, the quantities $x^2 + 10x + 25$ and $x^2 + 25$ are not equal. Therefore, the two expressions are not equivalent for $x \neq 0$.

Mathematical Focus 2

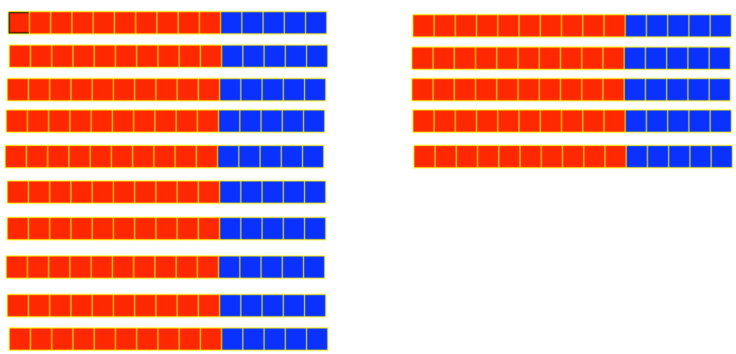
Interpreting x + 5 *as the sum of two units and using a linear operator model of multiplication (with additivity and homogeneity), one can argue that* $(x + 5)^2 = x^2 + 10x + 25$.

It is useful to start with a specific numerical example. Imagine 15 units that are partitioned (broken) into a unit of 10 and a unit of 5, as shown below:

Imagine iterating (mentally repeating) this quantity ten times to produce ten 10s and ten 5s, as shown below:



This iteration is equivalent to scalar multiplication, which requires homogeneity of the linear operator. Now, imagine continuing the iterating activity and iterate the initial 15 units five more times. This mental action would produce five new 10s and five new 5s:



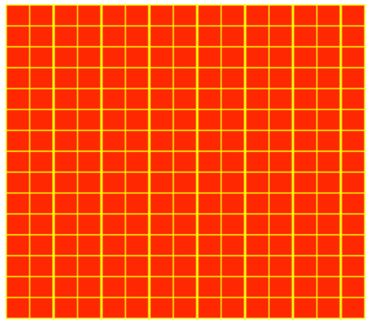
By keeping track of our initial partition while making first 10 copies and then 5 more copies, we arrive at ten 10s, ten 5s, five 10s, and five 5s, rather than only ten 10s and five 5s (the number suggested by the prompt). Here, we have interpreted the symbolism (10 + 5)(10 + 5) in terms of iterations of units (mentally using both the homogeneity and the additivity of the linear operator). The first (10 + 5) is the number of iterations, and the second (10 + 5) is the number of initial 15 units we partitioned.

This model can be extended to any nonzero natural number x, by allowing $(x + 5)$ to be an unknown number of units followed by 5 units and also an unknown number of iterations followed by 5 iterations.

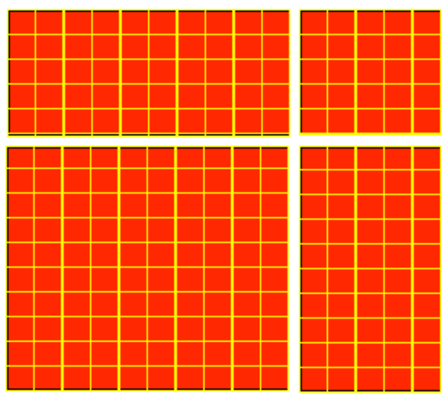
Mathematical Focus 3

The model of multiplication as an area can be used to demonstrate that $(x + 5)^2 = x^2 + 10x + 25$.

Again, it is useful to start with a specific example of an area whose dimensions are 15 units by 15 units, or 15^2 square units.



Imagine partitioning the linear dimensions into a unit of 10 and a unit of 5, and use that partition to create a partition of the area, as shown below:



A conservation-of-area argument shows that a square of dimension (10 + 5) by (10 + 5) yields four rectangular areas: one that is 10 by 10, one that is 5 by 5, and two that are 10 by 5. This model does not reveal why area is relevant to a discussion of multiplication, so one should begin by defining multiplication as the product of two lengths. The model can be extended to unknowns by imagining one part of the original square to be of unknown length.

Mathematical Focus 4

When testing values for each of the expressions $(x + 5)^2$ *and* $x^2 + 10x + 25$, *a single counterexample reveals that they are not equivalent.*

One can try to verify the conjectured equation by choosing specific values for *x* to substitute into the expression on each side. It is necessary to understand what it means to justify a conjecture and that it takes only one counterexample to disprove a conjecture. In other words, two algebraic expressions are equivalent only if they produce the same values over the domains of interest. Under the convention that the set of all real numbers is the domain for a polynomial, substituting a real number into one polynomial expression and evaluating the result should produce the same value as substituting that number into the other expression and evaluating the result. One example (e.g., $x = 0$) is not sufficient to prove equivalence of the expressions, but one counterexample is sufficient to disprove a statement of equivalence (e.g., $x = 1$). Since there exist values of x such as 1 and 2 for which $(x + 5)^2$ and $x^2 + 25$ are not equal, we can conclude that they are not equivalent expressions, as equivalent expressions would be equal for all real values of x.

Mathematical Focus 5

By graphing the two expressions as functions, one can observe that they yield different graphs and are therefore not equivalent.

Imagining the two expressions as functions yields $f(x) = (x+5)^2$ and $g(x) = x^2 + 25$. Generating the two graphs shows that $f(x)$ is a translation of the parabola $h(x) = x^2$ by five units to the left, and $g(x)$ is a translation of $h(x)$ by twenty-five units up. See Situation 30 for details of these translations.