Situation 32: Radicals
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Center for Proficiency in Teaching Mathematics
9/29/05—Amy Hackenberg

Prompt

A mathematics teacher, Mr. Fernandez, is bothered by his ninth grade algebra students' responses to a recent quiz on radicals—simplifying fractions with radicals in the denominator, adding radicals, and multiplying radicals. Some of them got mostly right answers but many students added root 2 and root 3 and got root 5, and others simplified the reciprocal of root 3 to root 3 divided by 9. Mr. Fernandez is disturbed over the incorrect answers but even more disturbing to him is the sense that none of this work is meaningful for his students—even for the ones who know the rules.

What mathematical knowledge does Mr. Fernandez need so that he might change his approach to radicals with his students?

Commentary

Mathematical Foci[1]

Mathematical Focus 1—roots of whole numbers

The teacher can approach work on radicals by using a quantitative approach. That is, he can focus primarily on radicals as lengths, and then work with radicals becomes geometrical problem solving at first, with only some numeric or algebraic calculation. Use of a tool like Geometer's Sketchpad will be helpful in this regard, although some of what's mentioned below can occur without GSP.

First question for students: Starting with a square of area 1 square unit (see Figure 1), can you make a square of area 2 square units?

[1] The following discussion is adapted from Leslie P. Steffe's course, EMAT 7080.

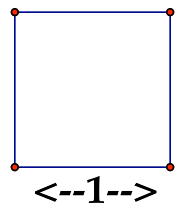


Figure 1

This problem can be solved in multiple ways, see Figure 2 below.

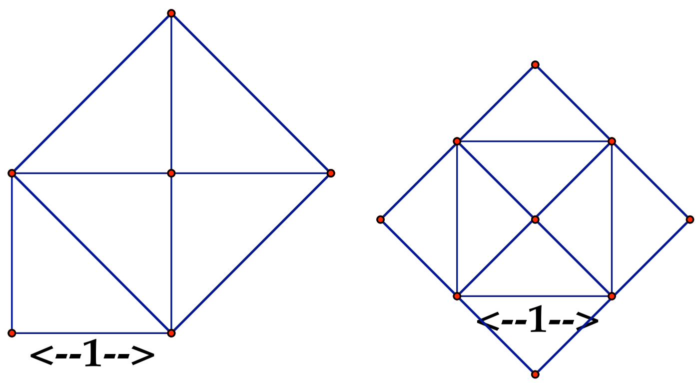
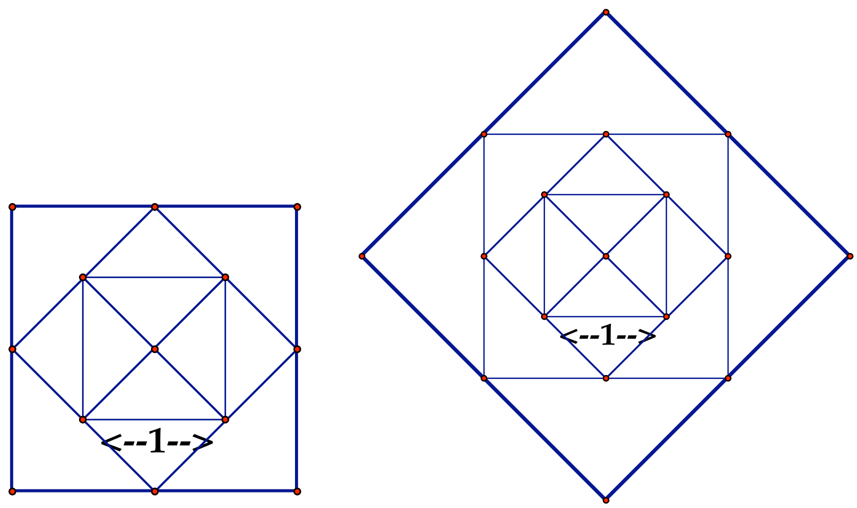


Figure 2

If a square with area 1 square unit has side root 1, which is 1, then a square with area 2 has side root 2. Some investigation here could occur with whether or not root 2 is “like the kind of numbers that we know about,” i.e., whole numbers or fractions. (Most students will agree it cannot be a whole number because they can’t think of a whole number that, when multiplied by itself, is 2. Fractions are another story but can be approached as noted in Focus 3).

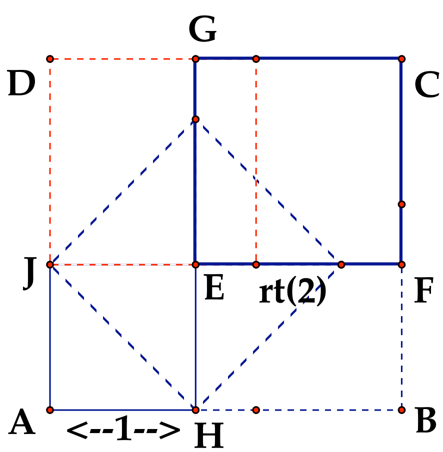
Can student make squares of other areas using some of the techniques they have tried so far in making the square of area 2 (e.g., drawing diagonals and using the isosceles right triangles that are formed, circumscribing squares, etc.)? If the circumscribed technique (right side of Figure 2) is continued, squares with areas 4, 8, 16, etc., can be made, and so lengths that are root 4 (a whole number), root 8 (not a whole number), root 16, etc. Note that all of this work can also be confirmed by the Pythagorean Theorem, examining the largest isosceles triangles formed at each iteration.



largest square has area 4. **largest square has area 8.**

Figure 3

What about a square of area 3? How can this square be made? See Figure 4 below and **page 1** of the GSP sketch SIT C 092905 **radicals.gsp**).



Note: to move the square of area 2 so that it touches the square with area 1 at one point as shown, the blue dashed square was rotated 45 degrees to make the red dashed square and then translated 1 unit to the right. Together, the areas of AHEJ and EFCG have area 3 square units.

Figure 4

This sketch shows one way to "combine" a square of area 1 (AHEJ) and a square of area 2 (EFCG), and ask students to determine how they can use the diagram to produce a square of area 3. Note that using the sketch to make a square of area 3 is an example of the Pythagorean Theorem.

The biggest square, ABCD, consists of an area of 1 square unit, an area of 2 square units, and two rectangles that are each 1 unit by root 2 units. When the areas of the two rectangles are cut apart into 4 triangular areas (with legs 1 and root 2 units) and separated, the remaining area in ABCD should also have area 1 + 2 = 3 (see Figure 5).

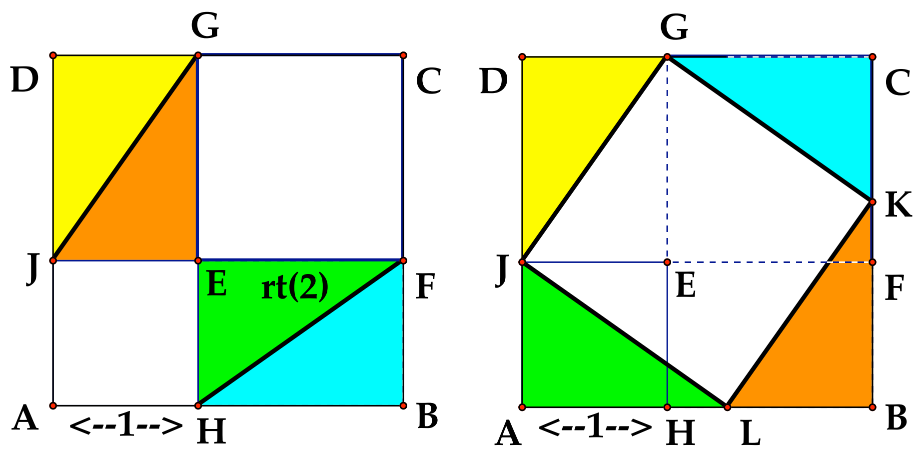


Figure 5

It remains to justify that this remaining area (GKLJ in Figure 5) is a square, which can be done using the 4 congruent right triangles (i.e., GKLJ is at least a rhombus and then it must have at least one right angle because, for example, angles AJL and DJG are complementary). So the sides of square GKLJ can be said to be root 3. Then the teacher can ask students to use what they know to make a square with area 5, then a square with areas 6, 7, and beyond. Note that here the Pythagorean Theorem is used directly as a theorem about areas to make these area relationships and resulting lengths of root 5, root 6, root 7, etc. Students can be challenged to locate all of these lengths on a number line in order to “fill in” the number line and to develop a sense of the relationship of these radicals to numbers like the whole numbers and fractions, which they already know something about (**see pages 2, 3, and 4** of the GSP sketch VIG C 092905 radicals.gsp).

Mathematical Focus 2—roots of fractions

The iterative approach to generating squares with areas that are powers of 2, described in Mathematical Focus 1, can go in the “other direction” to generate squares with areas that are fractions (i.e., starting with negative powers of 2). This exploration can lead to students developing radicals of some fractions (negative powers of 2).

The question of how to construct the square root of 1/3 is not trivial! One possibility is to take a square of area 3 and subdivide it horizontally and vertically into thirds, thereby creating 9 squares with area 1/3. One side of these squares must be the square root of 1/3, and is also root 3 divided by 3 (thereby showing that the two must be equal). A similar approach can be taken to generate roots of other unit fractions.

Then, combinations of methods already described can be used to generate roots of non-unit fractions. For example, how can the square root of 2/3 be generated? One way is to use the circumscribing method on the square with area 1/3 to create a square with area 2/3.

Mathematical Focus 3—is root 2 a fraction?

Although teachers may know a formal proof to show that this is not the case, the reasoning behind the proof—and ways to bring that reasoning out of students' thinking—is a different kind of mathematical knowledge than just knowing a formal proof (I'll argue).

One way to approach this question with students is to assume that root 2 is a fraction, which can be written in lowest terms as a/b (a, b are non-zero whole numbers). If this is the case, then $2 = a^2/b^2$, and $2b^2 = a^2$. So this implies that a^2 is even and so a is even (students will have to justify why!). But that means that there will be two factors of 2 in a^2, so there must be two factors of 2 in $2b^2$. That means that b^2 has to have one factor of 2 (can it?). Since if b^2 has a factor of 2, it must have two factors of 2, and so b must have a factor of 2, then a and b both are even—but we assumed that the fraction a/b was written in lowest terms! So root 2 must not be able to be written as a fraction we know about, a/b in lowest terms where a and b are both non-zero whole numbers.

Mathematical Focus 4—addition and multiplication of radicals

One of the big issues in the situation is why intuitions about addition, based on addition of whole numbers, do *not* hold with fractions or radicals, while intuitions about multiplication (again based on multiplication of whole numbers) *do*. So a major reason for taking a quantitative approach to work with radicals as outlined above is to investigate this issue, as I briefly describe here.

After radicals have been constructed as the lengths of sides of squares, questions can arise about how to combine these lengths. Lengths can certainly be added by joining them contiguously, but, since root 2 and root 3 cannot be written as whole numbers or fractions, we have no way to know whether we can notate their combination with a single graphic item, the way we can combine 2.5 and 3.75 into 6.25. So (at least for the moment), root 2 + root 3 is exactly that, root 2 + root 3. Furthermore, using lengths it is possible to develop intuitions about root 2 and root 3 NOT being equal to root 5 (see **page 5** of VIG C 092905 **radicals.gsp**). However, we can combine multiple lengths of root 2 by determining how many root 2's we have (e.g., 5 root 2).

Multiplication of two radicals like root 2 and root 3 can be thought about using similar triangles (see **page 6** of VIG C 092905 **radicals.gsp**). Work with commutativity and associativity can occur by thinking about finding volumes of rectangular prisms that have radicals as lengths of sides (**see page 7** of VIG C 092905 **radicals.gsp**).