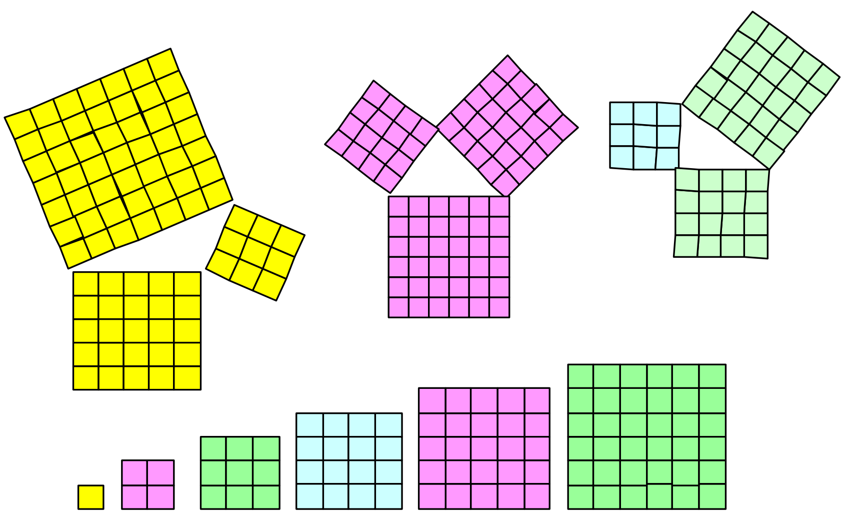
MAC-CPTM Situations Project

Situation 36: Pythagorean Theorem

Prepared at Pennsylvania State University
Mid-Atlantic Center for Mathematics Teaching and Learning
June 28, 2005—Patrick Sullivan
27 September 2009 – Kathy Heid, Maureen Grady, Shiv Karunakaran

Prompt

In both an Algebra I course and an Advanced Algebra course, students were given transparency cutouts of graph paper squares with side lengths from one unit to twenty-five units. Students were asked to create triangles whose sides had the side-lengths of three of the squares. Students began to notice the squares that would create right triangles and the relationship involving the area of those squares. A student asked, "Does this work for every right triangle?"



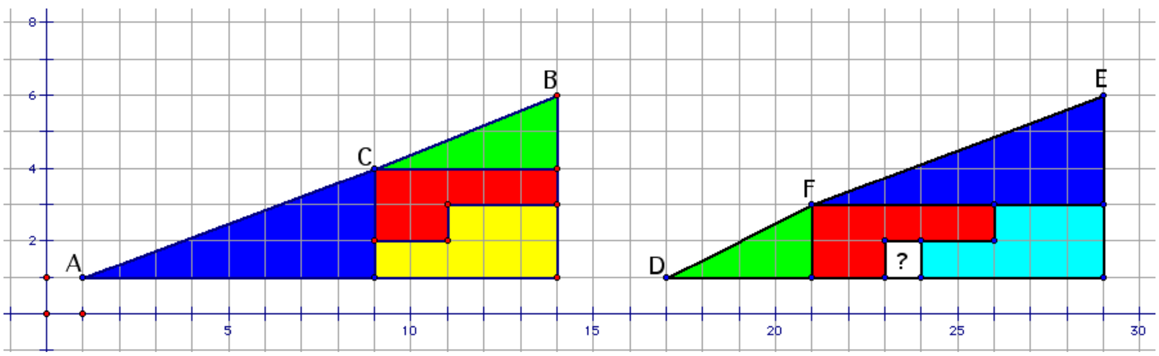
Commentary

The Pythagorean Theorem relates the squares of the lengths of the sides of a right triangle. The Law of Cosines establishes a more general relationship between the same quantities that holds for all triangles. Using algebra and geometry together we can prove the Pythagorean Theorem and the Law of Cosines.

Mathematical Focus 1

Visual inspection alone is insufficient for drawing mathematical conclusions
The generalization drawn by the student is based on what the student observed using physical models. Although such observations are important for mathematical discovery they cannot replace mathematical proof. The diagram that follows is an example of a case in which the physical representation is illusory.

The diagram shown below makes it appear that two right triangles with the same base and height have different areas. However, close examination reveals that neither of the figures is actually a right triangle. Calculating the slopes of segments AC, CB DF, and FE reveals that points A, C, and B are not collinear and that points D, F, and E are not collinear. The figures shown are quadrilaterals; one convex and once concave.



The existence of many such illusions provides a compelling rationale for the importance of mathematical proof.

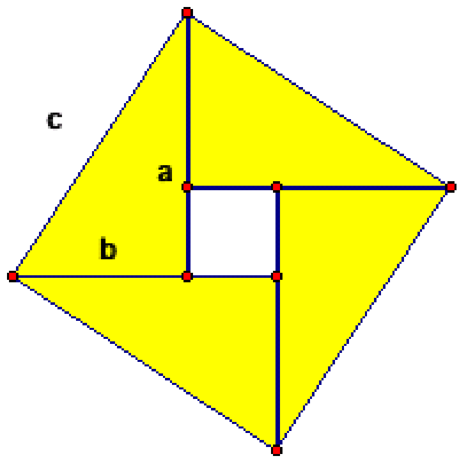
Mathematical Focus 2

Algebraic meanings of area formulas for geometric configurations involving right triangles can be used to prove the Pythagorean Theorem.

The question posed by the student seems to warrant the need for a justification that encompasses all cases. In other words, how would we go about proving the Pythagorean Theorem? This focus presents only two of the many proofs of the Pythagorean Theorem. There are various resources available that have compiled a lot of these proofs. One such resource is *The Pythagorean Proposition* by Elisha Scott Loomis. Also, Project Mathematics! has developed a video devoted to the Pythagorean Theorem. The video is available online at http://www.projectmathematics.com/pythag.htm.

The methods of proof we have chosen focus on utilizing both algebra and geometry.

Begin with a right triangle that has side lengths a and b and hypotenuse of length c. Make three copies of the same triangle rotated 90^0, 180^0, 270^0. Place together so that they form a square with side c as shown. [The vertices of the quadrilateral formed have measure 90



Inner square has side length a-b

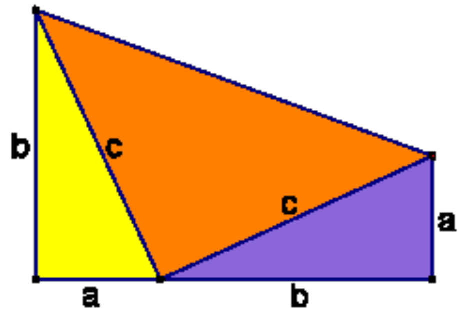
because the acute angles of a right triangle are complementary and each angle of the quadrilateral is formed by non-corresponding acute angles of congruent right triangles.]
Each of these right triangles has an area of $\frac{ab}{2}$.

The interior square has side length (*a* - *b*). Thus, the area of the inner square is $(a - b)^2$ and the area of the four triangles is 4 * $\frac{ab}{2}$. The area of the outer square is c^2.
Knowing that the area of the outer square is congruent to the area of the inner square plus the area of the four right triangles, one can conclude:

$$c^2 = (a-b)^2 + 2ab$$
$$c^2 = a^2 - 2ab + b^2 + 2ab$$
$$c^2 = a^2 + b^2$$

Therefore, for any right triangle, the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs of a right triangle.

An alternative proof, thought to have been developed by U. S. president James Garfield in 1876, involves both geometry and algebra. Consider the following arrangement of two copies the same right triangle.



Note that the isosceles triangle formed with legs of length c is a right triangle. This is true because the acute angles of a right triangle are complementary and the adjacent angles are non-corresponding acute angles of congruent right triangles. Now consider the area of the trapezoid formed by the three triangles. This area can be calculated in two different ways.

1. Using the formula for the area of a trapezoid,

$$\text{Area of trapezoid} = \left(\frac{1}{2}\right)\cdot(\text{height of trapezoid})\cdot(\text{sum of the lengths of the bases})$$
$$= \left(\frac{1}{2}\right)\cdot(a+b)\cdot(a+b) = \left(\frac{1}{2}\right)\cdot(a^2+2ab+b^2)$$

2. Using the area of the three triangles that constitute the trapezoid

$$\text{Area of trapezoid} = (2)\cdot(\text{area of right triangle with sides } a, b, \text{and } c) + (\text{area of isosceles right triangle})$$
$$= (2)\cdot\left(\frac{1}{2}\right)\cdot(a)\cdot(b) + \left(\frac{1}{2}\right)\cdot(c)\cdot(c) = \left(\frac{1}{2}\right)\cdot(2ab+c^2)$$

Equating the two expressions for the area of the trapezoid, we get

$$\left(\frac{1}{2}\right)\cdot(a^2+2ab+b^2) = \left(\frac{1}{2}\right)\cdot(2ab+c^2) \Rightarrow (a^2+2ab+b^2) = (2ab+c^2)$$
$$\Rightarrow a^2+b^2=c^2$$

Mathematical Focus 3

The Law of Cosines can be used to prove that if the lengths of the sides of a triangle are a, b, and c and $a^2+b^2=c^2$ then the triangle is a right triangle.

The observation that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse leads to the question of whether the relationship can apply to other types of triangles. So the question is one of whether every triangle with sides a, b, and c (where c is the longest side) for which $a^2 + b^2 = c^2$ is a right triangle.

Using the Law of Cosines (proven in Mathematical Foci 4 and 5), for any triangle with sides a, b, and c: $c^2 = a^2 + b^2 - 2ab \cos C$. If $a^2 + b^2 = c^2$, then $c^2 = c^2 - 2ab \cos C$, Therefore$- 2ab \cos C = 0$. If $2ab \cos C = 0$, then C is a 90° angle because a and b are non-zero and when $\cos C = 0$, C=90°. So triangle ABC is a right triangle.

Mathematical Focus 4

The Law of Cosines can be used to describe a relationship between a, b, and c for any triangle with sides of length a, b, and c.

If, for triangles with sides *a*, *b*, and *c* (where *c* is the length of the longest side), $a^2 + b^2 = c^2$ holds for right triangles and only right triangles, the question arises as to whether a similar relationship holds for other types of triangles. The Law of Cosines (proven later in this Focus), $c^2 = a^2 + b^2 - 2ab \cos C$ provides such a relationship.

If angle C is obtuse, then cos C will have a negative value and $a^2 + b^2 - 2ab \cos C$ will be greater than $a^2 + b^2$. So $c^2 > a^2 + b^2$.

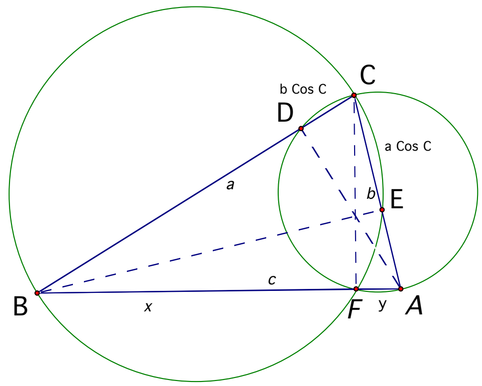
If angle C is acute, then cos C will have a positive value and $a^2 + b^2 - 2ab \cos C$ will be less than $a^2 + b^2$. So $c^2 < a^2 + b^2$.

The Law of Cosines can be proved using the Pythagorean Theorem. The proof involves three cases, based on the shape of different triangles.

Case 1. Triangle ABC is acute.

Side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. Construct the circles with diameters BC and AC, calling them circles a and b, respectively.

Let D be the intersection of circle b and segment BC, let E be the intersection of circle a and segment AC, and let F be the intersection of circle *a* and circle *b*. We know that F lies on segment AB because ∡CFB and ∡CFA are right angles (they are inscribed in semicircles). Let x = BF and y = FA.



ΔCFA and ΔCFB are right triangles (with right angles at vertex F),
so $y^2+(CF)^2=b^2$ and $x^2+(CF)^2=a^2$.
Therefore, one can conclude that $b^2- y^2=a^2- x^2$
Substituting y=c-x and simplifying yields: $b^2- (c-x)^2=a^2- x^2$

$\Rightarrow b^2- c^2+2cx-x^2=a^2- x^2$

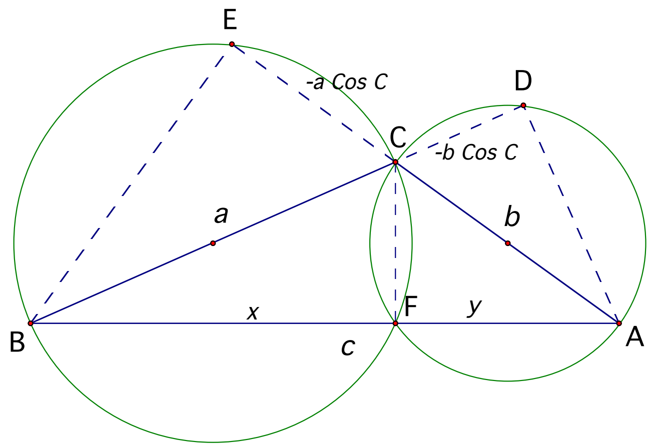
$\Rightarrow b^2= a^2+ c^2-2cx$

Substituting x=aCos*B* (because ΔCFB is a right triangle) yields: $b^2= a^2+ c^2-2ac\cos B$

Case 2. Triangle ABC is obtuse and angle C is obtuse.

Side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. Again, construct the circles with diameters BC and AC, calling them circles a and b, respectively.

Let D be the intersection of circle b and line BC, let E be the intersection of circle a and line AC, and let F be the intersection of circle *a* and circle *b*. We know that F lies on segment AB because ∡CFB and ∡CFA are right angles (they are inscribed in semicircles). Let x = BF and y = FA.



ΔCFA and ΔCFB are right triangles (with right angles at vertex F), so $y^2+(CF)^2=b^2$ and $x^2+(CF)^2=a^2$.

Therefore, one can conclude that $b^2- y^2=a^2- x^2$

Substituting y=c-x and simplifying yields: $b^2- (c-x)^2=a^2- x^2$

$\Rightarrow b^2- c^2+2cx-x^2=a^2- x^2$

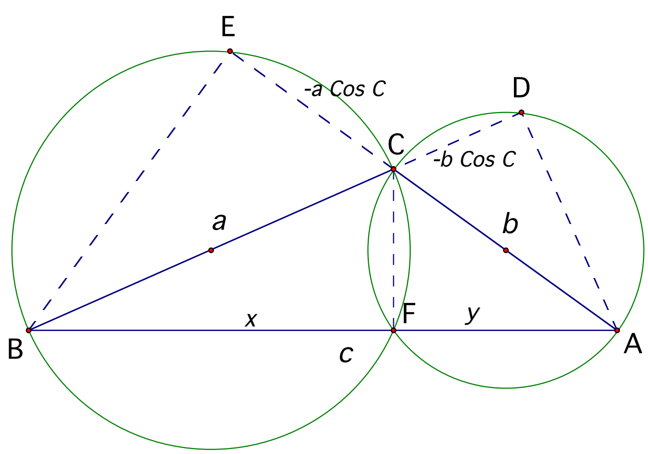
$\Rightarrow b^2= a^2+ c^2-2cx$

Substituting x=aCos*B* (because ΔCFB is right) yields: $b^2= a^2+ c^2-2ac\text{Cos}B$

Case 3. Triangle ABC is obtuse and angle C is acute.

Side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. Construct the circles with diameters BC and AC, calling them circles a and b, respectively.

Let D be the intersection of circle b and line BC, let E be the intersection of circle a and segment AC, and let F be the intersection of circle *a* and circle *b*. We know that F lies on line AB because ∡CFB and ∡CFA are right angles (they are inscribed in semicircles) and because through a point on a line there is exactly one perpendicular. Let z = BF.



ΔCFA and ΔCFB are right triangles (with right angles at vertex F), so $y^2+(CF)^2=b^2$ and $x^2+(CF)^2=a^2$.

Therefore, one can conclude that: $b^2- y^2=a^2- x^2$

Substituting y=c-x and simplifying yields: $b^2- (c-x)^2=a^2- x^2$

$\Rightarrow b^2- c^2+2cx-x^2=a^2- x^2$

$\Rightarrow b^2= a^2+ c^2-2cx$

Substituting x=aCos*B* (because ΔCFB is right) yields: $b^2= a^2+ c^2-2ac\text{Cos}B$

Mathematical Focus 5

The Law of Cosines can be proven without reference to the Pythagorean Theorem using the Power of Point Theorem.

Proof of Law of Cosines involves three cases, based on the shape of different triangles.

Case 1. Triangle ABC is acute.

Side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. Construct the circles with diameters BC and AC, calling them circles a and b, respectively.

Let D be the intersection of circle b and segment BC, let E be the intersection of circle a and segment AC, and let F be the intersection of circle *a* and circle *b*. We know that F lies on segment AB because ∡CFB and ∡CFA are right angles (they are inscribed in semicircles). Let x = BF and y = FA.

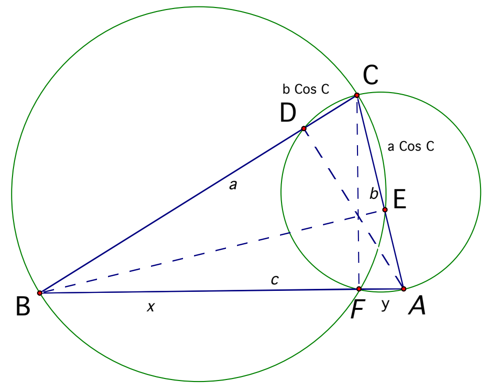
Then ΔCDA and ΔCEB are right triangles, so CE = a Cos C, and DC = b Cos C. Applying the two-secants case of the Power of the Point Theorem[1] to point B with respect to the circle *b*, we get $a(a-b\cos(C)) = xc$. Similarly, applying the two-secants case of the Power of the Point Theorem to point A with respect to the circle *a*, we have $b(b-a\cos(C)) = yc$. Adding the expressions on corresponding sides of the two equations yields

$$a(a-b\cos(C)) + b(b-a\cos(C)) = xc + yc$$

$$a^2 - ab\cos(C) + b^2 - ab\cos(C) = c(x+y)$$

$$a^2 + b^2 - 2ab\cos(C) = c^2$$

[1] If two secants are drawn to a circle from an outside point, the product of one secant and its external segment is equal to the product of the other secant and its external segment.



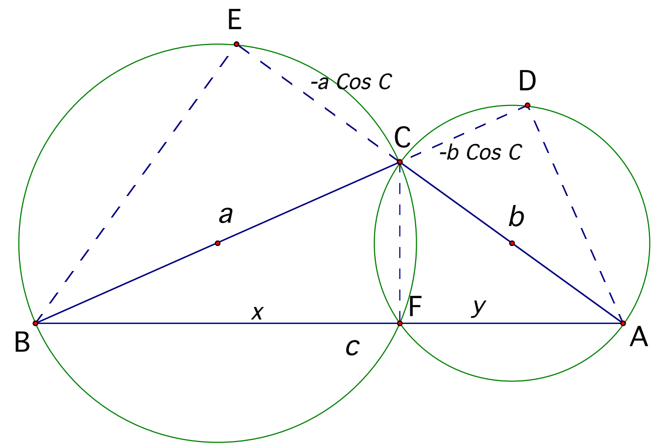
Case 2. Triangle ABC is obtuse and angle C is obtuse.

Side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. Again, construct the circles with diameters BC and AC, calling them circles a and b, respectively.

Let D be the intersection of circle b and line BC, let E be the intersection of circle a and line AC, and let F be the intersection of circle *a* and circle *b*. We know that F lies on segment AB because ∡CFB and ∡CFA are right angles (they are inscribed in semicircles). Let x = BF and y = FA.

Then ΔCDA and ΔCEB are right triangles, The lengths of CD and CE are *a* cos(180 - C) and *b* cos(180 - C), respectively. So CD = -*a* cos(C)and CE = -*b* cos(C). Following through with the same argument as above we get…

$$a(a - b\cos(C)) + b(b - a\cos(C)) = xc + yc$$
$$a^2 - ab\cos(C)) + b^2 - ab\cos(C) = c(x + y)$$
$$a^2 + b^2 - 2ab\cos(C) = c^2$$



Case 3. Triangle ABC is obtuse and angle C is acute.

Side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. Construct the circles with diameters BC and AC, calling them circles a and b, respectively.

Let D be the intersection of circle b and line BC, let E be the intersection of circle a and segment AC, and let F be the intersection of circle *a* and circle *b*. We know that F lies on line AB because ∡CFB and ∡CFA are right angles (they are inscribed in semicircles) and because through a point on a line there is exactly one perpendicular. Let z = BF.

Since angle C is acute and angle B is obtuse, the altitude from B cuts side AC- call the point of intersection D. The altitudes for A and C, however, lie outside the triangle, and meet the lines in which sides CB and AB lie at the points F and E, respectively. Let the length of BE be represented by z. Since the length of DC is a*cos(C) and the length of EC is b*cos(C) we have the power of A with respect to circle BC,

$b(b - a\cos(C)) = (c + z)c$.

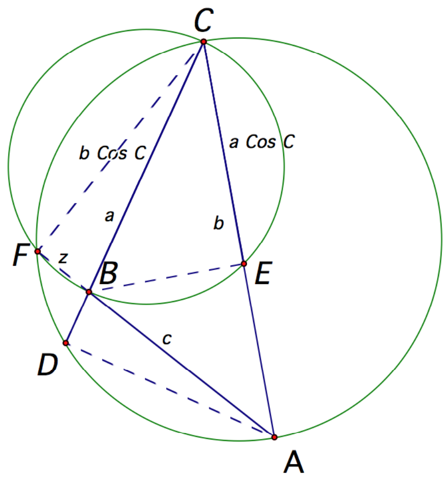
Similarly, using the two-chords case of the Power of a Point, we have for the power of B with respect to circle AC $a(b\cos(C) - a) = zc$.

Subtracting the expressions on corresponding sides of the two equations yields:

$$b(b - a\cos(C)) - a(b\cos(C) - a) = (c + z)c - zc$$

$$b^2 - ab\cos(C)) - ab\cos(C) + a^2 = c^2 + zc - zc$$

$$a^2 + b^2 - 2ab\cos(C) = c^2$$



Post-Commentary:

The foci in this situation serve to point out the connections between three important theorems in plane geometry: the Pythagorean Theorem, the Law of Cosines, and the Power of Point Theorem. The Pythagorean Theorem can be proved using the Law of Cosines, the Law of Cosines can be proved using the Pythagorean Theorem, and the Power of Point Theorem can used to prove the Law of Cosines and, thus, the Pythagorean Theorem as well.

References

Loomis, E. S. (1968). *The Pythagorean proposition*. (2nd ed). Washington, D.C : The National Council of Teachers of Mathematics

Proofs of Pythagorean Theorem and Law of Cosines, retrieved from www.cut-the-knot.org/pythagoras/index.html on June 21, 2005.

Proof of Power of a point theorem, retrieved from http://www.cut-the-knot.org/pythagoras/PPower.shtml on July 18, 2005.

END OF SITUATION – END OF SITUATION – END OF SITUATION