Situation 37: Multiplying Monomials & Binomials

(includes material from Situation 31)

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Prompt

The following scenario took place in a high school Algebra 1 class. Most of the students were sophomores or juniors repeating the course. During the spring semester, the teacher had them do the following two problems for a warm-up:

1) Are the two expressions, $(x^3y^5)^2$ and x^6y^{10}, equivalent? Why or why not?

2) Are the two expressions, $(a+b)^2$ and a^2+b^2, equivalent? Why or why not?

Roughly a third of the class stated that both pairs of expressions were equivalent because of the Distributive Property.

Commentary

This situation highlights the difference between multiplying monomials and multiplying binomials. The students' incorrect responses to the warm-up problem demonstrate a probable misunderstanding of this difference. The students appear to be misusing the Distributive Property, applying a procedure, "take the number on the outside of the parentheses and multiply it by what is inside of the parentheses," where the procedure is not applicable. Ironically, the students' error in warm-up #2 may have occurred because they did not use the Distributive Property. Therefore this Situation necessitates a proper understanding of the Distributive Property (including when it does and does not apply) as well as the rules for exponents. For a detailed explanation on the rules on multiplying monomials, please refer to Situation 21. For multiplying binomials, there are 3 foci to illustrate the proper use of exponents in binomials. Focus 1 contains an explanation of a proper use of the Distributive Property, including a geometric illustration of the square of a binomial. Focus 2 presents a graphical approach to showing why we believe that $\left(x^3y^5\right)^2$ and x^6y^{10} are equivalent, and why we know $(a+b)^2$ and a^2+b^2 are not equivalent. Finally, Focus 3 offers a numerical approach and provides a counterexample to prove that $(a+b)^2$ and a^2+b^2 are not equivalent.

Mathematical Foci

Mathematical Focus 1: Distributive Property

The Distributive Property of Multiplication over Addition (hereafter referred to as the Distributive Property) states that $a(b + c) = ab + ac$ and $(b + c)a = ba + bc$. This applies to multiplication being distributed over addition. Exponentiation (raising a quantity to an exponent/power) involves only one operation. However, the Distributive Property involves two operations: multiplication and addition. Therefore the Distributive Property does not apply to exponentiation. In this particular situation, then, the Distributive Property cannot be used to say that $(a+b)^2$ and a^2+b^2 are equivalent.

The Distributive Property is relevant in this situation, however, for problem 2:

$(a+b)^2$ is a product of two binomials, and the Distributive Property must be used to multiply correctly:

$$(a+b)(a+b) = (a+b)(a) + (a+b)(b) = a^2 + ba + ab + b^2 = a^2 + 2ab + b^2$$

Unless *a* and/or *b* is equal to zero, $a^2 + 2ab + b^2$ is not equivalent to $a^2 + b^2$.

We can illustrate this property geometrically through the use of Geometer's Sketchpad. We will be referring to the Sketchpad document in this link. In particular, we want to illustrate $(a + b)^2$ using the area of a square with side length $a + b$.

What will be obvious is that when we expand $(a+b)^2$, there does exist a^2 and b^2, but there is additional area (specifically, two rectangles with area *ab*), showing that $(a + b)^2 = a^2 + \underline{2ab} + b^2$, and not simply $a^2 + b^2$.

Mathematical Focus 2: Graphical Approach

For this focus we will express $(x^3y^5)^2$ and x^6y^{10} as functions of *z*. That is, we will let *z* be a function of two independent variables, *x* and *y*. When graphing functions of two variables, a three-dimensional graph represents the functions visually. The graphs of $z = (x^3y^5)^2$ and $z = x^6y^{10}$, shown in Figures 1 and 2, appear to be the same graphs, indicating that $(x^3y^5)^2$ and x^6y^{10} are equivalent expressions.

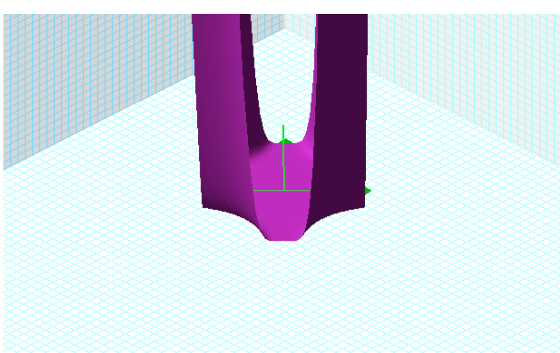

Figure 1: $z = (x^3y^5)^2$

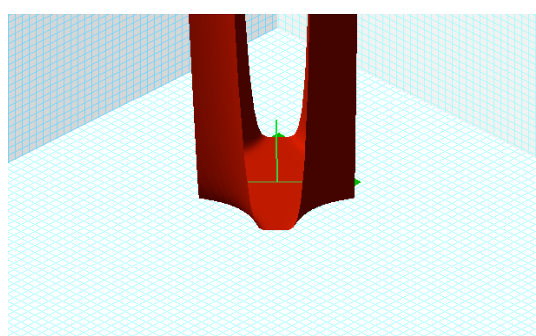

Figure 2: $z = x^6y^{10}$

When graphed on the same system of three-dimensional axes, it is difficult to distinguish between the two functions.

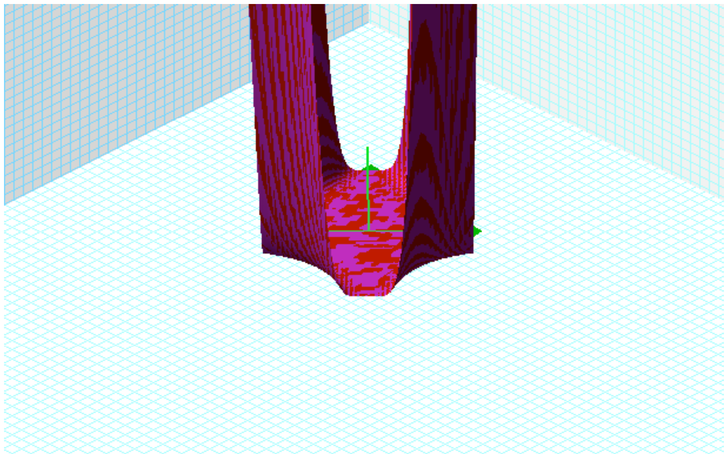

Figure 3: $z = \left(x^3 y^5\right)^2$ and $z = x^6 y^{10}$ graphed on the same set of axes

However, when the graphs of $z = (x + y)^2$ (in rainbow) and $z = x^2 + y^2$ (in red) are graphed on the same axes, it is clear that the graphs are not the same; therefore the equations are not equivalent.

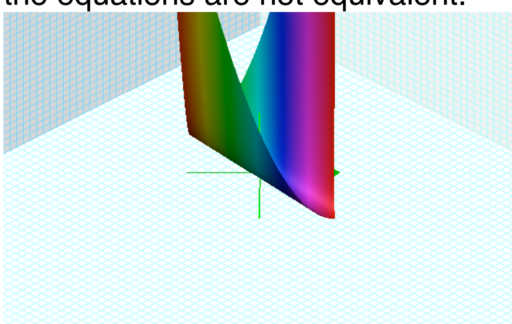

Figure 4: $z = (x + y)^2$

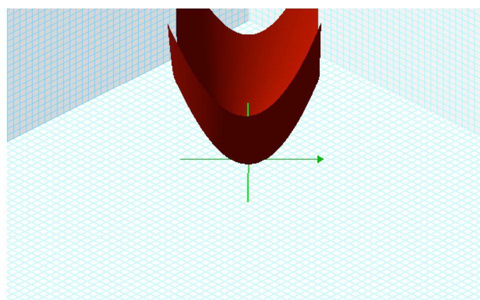

Figure 5: $z = x^2 + y^2$

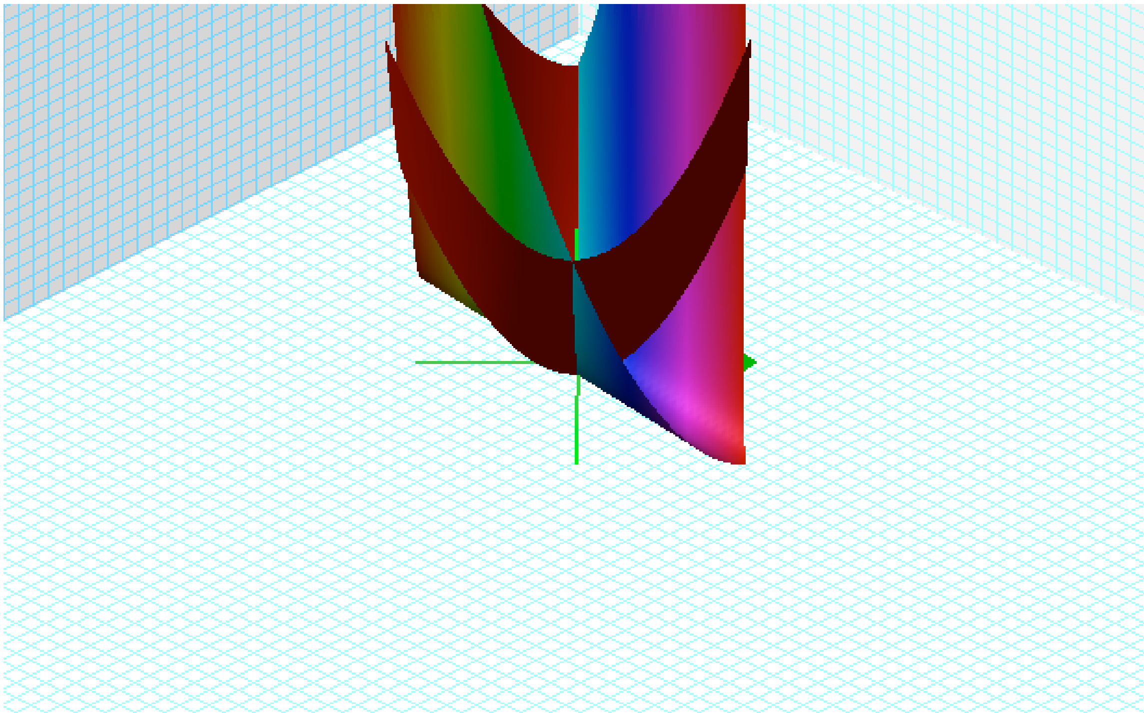

Figure 6: $z = (x+y)^2$ and $z = x^2 + y^2$

Mathematical Focus 3: Numerical Approach

A simple test of a few values is a way to give evidence that $(x^3y^5)^2$ and x^6y^{10} are equivalent, while $(a+b)^2$ and a^2+b^2 are not. For problem 1, we give an example of $(x^3y^5)^2 = x^6y^{10}$ for certain values (*x* = 2 and *y* = 3). This is merely evidence, and not a proof that $(x^3y^5)^2$ and x^6y^{10} are equivalent because we cannot test all possible values for *x* and *y*. For Problem 2, however, we have provided a counterexample to show that $(a+b)^2$ cannot be equivalent to a^2+b^2.

Problem 1:

Let *x* = 2 and *y* = 3.

$(x^3y^5)^2$	x^6y^{10}
$= (2^3 \cdot 3^5)^2$	$= 2^6 \cdot 3^{10}$
$= (8 \cdot 243)^2$	$= 64 \cdot 59049$
$= (1944)^2$	$= 3779136$
$= 3779136$	

Problem 2:

Let $a = 2$ and $b = 3$

$(a+b)^2$	$a^2 + b^2$
$= (2 + 3)^2$	$= 2^2 + 3^2$
$= (5)^2$	$= 4 + 9$
$= 25$	$= 13$

One example (as we have provided for Problem 1) is not sufficient to prove equivalence of the expressions, but one counterexample is sufficient to <u>disprove</u> a statement of equivalence. Since (as we have provided for Problem 2) there exist values of *a* and *b* for which $(a+b)^2$ and $a^2 + b^2$ are <u>not</u> equal, we can safely say that they are not equivalent expressions, as equivalent expressions would be equal for all real values of *a* and *b*.