MAC-CPTM Situations Project

Situation 37: Multiplying Monomials & Binomials

(includes material from Situation 31)

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Prompt

The following scenario took place in a high school Algebra 1 class. Most of the students were sophomores or juniors repeating the course. During the spring semester, the teacher had them do the following two problems for a warm-up:

1) Are the two expressions, $\left(x^3y^5\right)^2$ and x^6y^{10}, equivalent? Why or why not?

2) Are the two expressions, $(a+b)^2$ and a^2+b^2, equivalent? Why or why not?

Roughly a third of the class stated that both pairs of expressions were equivalent because of the Distributive Property.

Commentary

This situation highlights differences between multiplying monomials and multiplying binomials. The students' incorrect responses to the warm-up problem demonstrate a probable misunderstanding of important differences. The students appear to be misusing the Distributive Property by applying a procedure, "take the number on the outside of the parentheses and multiply it by what is inside of the parentheses," when the procedure does not apply. Ironically, the students' difficulty with warm-up #2 may have occurred because they did not use the Distributive Property. The following foci demonstrate several approaches for exploring the mathematics involved in the two warm-up problems, including application of the properties of real numbers, geometric representations, graphical representations and numerical exploration.

Mathematical Foci

Mathematical Focus 1:

Equivalence of expressions can be explained by application of the properties of real numbers

Applied to real numbers, the Distributive Property of Multiplication over Addition (hereafter referred to as the Distributive Property) states that $a(b + c) = ab + ac$ and $(b + c)a = ba + bc$ for all real numbers a, b, and c. This applies to multiplication being distributed over addition, and the property does not generalize to all configurations of the form $a*(b@c)$ or $(b@c)*a$ where * and @ are operations that apply to a, b, and c. The Distributive Property does not apply to exponentiation over addition. In this particular situation, then, the Distributive Property cannot be used to say that
$(a+b)^2$ and a^2+b^2 are equivalent.

The Distributive Property is relevant in this situation, however, for problem 2:

$(a+b)^2$ is a product of two binomials, and the Distributive Property can be used to determine the product of binomials:

$$(a+b)(a+b) = (a+b)(a)+(a+b)(b) = a^2+ba+ab+b^2 = a^2+2ab+b^2$$

Unless a and/or b is equal to zero, $a^2+2ab+b^2$ is not equivalent to a^2+b^2.

The Commutative Property of Multiplication states that ab=ba and the

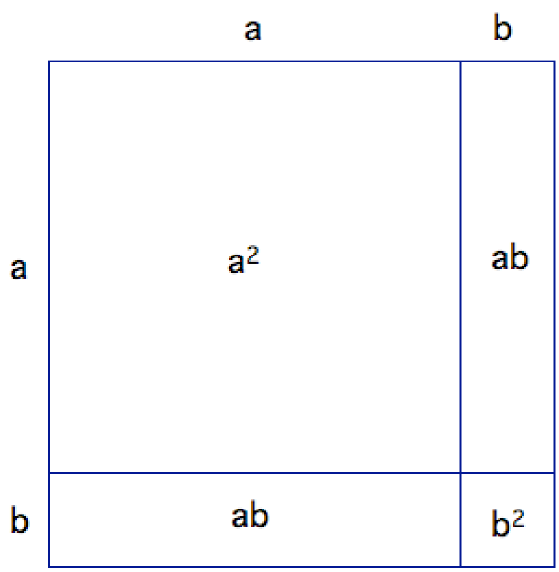
Associative Property of Multiplication states that (ab)c=a(bc). Both of these properties are used in simplifying the expression $\left(x^3y^5\right)^2$.

$$
\begin{aligned}
(x^3y^5)(x^3y^5) &= x^3(y^5(x^3y^5)) && \text{Associative Property of Multiplication}\\
&= x^3((y^5x^3)y^5) && \text{Associative Property of Multiplication}\\
&= x^3((x^3y^5)y^5) && \text{Commutative Property of Multiplication}\\
&= x^3(x^3(y^5y^5)) && \text{Associative Property of Multiplication}\\
&= (x^3x^3)(y^5y^5) && \text{Associative Property of Multiplication}\\
&= x^6y^{10} && \text{Property of Exponents } (x^ax^b = x^{a+b})
\end{aligned}
$$

Mathematical Focus 2:

The Distributive Property can be illustrated geometrically through the use of an area model and a right triangle inscribed in a circle.

One way to illustrate $(a + b)^2$ is to examine the area of a square with side length $a + b$. The square with side length $a + b$ can be partitioned into four rectangles (two squares having area a^2 and b^2, respectively, and two rectangles, both having area ab).



Therefore, by appealing to the area model, $(a + b)^2 = a^2 + \underline{2ab} + b^2$, and not simply $a^2 + b^2$.

We can also prove that $(a+b)^2 = a^2 + 2ab + b^2$ *using a right triangle inscribed in a circle.*

Let $a > 0$ and $b > 0$, and let $a > b$. Construct the segment $\overline{AC}$, so that AC = $a + b$. Locate the point D on the segment $\overline{AC}$, so that AD = a and DC = b (see Fig. 1). Construct a circle (O,R) centered at O with a diameter $\overline{AC}$. Therefore the radius of the circle R has length $(a + b)/2$. Construct the segment $\overline{DB}$, so that $\overline{DB} \perp \overline{AC}$, and the point B is on the circle (O, R).

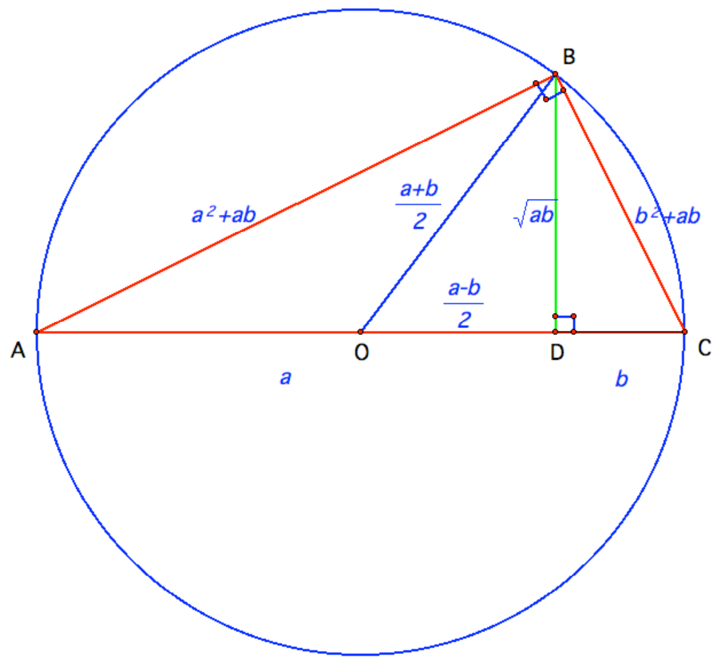


Figure 1.

I. First, we will prove that the length of $\overline{BD}$ is the geometric mean of the length of $\overline{AD}$ and $\overline{DC}$.

Consider triangles $\triangle ABC$, $\triangle ABD$ and $\triangle BDC$ Since, triangle $\triangle ABC$ is inscribed in the circle (O,R) and side $\overline{AC}$ is a diameter of the circle, the angle $\angle ABC$ is the right angle Since $\overline{DB} \perp \overline{AC}$, the angles $\angle ADB$ and $\angle BDC$ are right angles. Since two angles of $\triangle ABC$ and $\triangle ABD$ are congruent, the triangles are similar triangles and the length of their corresponding sides are proportional.

So, $\dfrac{DB}{AD} = \dfrac{BC}{AB}$. Since two angles of $\triangle ABC$ and $\triangle BDC$ are congruent, the triangles are similar triangles and the lengths of their corresponding sides are proportional. So, $\dfrac{BC}{AB} = \dfrac{DC}{DB}$, and, by the transitive property of equality: $\dfrac{DB}{AD} = \dfrac{DC}{DB}$. Since, AD = a and DC = b, then DB = $\sqrt{ab}$.

II. Second, we will prove that $(a+b)^2 = a^2 + 2ab + b^2$

Consider $\triangle ADB$. The angle $\angle ADB$ is the right angle, therefore, by the Pythagorean theorem, $AB^2 = AD^2 + DB^2$
Consider $\triangle BDC$. The angle $\angle BDC$ is the right angle, therefore, by the Pythagorean theorem, $BC^2 = DB^2 + DC^2$
Consider $\triangle ABC$. The angle $\angle ABC$ is the right angle, therefore, by the Pythagorean theorem, $AB^2 + BC^2 = AC^2$
Then

$$\begin{aligned} AC^2 &= AB^2 + BC^2 = (AD^2 + DB^2) + (DB^2 + DC^2) \\ &= a^2 + 2\left(\sqrt{ab}\right)^2 + b^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

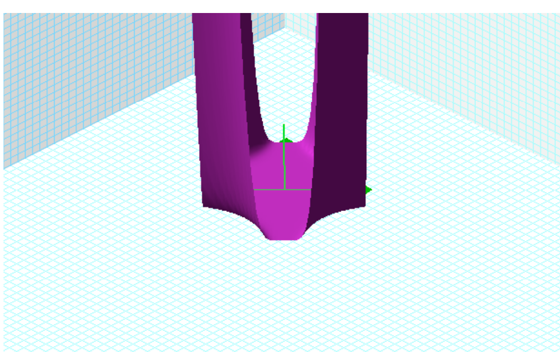
III. This proof can expanded to show that $(a-b)^2 = a^2 - 2ab + b^2$.
Consider triangle $\triangle OBD$. The angle $\angle ODB$ is a right angle, therefore, by the Pythagorean theorem, $OD^2 = OB^2 - BD^2$. Since, OD=$\frac{a-b}{2}$, OB=$\frac{a+b}{2}$, and BD=$\sqrt{ab}$, then

$\frac{(a-b)^2}{4} = \frac{(a+b)^2}{4} - (ab)$ so $(a-b)^2 = (a+b)^2 - 4ab$. Since $(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 + 2ab - b^2 - 4ab$, and therefore $(a-b)^2 = a^2 - 2ab + b^2$.

Mathematical Focus 3

The equivalence of expressions involving two variables can be explored using three-dimensional graphs.

For this focus we will express $(x^3y^5)^2$ and x^6y^{10} as functions of z. That is, we will let z be a function of two independent variables, x and y. The graph of a function of two variables is a three-dimensional graph. The graphs of $z = \left(x^3y^5\right)^2$ and $z = x^6y^{10}$, shown in Figures 1 and 2, appear to be the same graphs, providing some evidence that $(x^3y^5)^2$ and x^6y^{10} may be equivalent expressions.



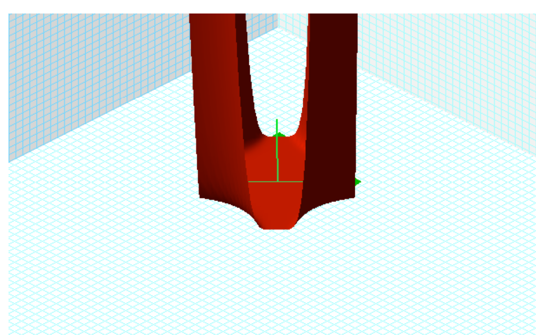


Figure 1: $z = \left(x^3y^5\right)^2$ Figure 2: $z = x^6y^{10}$

When graphed on the same system of three-dimensional axes, it is difficult to distinguish between the two functions.

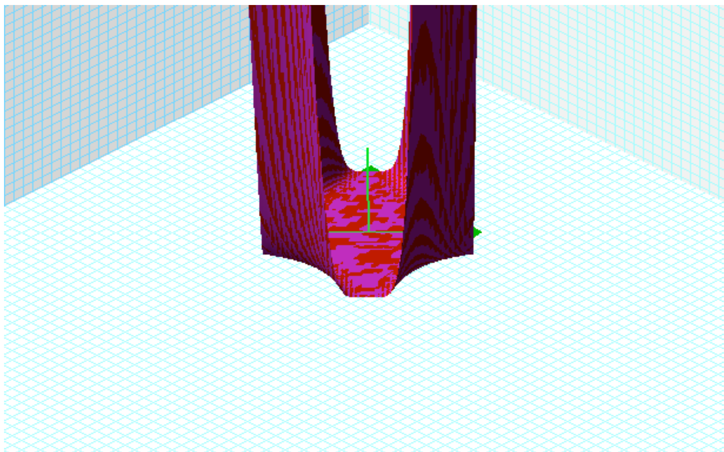


Figure 3: $z = \left(x^3y^5\right)^2$ and $z = x^6y^{10}$ graphed on the same set of axes

However, when the graphs of $z = (x + y)^2$ (in rainbow) and $z = x^2 + y^2$ (in red) are graphed on the same axes, it is clear that the graphs are not the same; therefore the equations are not equivalent.

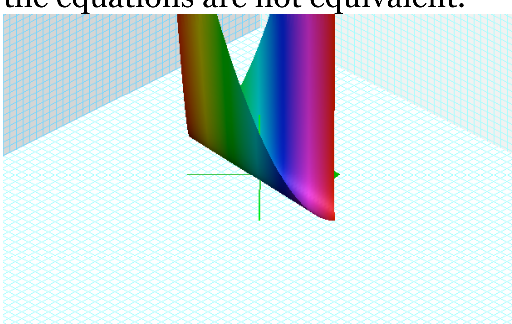


Figure 4: $z = (x + y)^2$

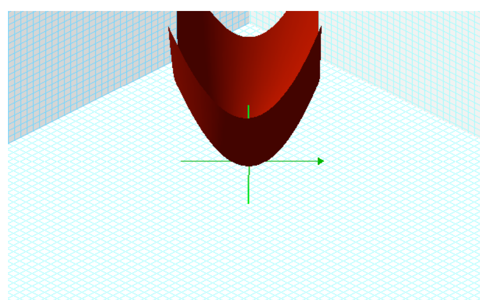


Figure 5: $z = x^2 + y^2$

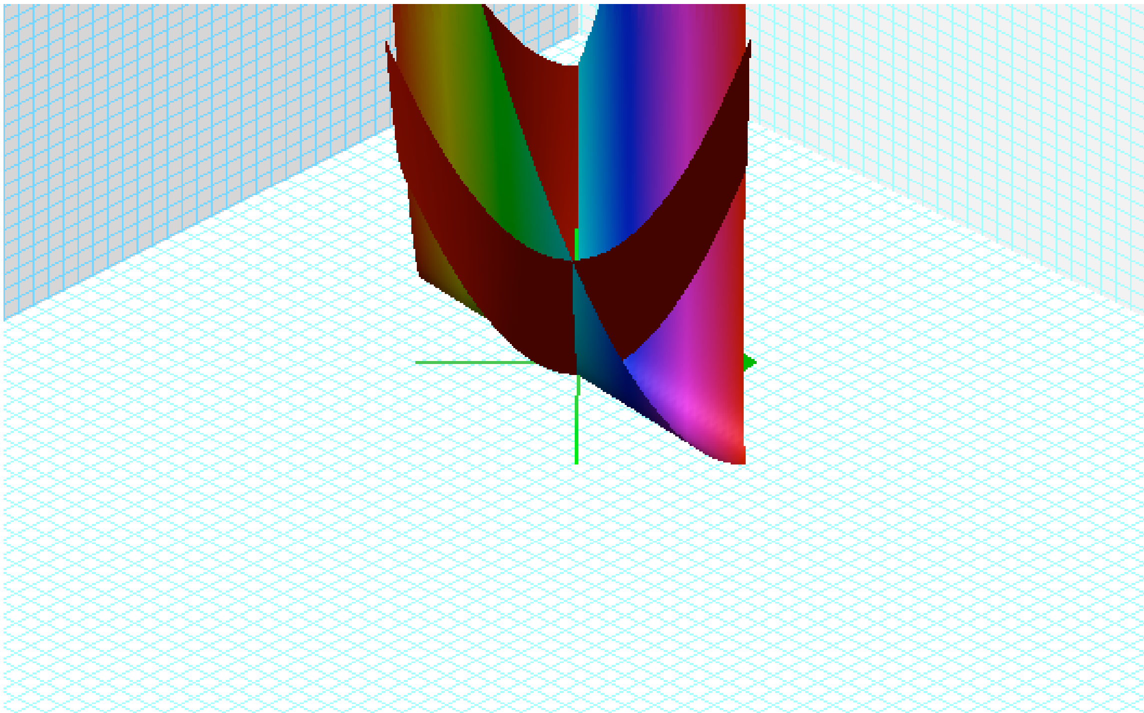


Figure 6: $z=(x+y)^2$ and $z=x^2+y^2$

Mathematical Focus 4

A numerical approach can provide some evidence about the equivalence of expressions.

A simple test of a few values is a way to give evidence that $(x^3y^5)^2$ and x^6y^{10} may be equivalent, while $(a+b)^2$ and a^2+b^2 are not. For problem 1, we give an example of $(x^3y^5)^2 = x^6y^{10}$ for certain values ($x = 2$ and $y = 3$). This is merely evidence, and not a proof that $(x^3y^5)^2$ and x^6y^{10} are equivalent because we cannot test all possible values for x and y. For Problem 2, however, we have provided a counterexample to show that $(a+b)^2$ cannot be equivalent to a^2+b^2.

Problem 1:

Let $x = 2$ and $y = 3$.

$(x^3y^5)^2$	x^6y^{10}
$= (2^3 \bullet 3^5)^2$	$= 2^6 \bullet 3^{10}$
$= (8 \bullet 243)^2$	$= 64 \bullet 59049$
$= (1944)^2$	$= 3779136$
$= 3779136$	

Problem 2:

Let $a = 2$ and $b = 3$

$(a+b)^2$	$a^2 + b^2$
$= (2 + 3)^2$	$= 2^2 + 3^2$
$= (5)^2$	$= 4 + 9$
$= 25$	$= 13$

One example (as we have provided for Problem 1) is not sufficient to prove equivalence of the expressions, but one counterexample is sufficient to <u>disprove</u> a statement of equivalence. Since (as we have provided for Problem 2) there exist values of a and b for which $(a+b)^2$ and $a^2 + b^2$ are <u>not</u> equal, we can safely say that they are not equivalent expressions, as equivalent expressions would have equal values for all real values of a and b.