Situation 38: Irrational Length

Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
Date last revised: July 16, 2005 – Started by: Jeanne Shimizu

Prompt

A secondary pre-service teacher was given the following task to do during an interview:

> Given: square ABCD.
> Construct a square whose area is half the area of square ABCD.
> (Note: The pre-service teacher was not given a drawing or any dimensions for ABCD.)

The student chose the dimensions of ABCD to be 1 unit by 1 unit and approached the problem in two ways.

Method 1: Reasoning with a figure
She divided ABCD into smaller squares as shown in Figure 1a and noted each small square has area $\frac{1}{4}$. Sketching a new square as in Figure 1b, she claimed a $\frac{3}{4} X \frac{3}{4}$ square has area $\frac{9}{16}$. She concluded that the square she wants (sketched in Figure 1c) has a side length somewhere between one-half and three-fourths.

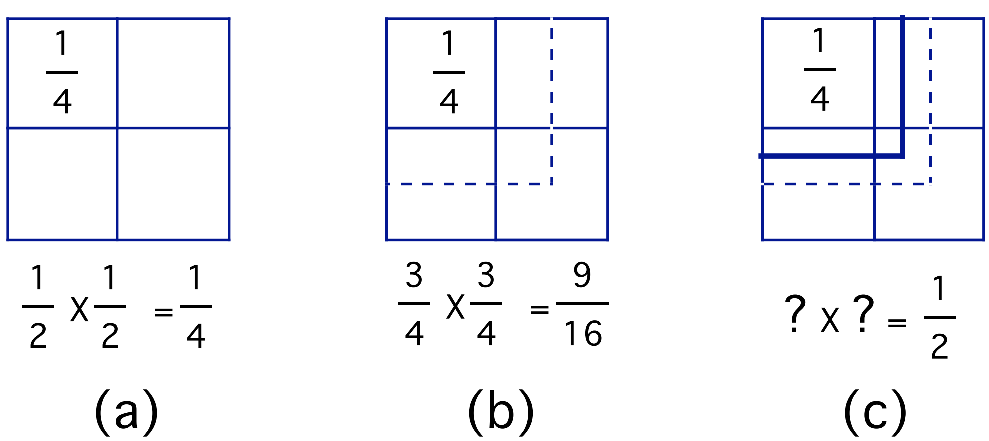

Figure 1.

Method 2: Reasoning from a formula

Assuming implicitly that the area of given square ABCD is 1 square unit, she noted that the desired area of the new square is one-half square unit. Using a formula for the area of a square, she produced $s^2 = \frac{1}{2}$ and then

$s = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

After a long pause, she pointed to $\frac{\sqrt{2}}{2}$ and said, "I don't know how long that is. So I can't draw the square."

Mathematical Foci

Mathematical Focus 1

A numerical approximation for $\frac{\sqrt{2}}{2}$ could be obtained using a calculator or by recalling that $\sqrt{2} \approx 1.414$. So, $\frac{\sqrt{2}}{2} \approx 0.707$.

Measuring approximating 0.707 units to create the square would be easy in some cases (e.g., if the length of a side of ABCD were 1 meter and the length of a side of the new square would be 707 millimeters).

Mathematical Focus 2

A segment of length, $\frac{\sqrt{2}}{2}$, could be found geometrically. The diagonals of a 1X1 square have length, $\sqrt{2}$. By bisecting a diagonal, we obtain a segment of length, $\frac{\sqrt{2}}{2}$. (See segment DQ in Figure 2.) Using segment DQ we may construct a square whose area is one-half that of square ABCD. (See DSTU in Figure 2.)

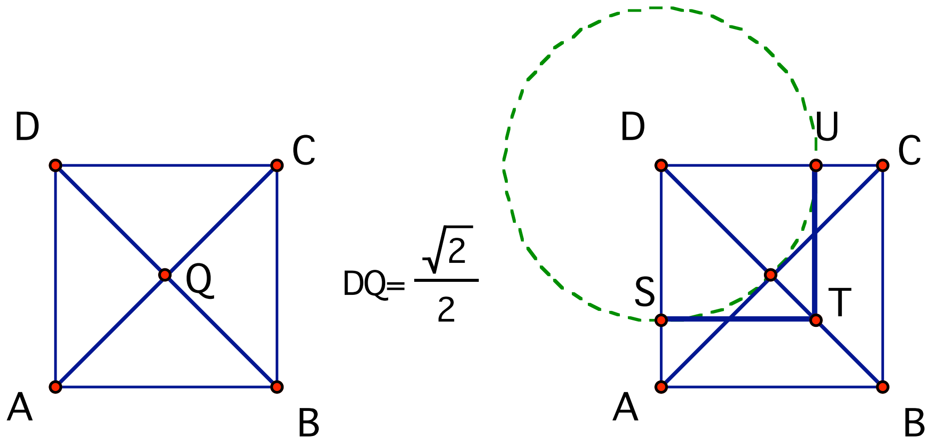

Figure 2.

Mathematical Focus 3

A segment of length $\frac{\sqrt{2}}{2}$ can be found using similar triangles.

The diagonals of a 1X1 square have length $\sqrt{2}$. By connecting the midpoints of segments AB and AD (points X and Y, respectively), we construct a pair of similar triangles, ΔABD and ΔAXY. See Figure 3. Because ΔABD ~ ΔAXY and $AX = AY = \frac{1}{2},\ XY = \frac{1}{2}BD = \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$.

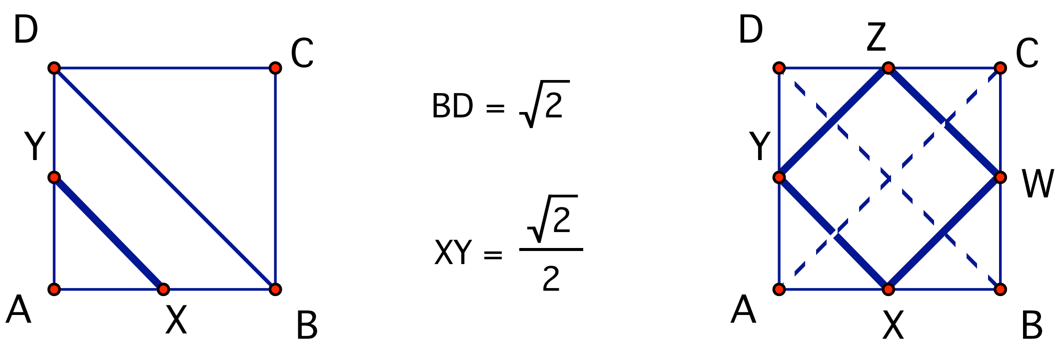

Figure 3.

By analogy, segments XW, WZ, and ZY have lengths $\frac{\sqrt{2}}{2}$. Since ΔXAY, ΔXBW, ΔWCZ, and ΔZDY are isosceles right triangles, the interior angles of YXWZ are right angles. So, YXWZ is a square whose area, $\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$, is one-half that of ABCD.

Mathematical Focus 4

An approximation of $\sqrt{2}$ can be found by using the square root algorithm:

```
          1.  4   1   4 ...
         )2. 00  00  00
          1
24        1  00
             96
281           4  00
              2  81
2824          1  19  00
              1  12  96
                  7  04
```

So, $\frac{\sqrt{2}}{2} \approx 0.707$ is the length of the sides of the desired square.

Mathematical Focus 5

A numerical approximation for $\frac{\sqrt{2}}{2}$ can be found by using Newton's Method.

Newton's Method

$$x_0 = an\ initial\ guess$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } n = 1,2,3,\ldots$$

Because we want to approximate the dimensions of a square whose area is $\frac{1}{2}$ square units, we want to approximate the positive real root of $x^2 - \frac{1}{2} = 0$.

We use $f(x) = x^2 - \frac{1}{2}$ and $f'(x) = 2x$ to obtain $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\ 2} - \frac{1}{2}}{2x_n}$

We know that our desired solution is between 0.5 and 0.75 (based on her work in Figure 1). These values are reasonable initial approximations. Figure 4 is a screen shot of ten successive approximations generated on Excel using the formulas shown in Figure 5.

n	x(n)
0	0.5
1	0.75
2	0.70833333
3	0.70710784
4	0.70710678
5	0.70710678
6	0.70710678
7	0.70710678
8	0.70710678
9	0.70710678
10	0.70710678

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9	0.70710678
10	0.70710678

Figure 4.

n	x(n)		n	x(n)
0	0.5		0	0.75
=A7+1	=B7-((B7)^2-0.5)/(2*B7)		=D7+1	=E7-((E7)^2-0.5)/(2*E7)
=A8+1	=B8-((B8)^2-0.5)/(2*B8)		=D8+1	=E8-((E8)^2-0.5)/(2*E8)
=A9+1	=B9-((B9)^2-0.5)/(2*B9)		=D9+1	=E9-((E9)^2-0.5)/(2*E9)
=A10+1	=B10-((B10)^2-0.5)/(2*B10)		=D10+1	=E10-((E10)^2-0.5)/(2*E10)
=A11+1	=B11-((B11)^2-0.5)/(2*B11)		=D11+1	=E11-((E11)^2-0.5)/(2*E11)
=A12+1	=B12-((B12)^2-0.5)/(2*B12)		=D12+1	=E12-((E12)^2-0.5)/(2*E12)
=A13+1	=B13-((B13)^2-0.5)/(2*B13)		=D13+1	=E13-((E13)^2-0.5)/(2*E13)
=A14+1	=B14-((B14)^2-0.5)/(2*B14)		=D14+1	=E14-((E14)^2-0.5)/(2*E14)
=A15+1	=B15-((B15)^2-0.5)/(2*B15)		=D15+1	=E15-((E15)^2-0.5)/(2*E15)
=A16+1	=B16-((B16)^2-0.5)/(2*B16)		=D16+1	=E16-((E16)^2-0.5)/(2*E16)

Figure 5.

There are other ways to generate these values. For example, we could use a TI-92 calculator in sequence mode to generate a table using

u1=u1(n-1)-((u1(n-1))^2-.5)/(2*u1(n-1))

ui1=.5

u2=u2(n-1)-((u2(n-1))^2-.5)/(2*u2(n-1))

ui2=.75

and

tblStart 1

Δtbl 1

Connection:

Students may find that The Math Forum's Dr. Math has a webpage devoted to finding square roots without a calculator. Dr. Math describes an algorithm in which the reader is instructed to follow a sequence of steps loosely described as:

1. Guess 2. Divide 3. Average

Continue steps 2 and 3 until the desired degree of accuracy is reached.

For example, to approximate the value of $\sqrt{2}$, we would as follows:

1. Guess. Let us use 1.
2. Divide. Since we want to find $\sqrt{2}$, we would compute $2 \div 1 = 2$.

3. Average. For this step we compute the arithmetic average of 2 and 1. $\frac{2+1}{2} = 1.5$
4. Divide. $2 \div 1.5 \approx 1.333$
5. Average. $\frac{1.5 + 1.333}{2} = 1.4165$
6. Divide. $2 \div 1.4165 \approx 1.4119$
7. Average. $\frac{1.4165 + 1.4119}{2} \approx 1.4142$
8. Divide. $2 \div 1.4142 \approx 1.4122$
9. Average. $\frac{1.4142 + 1.4122}{2} \approx 1.4132$

So, $\sqrt{2} \approx 1.41$ This process can be described symbolically as follows:

$$T(n+1) = \frac{\frac{2}{T(n)} + T(n)}{2}$$, where $T(0) = initial\ guess$

The mathematical basis for this recursive algorithm is Newton's Method applied using $f(x) = x^2 - 2$ for which we are interested in finding an approximation for a positive root.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^2 - 2}{2x_n}$$

$$= \frac{2x_n{}^2 - \left(x_n{}^2 - 2\right)}{2x_n} = \frac{x_n{}^2 + 2}{2x_n} = \frac{x_n + \frac{2}{x_n}}{2}$$

References

Math Forum's Dr. Math, "Square Roots Without a Calculator" http://mathforum.org/dr.math/faq/faq.sqrt.by.hand.html , July 13, 2005

Grzesina, A., A geometric view of the square root algorithm, http://mathcentral.uregina.ca/RR/database/RR.09.95/grzesina1.html , July 15, 2005

Homeschool Math, Why the square root algorithm works – the logic behind it, http://www.homeschoolmath.net/other_topics/sqr-algorithm-why-works.php , July 15, 2005

Anton, H. Calculus with Analytic Geometry, 4th ed. (1992) Wiley & Sons, Inc., New York

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