

MAC-CPTM Situations Project

Situation 39: Summing the Natural Numbers

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Prompt

The course was a mathematical modeling course for prospective secondary mathematics teachers. The discussion focused on finding an explicit formula for a sequence expressed recursively. During the discussion, a need to sum the integers from 1 to n was expressed. After several attempts to remember the formula, a student hypothesized that the formula contained n and $n + 1$. Another student said he thought it was $\frac{n(n+1)}{2}$, but was not sure. During the ensuing discussion, a third student asked, “How do we know that $\frac{n(n+1)}{2}$ is the sum of the integers from 1 to n , and won’t this formula sometimes give fractions?”

Commentary

The integers from 1 to n is an ordered sequence of natural numbers. The symmetry and ordered nature of the set allows for rearrangements and reconfigurations that facilitate finding a sum—under rearrangement of discrete entities, cardinality remains the same. A rule for the sum of any number of natural numbers can be developed from specific instances, and such a rule can be proved using mathematical induction. Just as the rule is a generalization, it too can serve as a specific case for a more broader generalization involving the sum of the n th powers of natural numbers, instead of the sum of the 1st powers of natural numbers.

Mathematical Foci

Mathematical Focus 1

Expressing the first n natural numbers as a sequence allows the use of formula for the sum of the sequence.

Consider the first n natural numbers, i.e., 1, 2, 3, ..., $n-1$, n . This series can be interpreted as an arithmetic sequence, since there is a common difference, 1, between any two consecutive terms. The formula for the sum of any arithmetic

series is $S_n = \frac{n(a_1 + a_n)}{2}$, where a_1 is the first term of the sequence and a_n is the n^{th}

term of the sequence. For the sequence of the first n natural numbers, it will always be the case that $a_1 = 1$ and $a_n = n$. Substituting the appropriate values yields the desired formula.

Mathematical Focus 2

Writing the sum or difference as a constant fosters an algebraic manipulation approach. The constant differences and symmetry of arithmetic sequences allow certain algebraic manipulations.

Let S be the sum of the first n natural numbers. Then $S = 1 + 2 + 3 + \dots + n-1 + n$. Because the sequence is arithmetic, writing the indicated sum twice (once in increasing orders and once in decreasing order) allows a pairing of terms each of whose sum is $n+1$. By the commutative and associative properties of addition and properties of equality,

$$S = 1 + 2 + 3 + \dots + n$$

$$S = n + (n-1) + (n-2) + \dots + 1$$

$$2S = (n+1) + (2+n-1) + (3+n-2) + \dots + (n+1)$$

$$2S = (n+1) + (n+1) + (n+1) + \dots + (n+1)$$

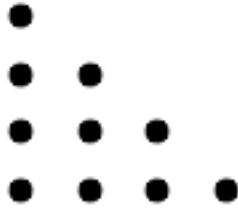
$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

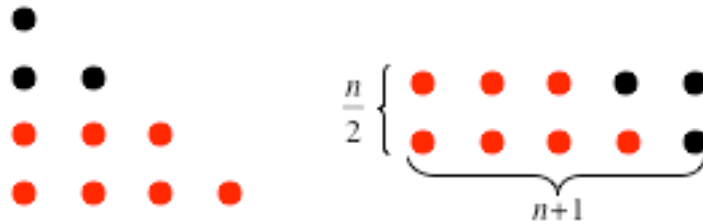
Mathematical Focus 3

Decomposition/recomposition of plane geometric figures preserves area. Under rearrangement of discrete entities, cardinality remains the same.

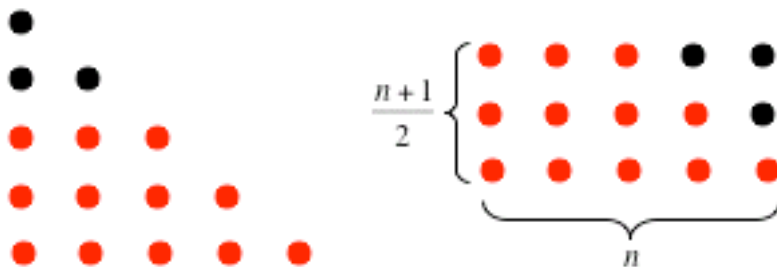
We can represent the first n natural numbers with a triangular array of dots, with each row of dots representing a number. The following array represents the sum of the first four natural numbers.



One way to verify the formula would be to rearrange the dots into a rectangular array. If n is even, we would have $\frac{n}{2}$ rows, as shown in the first figure below.



The remaining rows will be rotated and moved to complete a rectangular array having dimensions $\frac{n}{2} \times (n + 1)$ as shown in the second figure above. If n is odd, we would have $\frac{n + 1}{2}$ rows, as shown in the first figure below.

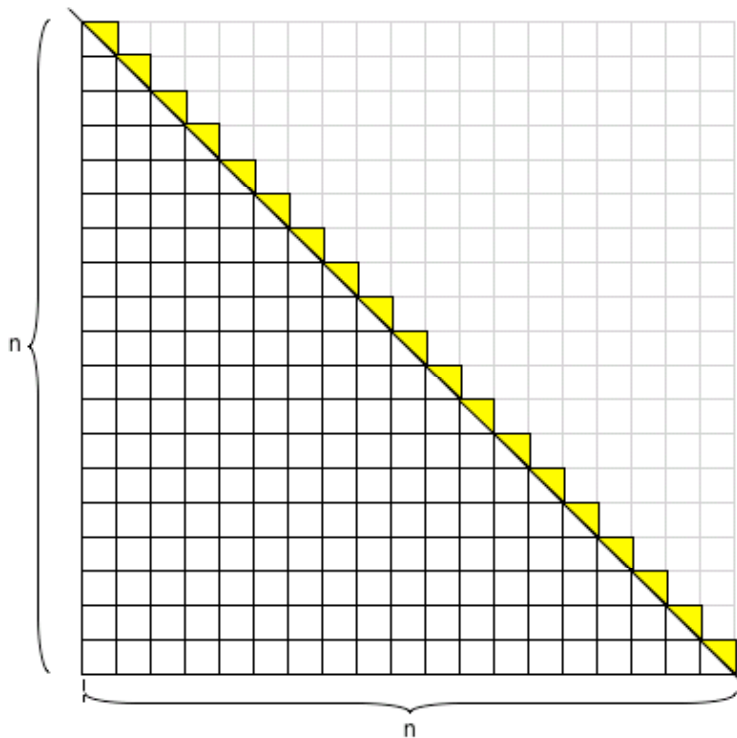


The remaining rows will be rotated and moved to complete a rectangle having dimensions $\frac{n + 1}{2} \times n$ as shown in the second figure above.

Thus, the sum of the first n natural numbers is merely the number of dots in the rectangle. In each of the odd and even cases above the number of dots is given by the number of rows multiplied by the number of columns. This gives $\frac{n(n + 1)}{2}$ in either case. Also, since this method only requires rearranging the dots, it is clear that it will always yield a natural number result.

This geometrical interpretation of the sum of the first n natural numbers could be done in a number of ways. Another way involves using unit squares instead of

dots (as shown below) and in this case the sum is represented as the sum of the areas of the unit squares. In this diagram, the square in which the triangular configuration of unit squares is embedded has area $n \cdot n$, the triangle has area $\frac{n \cdot n}{2}$, and the area of the half-unit-triangles above the triangle is $\frac{n}{2}$. So the area of the sum of the first n natural numbers is $\frac{n \cdot n}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$.



Mathematical Focus 4

The first principle of mathematical induction offers a method to verify a formula for the sum of the terms in a sequence.

We can use the first principle of mathematical induction to prove that our formula will work for all natural numbers. This proof does not derive the formula, but proves that the formula works once we have it.

The sum of the first n natural numbers can be expressed as $\sum_{i=1}^n i$.

Our inductive hypothesis is that $S_n = \frac{n(n+1)}{2}$.

When $n = 1$, we have $S_1 = 1 = \frac{1(1+1)}{2}$, which establishes our base case.

We assume that, for all numbers less than or equal to m , $S_m = \sum_{i=1}^m i = \frac{m(m+1)}{2}$. We must show that this also holds for $m + 1$, that is, we must show that,

$$\sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}.$$

Consider,

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^m i + m + 1 = \frac{m(m+1)}{2} + m + 1 = \frac{m(m+1)}{2} + \frac{2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$$

Thus the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

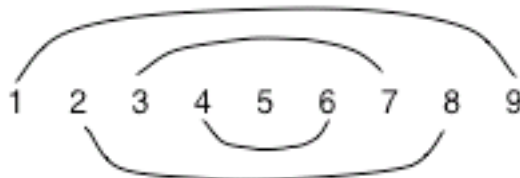
Mathematical Focus 5

Starting from specific examples, we can abstract to the generalized formula for the summation of the first n natural numbers.

As an alternate to directly dealing with the general case, we will first look at two specific examples. There are two basic cases for the natural number n , namely n could be an even or an odd number.

Suppose that $n = 16$. One way to add the numbers 1, 2, ..., 16, is to use both the commutative and associative laws of addition, and change the order and groupings of the numbers. In this example, the first grouping could be the largest number with the smallest number (i.e. $1 + 16$), next grouping the second largest number with the second smallest number (i.e. $2 + 15$), then grouping the third largest number with the third smallest number ($3 + 14$), and so forth until all possible groupings are created. This creates 8 groups, which each sum to 17. Thus the sum of the numbers from 1 to 17 is $8 \times 17 = 136$. We see that this is in line with the given formula of $\frac{n(n+1)}{2}$.

Now suppose that $n = 9$. We group the numbers as above using the following diagram.



Notice that since 9 is odd, 5 is not paired with another number. Thus we have 4 groups that sum to 10. So, the sum is given by $4 \times 10 + 5 = 45$. When we are adding an odd number of numbers, grouping as above will always result in even sums

since we add odd to odd and even to even. Even though we will always have to deal with one ungrouped number, this sum is still even and a natural number.

We can use the same technique used above to form a general view of what happens when we add the first n natural numbers. The general case is also divided into two cases (when n is odd and when n is even).

In the odd case, the groups would be $1 + n$, $2 + (n-1)$, $3 + (n-2)$, and so on. Each one of those sums is $n+1$. There are always $\frac{n-1}{2}$ of these groupings. However, there is always one term left out in the grouping, specifically $\frac{n+1}{2}$. As before we add these groups together,

$$\left(\frac{n-1}{2}\right)(n+1) + \left(\frac{n+1}{2}\right) = (n+1)\left(\left(\frac{n-1}{2}\right) + \frac{1}{2}\right) = (n+1)\left(\frac{n-1+1}{2}\right) = \frac{(n+1)n}{2}$$

Note that since n is an odd number, then $n+1$ is even. Thus $\frac{n+1}{2}$ is a natural number.

Now we consider the case when n is even.

$$1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

If we add the first and last terms then the second and second to last terms and continuing inward through the terms of the series. The sum of each grouping is $n+1$. There are a total of $\frac{n}{2}$ groups of sums that are added together. Therefore the sum of the series is $(n+1)\frac{n}{2}$.

Mathematical Focus 6

The sum of the first n natural numbers is a specific case of the sum of the first n k^{th} powers of natural numbers.

Polya (1981) describes how Pascal solved this problem and extended it to the general case of finding the sum of the k^{th} powers of the first n natural numbers. The method depends on expanding $(n+1)^{k+1}$.

Let S_1 be the sum of the first n natural numbers. To find S_1 , expand $(n+1)^2$ and write out its values for several numbers to see the pattern:

$$(n+1)^2 = n^2 + 2n + 1 .$$

$$\Rightarrow (n+1)^2 - n^2 = 2n + 1$$

$$(1) \quad 2^2 - 1^2 = 2 \cdot 1 + 1$$

$$(2) \quad 3^2 - 2^2 = 2 \cdot 2 + 1$$

$$(3) \quad 4^2 - 3^2 = 2 \cdot 3 + 1$$

And so on till we obtain,

$$(n) \quad (n+1)^2 - n^2 = 2n + 1$$

Then we can add all the equations (1) through (n) together. The sum of these equations will be $(n+1)^2 - 1 = 2(S_1) + n$. Then,

$$S_1 = \frac{(n+1)^2 - n - 1}{2} = \frac{(n+1)^2 - 1(n+1)}{2} = \frac{(n+1-1)(n+1)}{2} = \frac{n(n+1)}{2}.$$

To find the sum of the first n squares of natural numbers, S_2 , expand $(n+1)^3$ and write out its values for several numbers to see the pattern.

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$\Rightarrow (n+1)^3 - n^3 = 3n^2 + 3n + 1$$

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$

$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$

$$4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1$$

⋮

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

Again add all the equations together. The sum of these equations will be $(n+1)^3 - 1 = 3 \cdot S_2 + 3S_1 + n$. Substituting for S_1 and solving for S_2 gives us,

$$\begin{aligned} S_2 &= \frac{(n+1)^3 - 3 \frac{n(n+1)}{2} - n - 1}{3} = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{6} = \frac{(n+1)[2(n+1)^2 - 3n - 2]}{6} \\ &= \frac{(n+1)(2n^2 + n)}{6} = \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

This method can be extended to find the sum of the first n cubes, and then the first n fourth powers and so on, as long as the sums of all the previous powers have been found.

References

Polya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving*. John Wiley & Sons: New York.