MAC-CPTM Situations Project

Situation 39: Summing the Natural Numbers

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Prompt

The course was a mathematical modeling course for prospective secondary mathematics teachers. The discussion focused on finding an explicit formula for a sequence expressed recursively. During the discussion, a need to sum the integers from 1 to *n* was expressed. After several attempts to remember the formula, a student hypothesized that the formula contained n and $n + 1$. Another student said he thought it was $\frac{n(n+1)}{2}$, but was not sure. During the ensuing discussion, a third student asked, "How do we know that $\frac{n(n+1)}{2}$ is the sum of the integers from 1 to n, and won't this formula sometimes give fractions?"

Commentary

The set of integers from 1 to n is an ordered sequence of natural numbers. The symmetry and ordered nature of the set allows for rearrangements and reconfigurations that facilitate finding a sum–under rearrangement of discrete entities, cardinality remains the same. A rule for the sum of any number of natural numbers can be developed, or verified, from specific instances, and such a rule can be proved using a variety of methods that include mathematical induction.

Mathematical Foci

Mathematical Focus 1

Expressing the first n natural numbers as an arithmetic sequence allows for the use of the formula for the sum of the sequence. Since arithmetic sequences have both symmetry and constant differences, expressing the sum or difference of an arithmetic sequence as a constant affords certain algebraic manipulations.

Consider the first n natural numbers, i.e., 1, 2, 3, …, n-1, n. This progression can be interpreted as an arithmetic sequence, with a constant difference, 1, between any two consecutive terms. The formula for the sum of any arithmetic sequence is $S_n = \frac{n\left(a_1 + a_n\right)}{2}$, where a_1 is the first term of the sequence and a_n is the n^{th} term of the sequence. For the sequence of the first n natural numbers, it will always be the case that $a_1 = 1$ and $a_n = n$. Substituting the appropriate values yields the desired formula.

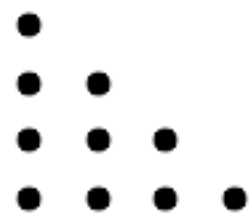
Let S be the sum of the first n natural numbers. Then $S = 1 + 2 + 3 + \ldots + n-1 + n$. Because the sequence of the first n natural numbers is arithmetic, writing the indicated sum, S, twice (once in increasing order and once in decreasing order) allows a pairing of terms, each of whose sum is n+1. By the commutative and associative properties of addition and properties of equality,

$$
\begin{aligned}
S &= 1 + 2 + 3 + \ldots + n \\
S &= n + \left(n-1\right) + \left(n-2\right) + \ldots + 1 \\
2S &= \left(n+1\right) + \left(2+n-1\right) + \left(3+n-2\right) + \ldots + \left(n+1\right) \\
2S &= \left(n+1\right) + \left(n+1\right) + \left(n+1\right) + \ldots + \left(n+1\right) \\
2S &= n\left(n+1\right) \\
S &= \frac{n\left(n+1\right)}{2}
\end{aligned}
$$

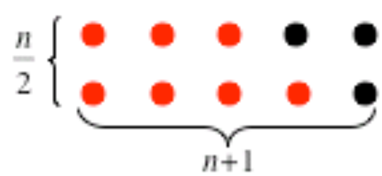
Mathematical Focus 2

Decomposition/recomposition of plane geometric figures preserves area. Geometric representations of the first n natural numbers provide opportunities to derive the formula for summing the first n natural numbers.

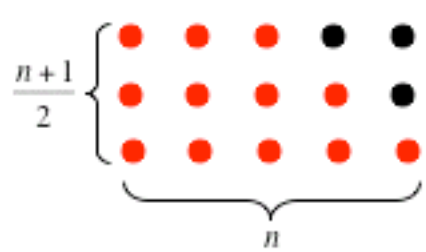
One representation of the first n natural numbers is a triangular array of dots, with each row of dots representing a number. The following array represents the sum of the first four natural numbers.



One way to derive or verify the formula for the sum of the first n natural numbers would be to rearrange the dots into a rectangular array. If n is even, there are $\frac{n}{2}$ rows, as shown in the first figure below.

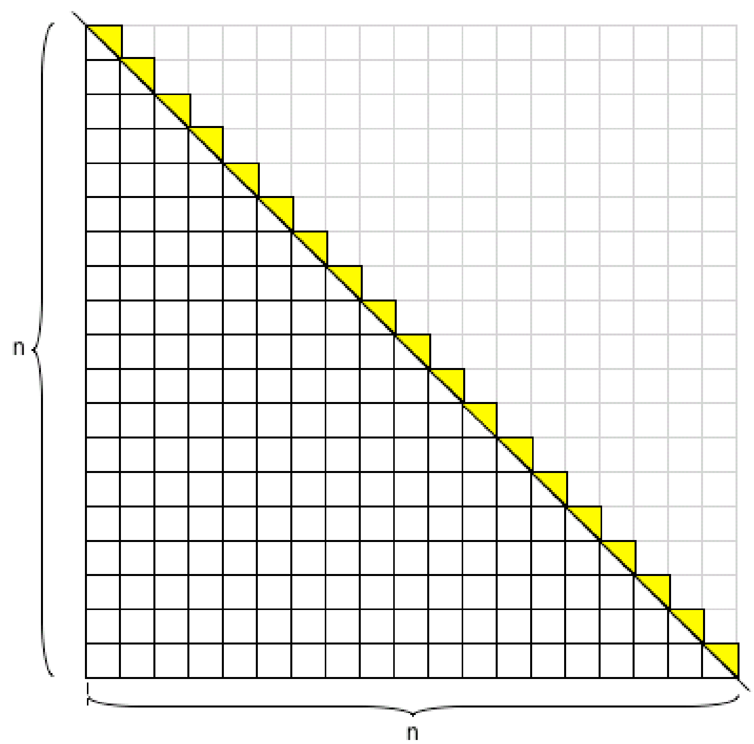


The remaining rows are rotated and moved to complete a rectangular array having dimensions $\frac{n}{2} \times (n + 1)$ as shown in the second figure above. If n is odd, there are $\frac{n+1}{2}$ rows, as shown in the first figure below.



The remaining rows are rotated and moved to complete a rectangle having dimensions $\frac{n+1}{2}$ × n as shown in the second figure above.
Thus, the sum of the first *n* natural numbers is merely the number of dots in the rectangle. In each of the odd and even cases above the number of dots is given by the number of rows multiplied by the number of columns. This gives $\frac{n(n+1)}{2}$ in either case. Also, since this method only requires rearranging the dots, it is clear that it will always yield a natural number result.

Another geometrical interpretation of the sum of the first *n* natural numbers involves using unit squares instead of dots (as shown below). In this case the sum of the areas of the unit squares represents the sum of the first *n* natural numbers. In this diagram, the square in which the triangular configuration of unit squares is embedded has area $n \cdot n$, the triangle has area $\frac{n \cdot n}{2}$, and the area of the half-unit-triangles shaded in yellow is $\frac{n}{2}$. So the sum of the areas of the unit squares is $\frac{n \cdot n}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$. In this way, the sum of the first *n* natural numbers is
$\frac{n \cdot n}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$.



Mathematical Focus 3

The first principle of mathematical induction offers one method for verifying a formula for the sum of the terms in a sequence.

The first principle of mathematical induction is one method for proving that the formula holds for all natural numbers. This proof does not derive the formula, but proves that the formula works once we have it.

The sum of the first n natural numbers can be expressed as $\sum_{i=1}^{n} i$.

The inductive hypothesis is that $S_n = \frac{n(n+1)}{2}$.

When $n = 1$, $S_1 = 1 = \frac{1(1+1)}{2}$, which establishes the base case.

Assume that, for all numbers less than or equal to m, $S_m = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$. It is now necessary to show that this also holds for $m + 1$; that is, that

$$\sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}.$$

Consider,

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + m + 1 = \frac{m(m+1)}{2} + m + 1 = \frac{m(m+1)}{2} + \frac{2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$$

Thus the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

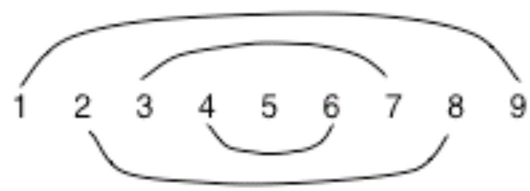
Mathematical Focus 4

Starting from specific examples, it is possible to abstract a generalized formula for the summation of the first n natural numbers.

As an alternate to directly dealing with the general case, consider two specific examples. There are two basic cases for the natural number *n*, namely *n* could be an even or an odd number.

Suppose that $n = 16$. One way to add the numbers 1, 2, …, 16, is to use both the commutative and associative properties of addition, and change the order and groupings of the numbers. In this example, the first grouping could be the largest number with the smallest number (i.e. 1 + 16), next grouping the second largest number with the second smallest number (i.e. 2 + 15), then grouping the third largest number with the third smallest number (3 + 14), and so forth until all possible groupings are created. This creates 8 groups, which each sum to 17. Thus the sum of the numbers from 1 to 16 is $8 \times 17 = 136$. This corresponds with the given formula of $\frac{n(n+1)}{2}$.

Now suppose that $n = 9$. Group the numbers as above using the following diagram.



Notice that since 9 is odd, 5 is not paired with another number. Thus there are 4 groups that sum to 10. So, the sum is given by $4 \times 10 + 5 = 45$. When adding an odd number of numbers, grouping as above will always result in even sums since odd numbers are added to odd numbers and even numbers to even numbers. Even though there will always be one ungrouped number, this sum is still even and a natural number.

The same technique used above allows for the abstraction to a general view of what happens when the first *n* natural numbers are summed. The general case is also divided into two cases (when *n* is odd and when *n* is even).

In the odd case, the groups would be 1 + n, 2 + (n-1), 3 + (n-2), and so on. Each one of those sums is n+1. There are always $\frac{n-1}{2}$ of these groupings. However, there is always one term left out in the grouping, specifically $\frac{n+1}{2}$. As before we add these groups together,

$$\left(\frac{n-1}{2}\right)(n+1)+\left(\frac{n+1}{2}\right)=(n+1)\left(\left(\frac{n-1}{2}\right)+\frac{1}{2}\right)=(n+1)\left(\frac{n-1+1}{2}\right)=\frac{(n+1)n}{2}$$

Note that since n is an odd number, then n+1 is even. Thus $\frac{n+1}{2}$ is a natural number, and therefore, $\frac{n(n+1)}{2}$ is a natural number.

Now consider the case when n is even.

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

Pairing the first and last terms, then the second and second to last terms, and continuing inward through the terms of the series, the sum of each grouping is n+1. There are a total of $\frac{n}{2}$ groups of sums that are added together. Therefore the sum of the series is $(n+1)\frac{n}{2}$. Note that since n is even, $\frac{n(n+1)}{2}$ is a natural number.

Post-Commentary

The sum of the first n natural numbers is a specific case of the sum of the first n k^{th} powers of natural numbers.

Just as the rule is a generalization, it too can serve as a specific case for a broader generalization involving the sum of the nth powers of natural numbers, instead of the sum of the 1^{st} powers of natural numbers.

Polya (1981) describes how Pascal solved this problem and extended it to the general case of finding the sum of the k^{th} powers of the first n natural numbers. The method depends on expanding $(n+1)^{k+1}$.

Let S_1 be the sum of the first n natural numbers. To find S_1, expand $(n+1)^2$ and write out its values for several numbers to see the pattern:

$(n+1)^2 = n^2 + 2n + 1$.

$\Rightarrow (n+1)^2 - n^2 = 2n + 1$

(1) $2^2 - 1^2 = 2 \cdot 1 + 1$

(2) $3^2 - 2^2 = 2 \cdot 2 + 1$

(3) $4^2 - 3^2 = 2 \cdot 3 + 1$

And so on till we obtain,

$(n) \quad (n+1)^2 - n^2 = 2n+1$

Then we can add all the equations (1) through (n) together. The sum of these equations will be $(n+1)^2 - 1 = 2(S_1) + n$. Then,

$$S_1 = \frac{(n+1)^2 - n - 1}{2} = \frac{(n+1)^2 - 1(n+1)}{2} = \frac{(n+1-1)(n+1)}{2} = \frac{n(n+1)}{2}.$$

To find the sum of the first n squares of natural numbers, S_2, expand $(n + 1)^3$ and write out its values for several numbers to see the pattern.

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$
$$\Rightarrow (n+1)^3 - n^3 = 3n^2 + 3n + 1$$
$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$
$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$
$$4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1$$
$$\vdots$$
$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

Again add all the equations together. The sum of these equations will be $(n+1)^3 - 1 = 3 \bullet S_2 + 3S_1 + n$. Substituting for S_1 and solving for S_2 gives us,

$$S_2 = \frac{(n+1)^3 - 3\frac{n(n+1)}{2} - n - 1}{3} = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{6} = \frac{(n+1)[2(n+1)^2 - 3n - 2]}{6}$$
$$= \frac{(n+1)(2n^2 + n)}{6} = \frac{n(n+1)(2n+1)}{6}$$

This method can be extended to find the sum of the first n cubes, and then the first n fourth powers and so on, as long as the sums of all the previous powers have been found.

References

Polya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving*. John Wiley & Sons: New York.