MAC-CPTM Situations Project

Situation 39: Summing the Natural Numbers

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Prompt

The course was a mathematical modeling course for prospective secondary mathematics teachers. The discussion focused on finding an explicit formula for a sequence expressed recursively. During the discussion, the students expressed a need to sum the natural numbers from 1 to n. After several attempts to remember the formula, a student hypothesized that the formula contained n and $n + 1$. Another student said he thought it was $\frac{n(n+1)}{2}$ but was not sure. During the ensuing discussion, a third student asked, "How do we know that $\frac{n(n+1)}{2}$ is the sum of the integers from 1 to n? And won't that formula sometimes give a fraction?"

Commentary

The set of integers from 1 to n is an ordered sequence of natural numbers. Symmetry and the ordered nature of the sequence allow for rearrangements that facilitate finding a sum; under any rearrangement of the elements of a discrete set, the cardinality remains the same, as does the sum of the elements. A formula for the sum of the natural numbers from 1 to any number can be developed from, and verified for, specific instances, and the formula can be proved using a variety of methods, including mathematical induction.

Mathematical Foci

Mathematical Focus 1

There are two possibilities for the natural number n*: It can be even, or it can be odd. In this way, when* n *is a natural number,* $\frac{n(n+1)}{2}$ *is also a natural number.*

If n is odd, then n + 1 is even and therefore divisible by 2. Hence $\frac{n(n+1)}{2}$ is a natural number. If n is even, then it is divisible by 2. And hence $\frac{n(n+1)}{2}$ is a natural number.

The argument can be expressed symbolically:

For n odd, let $n = 2m + 1$, with m a natural number, then
$$\frac{n(n+1)}{2} = \frac{(2m+1)(2m+1+1)}{2} = \frac{2(2m+1)(m+1)}{2} = (2m+1)(m+1).$$
For n even, let $n = 2k$, with k a natural number, then
$$\frac{n(n+1)}{2} = \frac{(2k)(2k+1)}{2} = (k)(2k+1).$$

Mathematical Focus 2

Specific examples suggest a general formula for the sum of the first n *natural numbers. Strategic choices for pair-wise grouping of numbers is critical to the development of the general formula.*

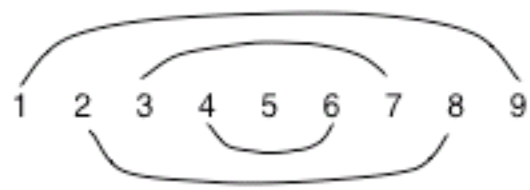
Case 1: n is even
Specific Example: $n = 16$
Suppose that $n = 16$. One way to add the numbers 1, 2, ..., 16, is to use the commutative and associative properties of addition to group the numbers in pairs. In this example, the first pair could be the largest number with the smallest number (i.e., 1 + 16); the next pair, the second largest number with the second smallest number (2 + 15); the third pair, the third largest number with the third smallest number (3 + 14), and so forth until all the numbers had been paired. These pairings create 8 groups, each of which sums to 17. Thus the sum of the numbers from 1 to 16 is $8 \times 17 = 136$.

General Case: n is even
The technique used for the specific case above suggests the derivation of a general formula. The sum to be found is as follows:

$$1 + 2 + 3 + \ldots + (n - 2) + (n - 1) + n$$

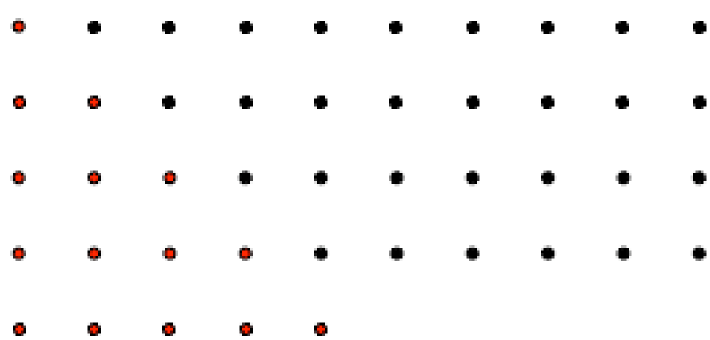
If n is even, then pairing the first and last terms, then the second and second-to-last terms, and continuing inward through the sequence yields $\frac{n}{2}$ pairs, each of which is $n+1$. Therefore the sum of the sequence is $(n+1)\frac{n}{2}$.

<u>Case 2: n is odd</u>
<u>Specific Example: $n = 9$</u>
Now suppose that $n = 9$. Pairing the numbers as above yields the following diagram:



Since 9 is odd, the 5 is not paired with another number. There are 4 groups, each of which sums to 10, plus the unpaired 5. So the sum is given by $4 \times 10 + 5 = 45$.

Alternatively, use an array to group the numbers 1-9. In the array below, the first row pairs 1 red dot with 9 black dots, the second row pairs 2 red dots with 8 black dots, the third row pairs 3 red dots with 7 black dots, and the fourth row pairs 4 red dots with 6 black dots. Because 5 is not paired with another number, there are only 5 red dots in the fifth row. Hence, the array contains four rows of 10 dots each and one row of 5 dots, giving a total of 45 dots in the array. Again, the sum of the first 9 natural numbers is $4 \times 10 + 5 = 45$.



<u>General Case: n is odd</u>
Consider sum of the first n natural numbers for n odd.

$$1 + 2 + 3 + \ldots + (n - 2) + (n - 1) + n$$

As before, the pairs are 1 + n, 2 + (n – 1), 3 + (n – 2), and so on. This time, there are $\frac{n-1}{2}$ pairs, each of which is n + 1, and one term, the middle term $\frac{n+1}{2}$, is not paired. Therefore the sum from 1 to n is

$$\left(\frac{n-1}{2}\right)(n+1) + \left(\frac{n+1}{2}\right) = (n+1)\left(\left(\frac{n-1}{2}\right) + \frac{1}{2}\right) = (n+1)\left(\frac{n-1+1}{2}\right) = \frac{(n+1)n}{2}.$$

Mathematical Focus 3

Because the first n *natural numbers form an arithmetic sequence, properties of such sequences can be used to find their sum. The terms of an arithmetic sequence have a kind of symmetry, and the difference between consecutive terms is constant. The commutative and associative properties of addition allow the terms to be regrouped so their sum can be calculated more efficiently.*

Let S be the sum of the first n natural numbers. Then $S = 1 + 2 + 3 + \ldots + n-1 + n$. When the terms for S are written twice, once in increasing order and once in decreasing order, and corresponding terms added, each sum is n + 1. By the commutative and associative properties of addition and properties of equality,

$$S = 1 + 2 + 3 + \ldots + n$$
$$S = n + (n-1) + (n-2) + \ldots + 1$$
$$2S = (n+1) + (2+n-1) + (3+n-2) + \ldots + (n+1)$$
$$2S = (n+1) + (n+1) + (n+1) + \ldots + (n+1)$$
$$2S = n(n+1)$$
$$S = \frac{n(n+1)}{2}$$

Therefore the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Stated another way, the sum of the first n natural numbers is equal to the arithmetic mean of 1 and n, $\left[\frac{(n+1)}{2}\right]$, multiplied by n, the number of natural numbers being summed. This approach was, according to legend, used by Gauss when he was asked as a schoolboy to find the sum of the first hundred natural numbers.

More generally, any arithmetic sequence can be written as $(a_1),(a_1+d),(a_1+2d),\ldots,(a_1+(n-1)d)$, where each term differs from the previous term by a constant difference, d. The sum of the terms of the sequence, $(a_1)+(a_1+d)+(a_1+2d)+\ldots+(a_1+(n-1)d)$, can be derived in the following way:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left[a_1 + (k-1)d\right]$$

$$= na_1 + d \cdot \sum_{k=1}^{n} (k-1)$$

$$= na_1 + d \cdot \sum_{k=1}^{n-1} (k)$$

$$= na_1 + d \cdot \frac{(n)(n-1)}{2}$$

$$= \frac{n}{2}\left[2a_1 + d(n-1)\right]$$

$$= \frac{n}{2}\left[a_1 + a_1 + d(n-1)\right]$$

$$= \frac{n}{2}\left[a_1 + a_{n]}\right]$$

In other words, the sum of an arithmetic sequence is the arithmetic mean of the first and last terms times the number of terms in the sequence.

The first n natural numbers—that is, 1, 2, 3, ... , n–1, n—form an arithmetic sequence with a constant difference of 1 and with $a_1 = 1$ and $a_n = n$. Substituting these values into the expression above yields the desired formula:

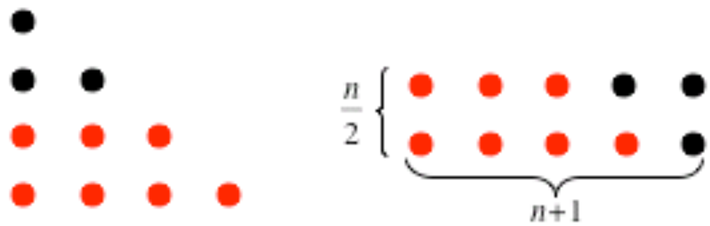
$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n(1+n)}{2} = \frac{n(n+1)}{2} . n$$

Mathematical Focus 4

Geometric arrays provide opportunities to derive the formula for the sum of the first n *natural numbers.*

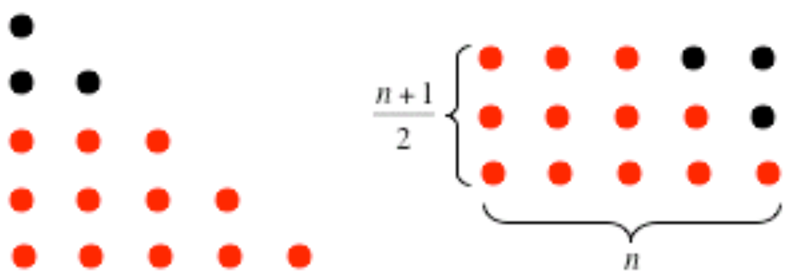
<u>(Model 1): Rearranging a Triangular Array of Dots</u>
One representation of the first n natural numbers is a triangular array of dots, having n rows, in which the number of dots in the first row is 1, and the number of dots in each successive row increases by 1, so that in the nth row there are n dots.

Consider the sum of the first n natural numbers when n is even. One way to derive the formula for the sum of the first n natural numbers is to rearrange the triangular array of dots into a rectangular array. When n is even, there will be $\frac{n}{2}$ rows and n + 1 columns in the rectangular array, as shown in the figure below.



There are $\frac{n}{2} \times (n + 1)$ dots in the rectangular array. Hence, the sum of the first n natural numbers for n even is $\frac{n(n+1)}{2}$.

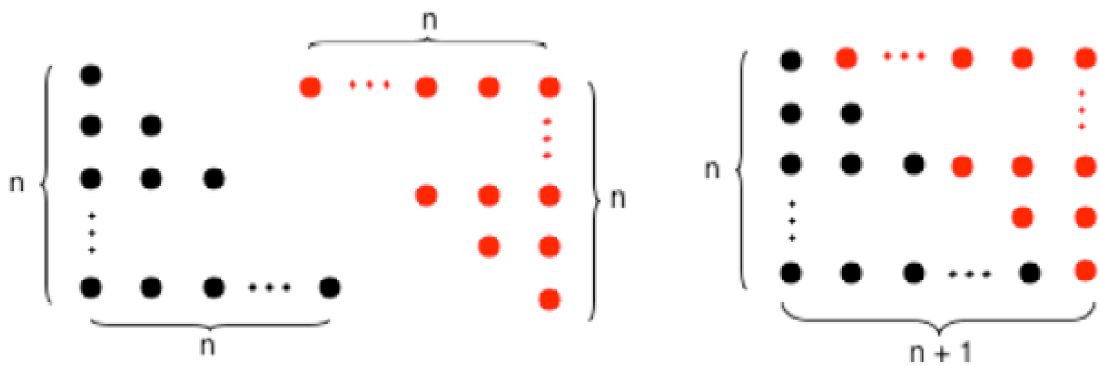
Now consider the sum of the first n natural numbers when n is odd. If n is odd, there are $\frac{n+1}{2}$ rows and n columns in the rectangular array, as shown in the figure below.



There are $\frac{n+1}{2} \times n$ dots in the rectangular array. Hence, the sum of the first n natural numbers for n odd is $\frac{n(n+1)}{2}$.

<u>(Model 2):Duplicating a Triangular Array of Dots</u>
As previously stated, one representation of the first n natural numbers is a triangular array of dots, having n rows, in which the number of dots in the first row is 1, and the number of dots in each successive row increases by 1, so that in the nth row there are n dots.

Copy the original triangular array, rotate it 180° clockwise, and join it with the original to form a rectangular array of dots. The rectangular array contains n rows and $n + 1$ columns, as shown below.



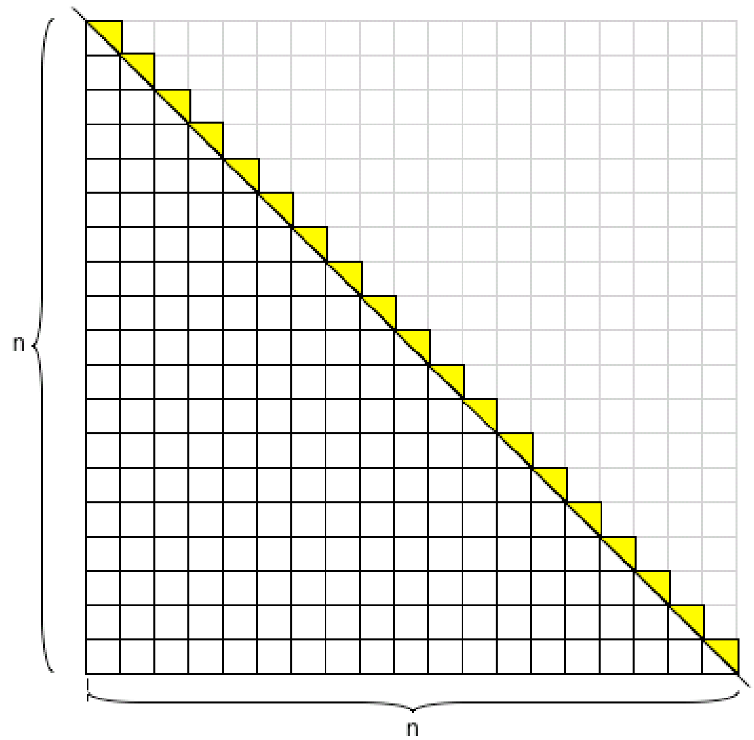
The number of dots in the rectangular array is $n(n+1)$. The rectangular array is composed of two triangular arrays, each containing a number of dots equal to the sum of the first n natural numbers. Because the rectangular array comprises two triangular arrays, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Mathematical Focus 5

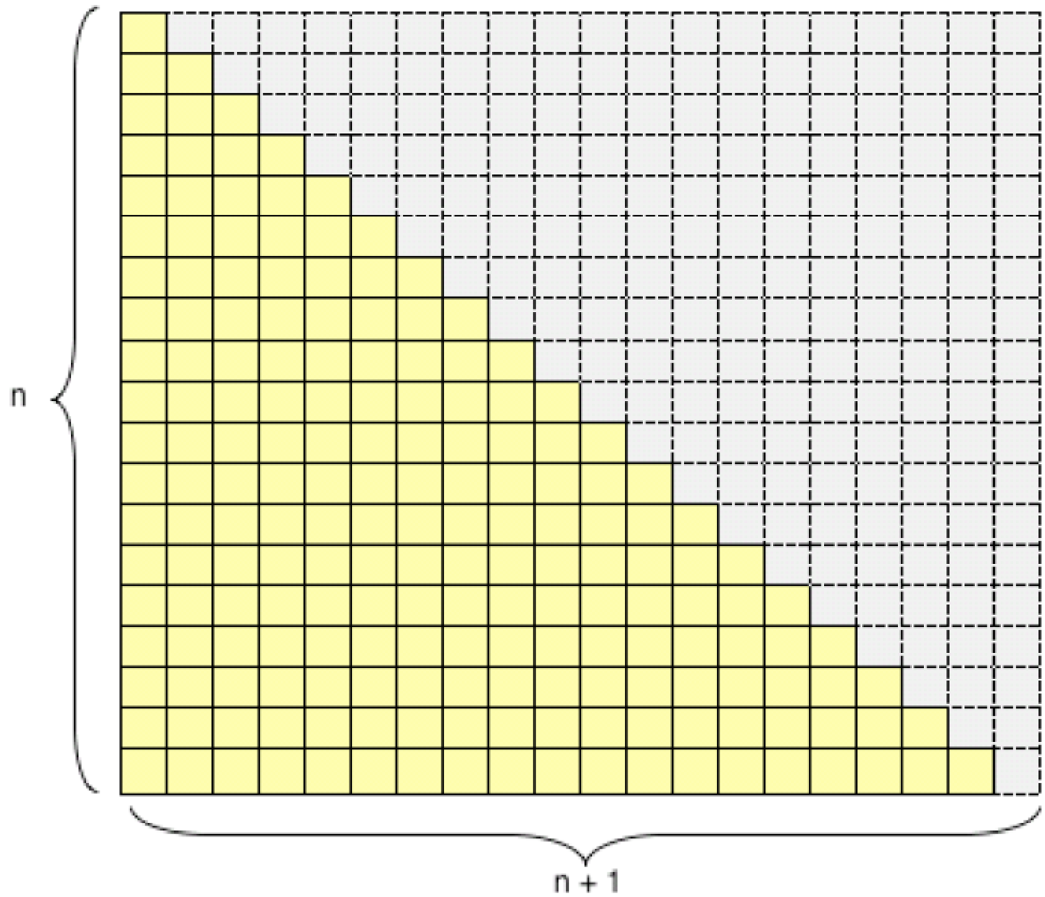
Decomposition and recomposition of plane geometric figures preserve area. Geometric figures provide opportunities to derive the formula for the sum of the first n *natural numbers.*

Another representation of the first n natural numbers is a staircase: a triangular array of unit squares, having n rows, in which the number of unit squares in the first row is 1, and the number of unit squares in each successive row increases by 1, so that in the nth row there are n unit squares. Both the number of unit squares in the triangular array and the sum of their areas represent the sum of the first n natural numbers.

(Model 1) Decomposing and Recomposing Unit Squares in a Staircase
As shown in the diagram below, the square in which the triangular array of unit squares is embedded has area $n \cdot n$, so the triangle has area $\frac{n \cdot n}{2}$, and the sum of the areas of the half-unit-triangles shaded in yellow is $\frac{n}{2}$. The sum of the areas of the unit squares in the triangular array, therefore, is $\frac{n \cdot n}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$, the sum of the first n natural numbers.



<u>(Model 2) Duplicating a Staircase</u>
Copy the original triangular array of unit squares, rotate it 180° clockwise, and join it with the original to form a rectangular array of unit squares. The dimensions of the rectangular array of unit squares are n by $n + 1$, and the sum of the areas of the unit squares in the rectangular array is $n(n + 1)$.

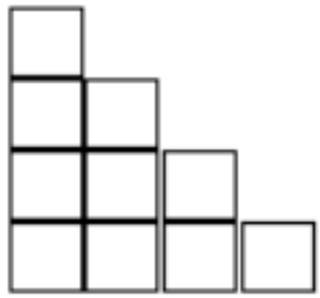
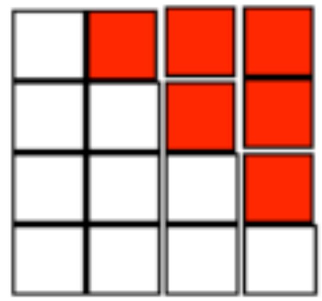


The sum of the areas of half of the unit squares in the rectangular array equals the sum of the areas of the unit squares in one triangular array, so the sum of the areas of the unit squares in one triangular array is $\frac{n(n+1)}{2}$. Because the sum of the areas of the unit squares in the triangular array is the sum of the first n natural numbers, the sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$.

(Model 3) Completing a Rectangular Array

A third unit-square-area model uses the idea of *completing* a rectangular array by adding additional unit squares to the rows in a staircase.

To complete a rectangular array of unit squares, add $n-1$ unit squares to the first row in the staircase, $n-2$ to the second row in the staircase, and so on, with $n-r$ unit squares being added to the rth row in the staircase. The resulting rectangular array has n rows of unit squares and n columns of unit squares—a total of n^2 unit squares—and the sum of the areas of those squares is n^2.

Because $S = 1 + 2 + 3 + \ldots + n$, the sum of the areas of the unit squares in the rectangular array is $S + (n-1) + (n-2) + \ldots + 1$. Because the sum of the areas of the unit squares is n^2, $S + (n-1) + (n-2) + \ldots + 2 + 1 = n^2$.
Adding n to both sides of the equation yields

$$S + (n-1) + (n-2) + \ldots + 2 + 1 + n = n^2 + n$$

Substitution yields $2S = n(n+1)$

And hence $S = \frac{n(n+1)}{2}$.

Mathematical Focus 6

The principle of mathematical induction offers one method for verifying a formula for the sum of the terms in a sequence.

For the sum of the first n natural numbers, S_n, the principle of mathematical induction can be used to prove that the formula $S_n = \frac{n(n+1)}{2}$ holds for all natural numbers. The proof does not derive the formula but proves that it holds.

The sum of the first n natural numbers can be expressed as $\sum_{i=1}^{n} i$.

The inductive hypothesis is that $S_n = \frac{n(n+1)}{2}$.

When $n = 1$, $S_1 = 1 = \frac{1(1+1)}{2}$, which establishes the base case.

Assume that for all natural numbers i less than or equal to m, $S_m = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$.

It is now necessary to show that the inductive hypothesis holds for $m + 1$; that is, that

$$\sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}.$$

The sum from 1 to $m + 1$ can be expressed as $m + 1$ added to the sum from 1 to m:

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + m + 1 = \frac{m(m+1)}{2} + m + 1 = \frac{m(m+1)}{2} + \frac{2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$$

By the principle of mathematical induction, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Post-Commentary 1
The sum of the first n *natural numbers is a specific case of the sum of the first* n k*th powers of natural numbers.*

Although the rule for the sum of the first n natural numbers is itself a generalization, it can be generalized further. The rule is the case $n = 1$ for the sum of the first n kth powers of natural numbers.

Polya (1981, ch. 3) describes how Pascal solved the problem of finding the sum of the kth powers of the first n natural numbers. This iterative method depends on expanding $(n+1)^{k+1}$ consecutively for $k = 1, 2, \ldots$.

Let S_n be the sum of the first n natural numbers. To find S_n, expand $(n+1)^2$ and write out its values for several numbers to see the pattern:

$(n+1)^2 = n^2 + 2n + 1$.

$\Rightarrow (n+1)^2 - n^2 = 2n + 1$

(1) $2^2 - 1^2 = 2 \cdot 1 + 1$

(2) $3^2 - 2^2 = 2 \cdot 2 + 1$

(3) $4^2 - 3^2 = 2 \cdot 3 + 1$

And so on, until one obtains

(n) $(n+1)^2 - n^2 = 2n + 1$

Add the equations (1) through (n) to get $(n+1)^2 - 1 = 2(S_1) + n$. Then

$$S_n = \frac{(n+1)^2 - n - 1}{2} = \frac{(n+1)^2 - 1(n+1)}{2} = \frac{(n+1-1)(n+1)}{2} = \frac{n(n+1)}{2}.$$

Similarly, one can find the sum of the first n squares of natural numbers by expanding $(n + 1)^3$ and writing out its values for several numbers to see the pattern. Summing the set of equations from (1) through (n) yields the sum of the first n squares as

$$\frac{n(n+1)(2n+1)}{6}.$$

The method can be extended to find the sum of the first n cubes, the first n fourth powers, and so on, as long as the sums of all the previous powers have been found.

Post-Commentary 2

In Mathematical Focus 4, triangular arrays of dots were used to represent the sum of the first n natural numbers. In other words, the sum of the first n natural numbers is a *triangular* number. A triangular number is a special case of a *figurate* number; that is, a number that can be represented by a regular geometrical arrangement of equally spaced points. The following links provide more information:
http://mathworld.wolfram.com/TriangularNumber.html
http://mathworld.wolfram.com/FigurateNumber.html

Reference

Polya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving*. John Wiley & Sons: New York.