Situation: Summing the Natural Numbers

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Prompt

The course was a mathematical modeling course for prospective secondary mathematics teachers.

The discussion focused on finding an explicit formula for a sequence expressed recursively. During the discussion, a need to sum the integers from 1 to n was expressed. After several attempts to remember the formula, a student hypothesized that the formula contained n and $n + 1$. Another student said he thought it was $n(n + 1)/2$, but was not sure. During the ensuing discussion, a third student asked, "How do we know that $n(n + 1)/2$ is the sum of the integers from 1 to n, and won't this formula sometimes give fractions?"

Commentary

This prompt illustrates a reason to make a relationship between two concepts: an explicitly defined formula that could effectively determine the sum of a series with actually working the entire sum. In creating this closed from, the question arises on the development of such a formula and its effectiveness for appropriate values.

The sum of the natural numbers is an example of determining the sum of an arithmetic series. In Focus 1, we are determining the sum of natural numbers by adding the terms of the series to itself and taking half the result. In this Focus 2, we are combining parts of Foci 2 and 3 of the first draft to emphasize that the sum of the arithmetic series is a triangular number. A triangular number is a number that can be represented by an equilateral triangle whose side length is a natural number. In this Focus 3, we determine the formula for the sum of an arithmetic series by geometric approaches by using rectangles or right triangles. In Focus 4, the formula for the sum of the series is proven true from proof by mathematical induction. In Focus 5, the terms of the series are added together two at a time to derive the formula for the sum of the series. I have added my proof of the groupings at the end of this focus. The last focus shows how the formula of the sum of natural numbers can be derived from a binomial expansion.

Mathematical Foci

Mathematical Focus 1

Let S be the sum of the first n natural numbers. Then $S = 1 + 2 + 3 + ... + n\text{-}1 + n$. By the commutative and associative properties of addition, $S + S = 1 + 2 + 3 + n\text{-}1 + n + 1 + 2 + 3 + ... + n\text{-}1 + n = (n + 1) + (n - 1 + 2) + (n\text{-}2 + 3) + ... + (2 + n - 1) + (1 + n) = n(n + 1)$. So $2S = n(n + 1)$, and $S = n(n + 1)/2$. Thus the sum of the first n natural numbers is $n(n + 1)/2$.

We know n and n+1 are consecutive integers, thus one is odd and the other even. Since the product of an even number and any other natural number is always even, our numerator is even, and thus divisible by two. Thus our formula always yields a natural number.

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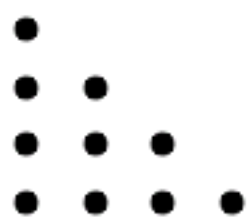
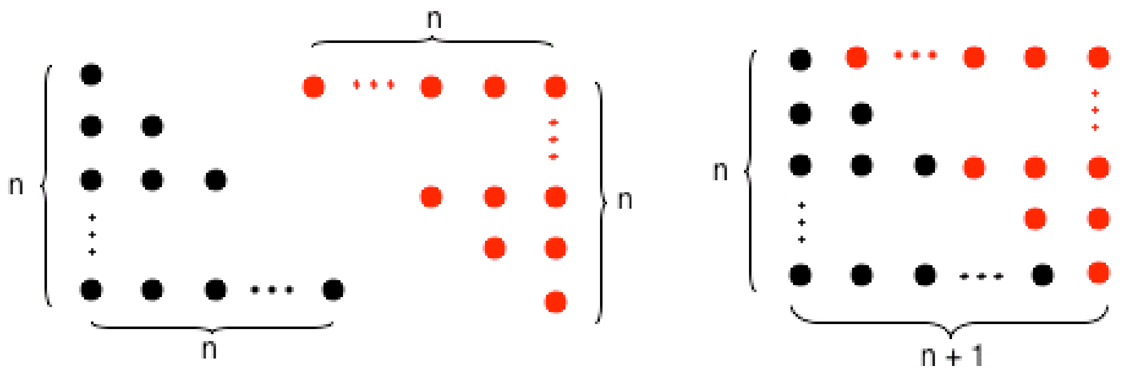
Mathematical Focus 2

We can represent the first n natural numbers with a triangular array of dots, with each row of dots representing a number. The following array represents the first four natural numbers.

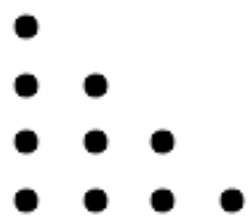


If we make a copy of that array and rotate it so that, when joined with the original we obtain a rectangle, we can see that the dimensions of the rectangular array for this specific case are 4 by 5, and in the general case are n by $n + 1$.



Thus we have 4*5 = 20 dots in the rectangular array above ($n(n + 1)$ dots in the general case), and since we have two identical triangular arrays making up the rectangular array, there are 20/2 = 10 dots in one triangular array ($n(n + 1)/2$ dots in one triangular array in the general case). Then since the triangular array represented the first n natural numbers, we have $n(n + 1)/2$ as the sum of the first n natural numbers. Since in every case we are working with two identical triangular arrays of dots representing natural numbers, it is clear that our formula will always yield a natural number.

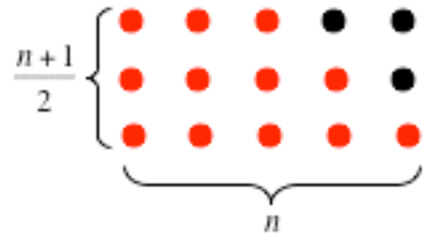
We can represent the first n natural numbers with a triangular array of dots, with each row of dots representing a number. The following array represents the first four natural numbers.



One way to verify the formula would be to rearrange the dots into a rectangle. If n is even, we would have $n/2$ rows, as shown in the first figure below.

The remaining rows will be rotated and moved to complete a rectangle having dimensions $n/2$ by $n + 1$ as shown in the second figure above.

If n is odd, we would have $(n + 1)/2$ rows, as shown in the first figure below.

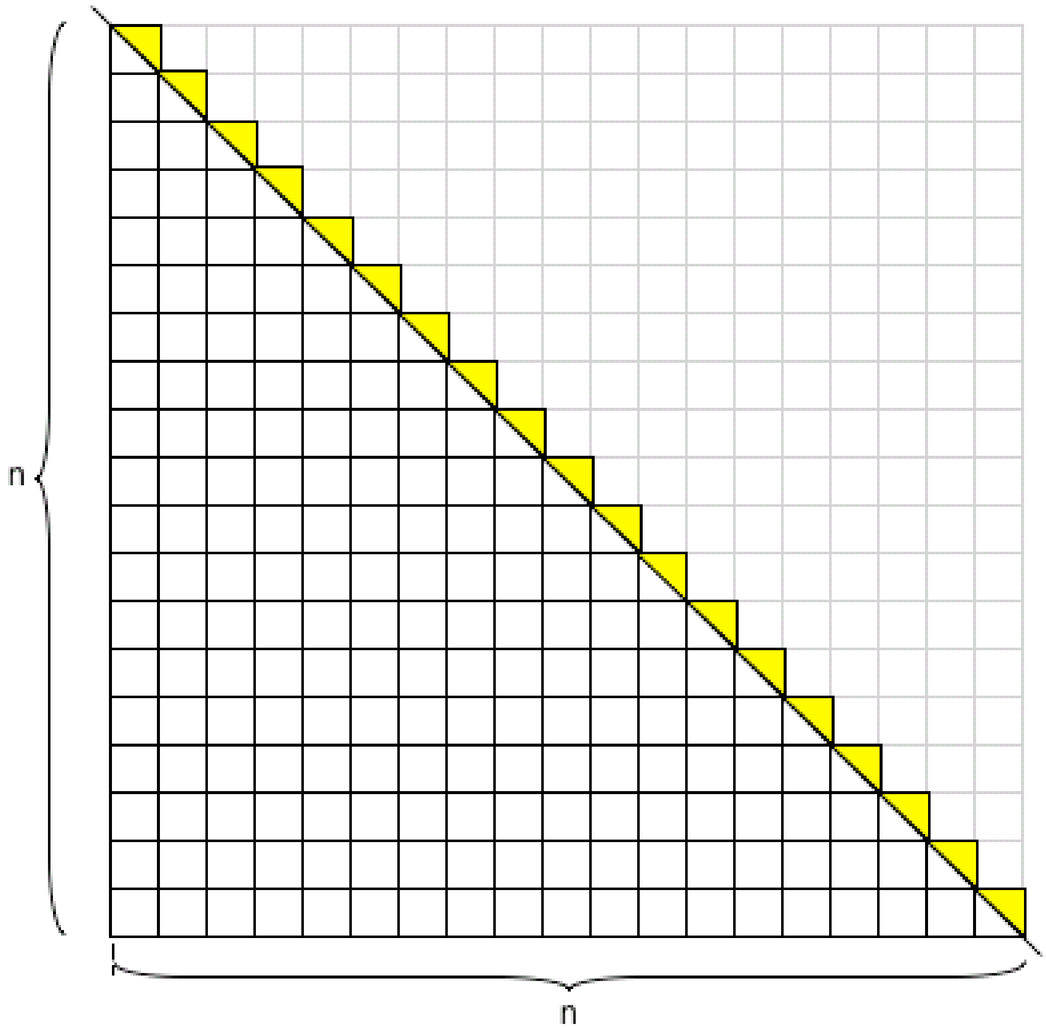


The remaining rows will be rotated and moved to complete a rectangle having dimensions (n + 1)/2 by n as shown in the second figure above.

Since this method only requires rearranging the dots, it is clear that it will always yield a natural number result.

Mathematical Focus 3

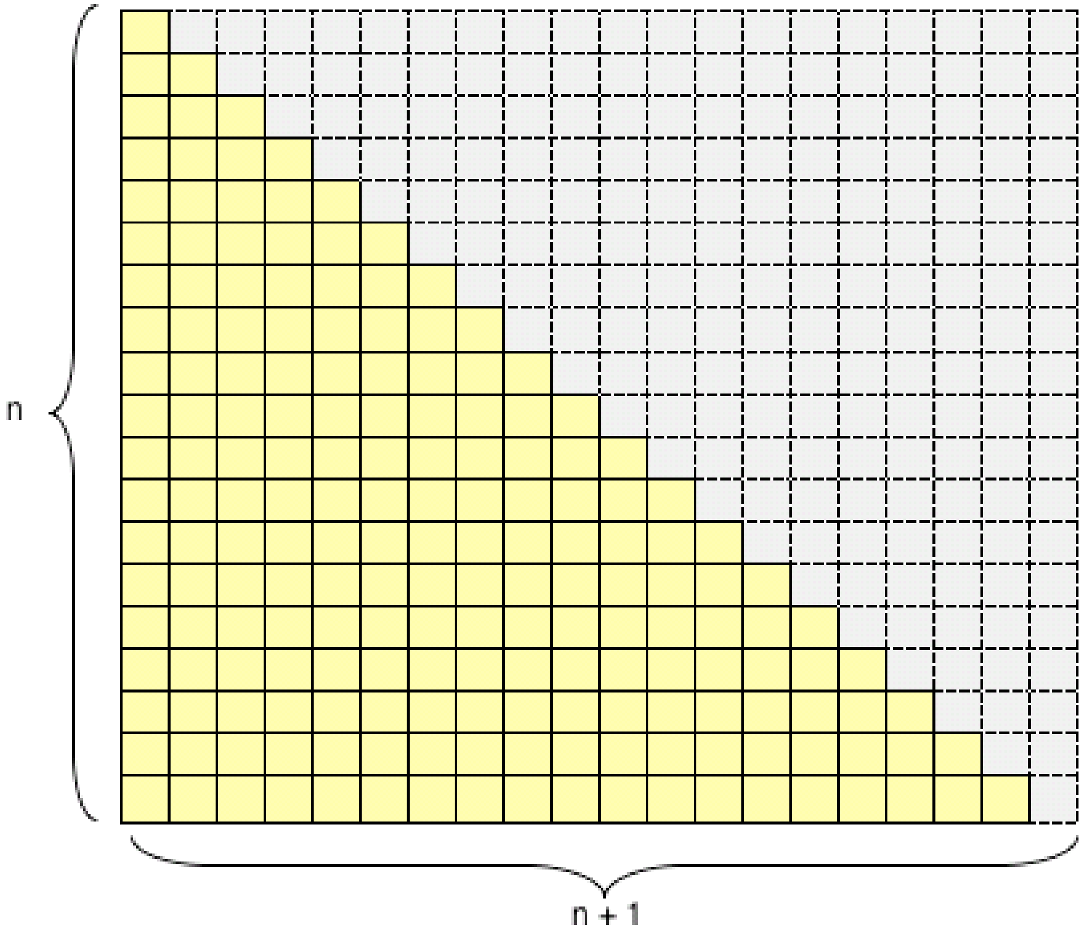
We can represent the first n natural numbers by unit squares arranged in n rows with each row of squares representing a number. The sum of the first n natural numbers is equal to the number of squares in the region, which, since the squares are unit squares, is equal to the area of the region.



The area of this region can be found by finding the area of the right triangle with legs of length n and adding the area of each of the small triangles left out of this triangle (highlighted above). The area of the large right triangle is $n^2/2$, and the area of each of the n small triangles is 1/2, so the combined area is $n^2/2 + n/2 = n(n + 1)/2$.

If n is odd, then n^2 is odd, so $n^2 + n$ is even and thus our formula yields a natural number. If n is even, then n^2 is even, so $n^2 + n$ is even and our formula once again yields a natural number

A second way to visualize this is to use unit squares in place of the dots as seen below. The unit squares also form a triangular shape that can be copied and rotated 180 degrees to form a rectangle. One-half the area of this n by $n + 1$ rectangle is the sum of the first n natural numbers.



Mathematical Focus 4

We can use mathematical induction to prove that our formula will always work. This proof does not derive the formula, but proves that the formula works once we have it.

Let sum be $\sum_{i=1}^{n} i$.

Our inductive hypothesis is that $S_n = \frac{n(n+1)}{2}$.

When $n = 1$, we have 1 = 1(2)/2 = 1, which establishes our base case.

We assume that, for all numbers less than or equal to m, $S_m = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$. We must show that this also holds for $m + 1$, that is, we must show that $\sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}$.

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + m + 1 = \frac{m(m+1)}{2} + m + 1 = \frac{m(m+1)}{2} + \frac{2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$$ as required.

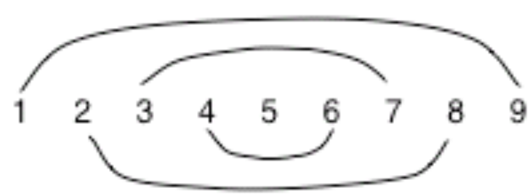
Thus the sum of the first *n* natural numbers is *n*(*n* + 1)/2.

We know *n* and *n*+1 are consecutive integers, thus one is odd and the other even. Since the product of an even number and any other is always even, our numerator is even, and thus divisible by two. Thus our formula always yields a natural number.

Mathematical Focus 5

Considering these questions could lead to an exploration of the sum of the first n natural numbers. For example, suppose that *n* = 16. One could begin by adding the numbers (i.e. 1 + 2 + 3 + … + 16). However, one way to add the numbers, using the both the commutative and associative laws of addition, could be to change the order and groupings of the numbers. In our example, the first grouping could be the largest number with the smallest number (i.e. 1 + 16), next grouping the second largest number with the second smallest number (i.e. 2 + 15), then grouping the third largest number with the third smallest number (3 + 14), and so forth until all possible groupings are created. We will have 8 groups that sum to 17, since 8 is half of sixteen. Thus the sum of the numbers from 1 to 17 is 8*17 = 136. We see that this is in line with the given formula of *n*(n+1)/2. However, this only shows that the formula seems to work and give a natural number when adding an even number of numbers.

Now suppose that *n* = 9. One could begin by adding the numbers (i.e. 1 + 2 + 3 + … + 9). However, we could also group the numbers as above using the following diagram.



Notice that since 9 is odd, 5 is not paired with another number. Thus we have 4 groups that sum to 10. Notice that 5 is half of our group sum of 10, so we have 4 1/2 groups of 10. 4 1/2 is half of 9, and 10 is 9 + 1, so our formula *n*(*n* + 1)/2 seems to hold. When we are adding an odd number of numbers, grouping as above will always result in even sums since we add odd to odd and even to even. Even though we will always have to deal with half of a group sum (as in the 5 in our example above), that sum is even, so the result is a natural number. We have obtained a natural number result of the formula when adding an odd number of numbers.

A proof of these groupings follows:

I want to use the same grouping technique that used with n = 9. This time it would be 1 + n, 2 + (n-1), 3 + (n-2), and so on. Each one of those sums is n+1. There are always n/2 – ½ of these groupings. However, there is always one term left out in the grouping: n/2 + ½. To add up all of these terms, I want to add all of the groupings and the leftover term. A little commutative property of multiplication and factoring will give me the desired formula.

$$\left(\frac{n}{2}-\frac{1}{2}\right)(n+1)+\left(\frac{n}{2}+\frac{1}{2}\right)$$

$$\left(\frac{n-1}{2}\right)(n+1)+\left(\frac{n+1}{2}\right)$$

$$(n-1)\left(\frac{n+1}{2}\right)+\left(\frac{n+1}{2}\right)$$

$$(n-1+1)\left(\frac{n+1}{2}\right)$$

$$n\left(\frac{n+1}{2}\right)=\frac{n(n+1)}{2}$$

Now to show that this answer would yield a natural number answer. If I look at the formula developed in this solution

$$\left(\frac{n}{2}-\frac{1}{2}\right)(n+1)+\left(\frac{n}{2}+\frac{1}{2}\right)$$

We could simplify the fractions to $\left(\frac{n-1}{2}\right)(n+1)+\left(\frac{n+1}{2}\right)$. Since n is an odd number then both n – 1 and n + 1 are even numbers. Dividing an even number by 2 will yield a natural number answer. The multiplying and adding of natural numbers will still result in a natural number.

Now to show a general case for an even number of numbers.

1 + 2 + 3 + … + (n – 2) + (n – 1) + n

If we add the first and last terms then the second and second to last terms and continuing inward through the terms of the series. Each sum of pairs is n + 1. There are a total of n/2 groups of sums that are added together. Therefore the sum of the series is $(n+1)\frac{n}{2}$, and through the commutative property we get $\frac{n(n+1)}{2}$.

To show that the result will yield a natural number, it goes to show that since n is an even number, and then dividing the even number by 2 will still give a natural number answer and multiplying the quotient by an odd number will still give a natural number answer.

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Mathematical Focus 6 & Extension

Polya (1981) describes how Pascal solved this problem and extended it to the general case of finding the sum of the k^{th} powers of the first n natural numbers. The method depends on expanding $(n + 1)^{k+1}$.

Let S_1 be the sum of the first n natural numbers. To find S_1, expand $(n + 1)^2$ and write out its values for several numbers to see the pattern:

$(n+1)^2 = n^2 + 2n + 1$ so $(n+1)^2 - n^2 = 2n + 1$.

$(n+1)^2 - n^2 = 2n+1$

(1) $2^2 - 1^2 = 2 \bullet 1 + 1$

(2) $3^2 - 2^2 = 2 \bullet 2 + 1$

(3) $4^2 - 3^2 = 2 \bullet 3 + 1$

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(n) $(n+1)^2 - n^2 = 2n+1$

Then we can add all the equations (1) through (n) together. The sum of these equations will be $(n+1)^2 - 1 = 2(S_1) + n$. Then

$$S_1 = \frac{(n+1)^2 - n - 1}{2} = \frac{(n+1)^2 - 1(n+1)}{2} = \frac{(n+1-1)(n+1)}{2} = \frac{n(n+1)}{2}.$$

To find the sum of the first n squares, S_2, expand $(n + 1)^3$ and write out its values for several numbers to see the pattern:

$(n+1)^3 = n^3 + 3n^2 + 3n + 1$, so $(n+1)^3 - n^3 = 3n^2 + 3n + 1$

$(n+1)^3 - n^3 = 3n^2 + 3n + 1$

$2^3 - 1^3 = 3 \bullet 1^2 + 3 \bullet 1 + 1$

$3^3 - 2^3 = 3 \bullet 2^2 + 3 \bullet 2 + 1$

$4^3 - 3^3 = 3 \bullet 3^2 + 3 \bullet 3 + 1$

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.
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$(n+1)^3 - n^3 = 3n^2 + 3n + 1$

Again add all the equations together. The sum of these equations will be $(n+1)^3 - 1 = 3 \bullet S_2 + 3S_1 + n$. Substituting for S_1 and solving for S_2 gives us

$$S_2 = \frac{(n+1)^3 - 3\frac{n(n+1)}{2} - n - 1}{3} = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{6} = \frac{(n+1)[2(n+1)^2 - 3n - 2]}{6}$$

$$= \frac{(n+1)(2n^2+n)}{6} = \frac{n(n+1)(2n+1)}{6}$$

This method can be extended to find the sum of the first *n* cubes, and then the first *n* fourth powers and so on, as long as the sums of all the previous powers have been found.

References

Polya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving.* John Wiley & Sons: New York.

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