MAC-CPTM Situations Project

Situation 40: Powers

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Prompt

During an Algebra I lesson on exponents, the teacher asked the students to calculate positive integer powers of 2. A student asked the teacher, "We've found 2^2 and 2^3. What about $2^{2.5}$?"

Commentary

The prompt centers on the extension of the domain of the exponent from integers to rational numbers. The foci explore the nature of exponents numerically, analytically, and graphically. The table with integral values in Focus 1 suggests a pattern for a curve for an extended domain that is illustrated in the graphical representation in Focus 3. Although Foci 1 and 3 provide an estimate, the analytical treatment in Focus 2 generates an exact value. These foci help expand the concept of exponentiation beyond repeated multiplication to accommodate the use of rational exponents.

Mathematical Foci

Mathematical Focus 1

One method for estimating the value of 2^x *where* $x \in §^{*}$ *uses linear interpolation and the properties of the function f with rule* $f(n)=2^n$, *where* $n \in ¢^{*}$.

One way to calculate an estimate of the value $2^{2.5}$ is to use linear interpolation between the two known values, 2^2 and 2^3. The use of linear interpolation for estimating function values assumes that the function is well-behaved in that interval. Using linear interpolation, then $2^{2.5}$ is approximately 6. However, it is important to understand that the value for $2^{2.5}$ will not be exactly halfway between 2^2 and 2^3. Whether the value of $2^{2.5}$ is greater than 6 or less than 6 is determined by the pattern of growth of the function.

The table below shows a pattern of increasing growth between successive values of 2^x and illustrates that exponential growth is different from a linear pattern of growth. In particular, nonlinearity implies that even though 2.5 is the average of 2 and 3, $2^{2.5}$ will not be the arithmetic mean of 4 and 8.

x	2^x
0	1
1	2
2	4
3	8
4	16

Since the differences between the successive values of 2^x are increasing, one can argue that linear approximations will overestimate the value of $2^{2.5}$. Therefore, $2^{2.5}$ should be less than 6.

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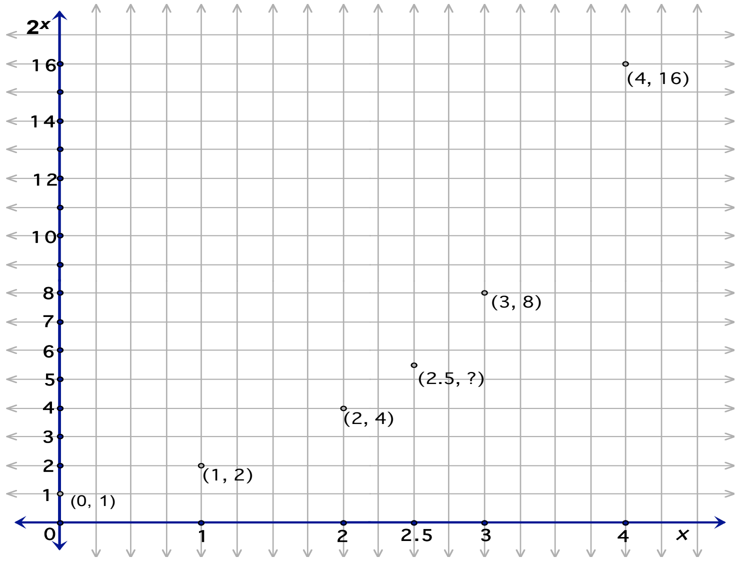


Specifically, if we look at this graph, we can begin to think of a way to connect the points that preserves the rate of growth in the table of values. This pattern of points suggests a graph that is concave up. A graph that is concave up increases or decreases at an increasing rate whereas the graph of a line increases or decreases at a constant rate. We can use this fact to conclude that the value of $2^{2.5}$ will be closer to 4 than to 8.

Mathematical Focus 2

By defining rational powers in terms of integral powers, the properties of integral exponents are extended to rational exponents.

By definition, $b^{m/n}$ is the number that when raised to the n^th power gives b^m as a result. So, $b^{m/n} = \sqrt[n]{b^m}$ for m∈¢*, n∈•, and b∈° +. By this definition, $\left(b^{m/n}\right)^n = b^m$. We can use this definition to extend the properties of integral exponents to rational exponents. Using this definition, $\left(b^r\right)^{\frac{1}{s}} = \sqrt[s]{\left(b^r\right)^1} = \sqrt[s]{b^r} = b^{\frac{r}{s}}$. We can establish other properties of rational exponents, such as the following:

1.

To show: $\left(b^{\frac{1}{s}}\right)^r = \left(b^r\right)^{\frac{1}{s}}$

$$\left(\left(b^{\frac{1}{s}}\right)^r\right)^s = \underline{\underline{\left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right) \cdot \left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right) \cdot \ldots \cdot \left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right)}}$$

where each factor of $\left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right)$ consists of r factors of $b^{\frac{1}{s}}$

and there are s factors of $\left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right)$ in the product.

$= b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}$ (in which there are rs factors of $b^{\frac{1}{s}}$)

$$= \underline{\underline{\left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right) \cdot \left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right) \cdot \ldots \cdot \left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right)}}$$

where each factor of $\left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right)$ consists of s factors of $b^{\frac{1}{s}}$

and there are r factors of $\left(b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdot \ldots \cdot b^{\frac{1}{s}}\right)$ in the product.

$= b^r$

So by our definition of rational exponents, $\left(b^r\right)^{\frac{1}{s}}$ is the number that when raised to the s^{th} power gives b^r as a result.

So $\left(b^{\frac{1}{s}}\right)^r = \left(b^r\right)^{\frac{1}{s}}$.

(Continued)

2.

To show: $\left(b^{\frac{p}{r}}\right)^{\frac{q}{s}} = b^{\frac{pq}{rs}}$

$$\left(b^{\frac{pq}{rs}}\right)^{rs} = b^{pq} = \left(b^{p}\right)^{q} = \left(\left(b^{\frac{p}{r}}\right)^{r}\right)^{q} = \left(\left(\left(b^{\frac{p}{r}}\right)^{\frac{q}{s}}\right)^{s}\right)^{r} = \left(\left(b^{\frac{p}{r}}\right)^{\frac{q}{s}}\right)^{rs}$$

$$\text{So}\left(\left(b^{\frac{pq}{rs}}\right)^{rs}\right)^{\frac{1}{rs}} = \left(\left(b^{\frac{p}{r}}\right)^{\frac{q}{s}}\right).$$

$$\text{So } b^{\frac{pq}{rs}} = \left(\left(b^{\frac{p}{r}}\right)^{\frac{q}{s}}\right).$$

Using similar techniques, we can prove properties such as:

3. $b^{\frac{p}{r}} \cdot b^{\frac{q}{s}} = b^{\frac{pq}{rs}}$

4. $(ab)^{\frac{p}{r}} = a^{\frac{p}{r}} \cdot b^{\frac{p}{r}}$

In the case of this prompt, $2^{1/2} = \sqrt{2}$. Using this representation, $\left(2^5\right)^{1/2} = \sqrt{2^5} = \sqrt{32} \approx 5.656$ or $(2^{\frac{1}{2}})^5 = (\sqrt{2})^5 \approx 1.414^5 \approx 5.656$. To rewrite $2^{2.5}$ in another form, $(2^{\frac{1}{2}})^5 = (\sqrt{2})^5 = \sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2} = 4\sqrt{2}$.

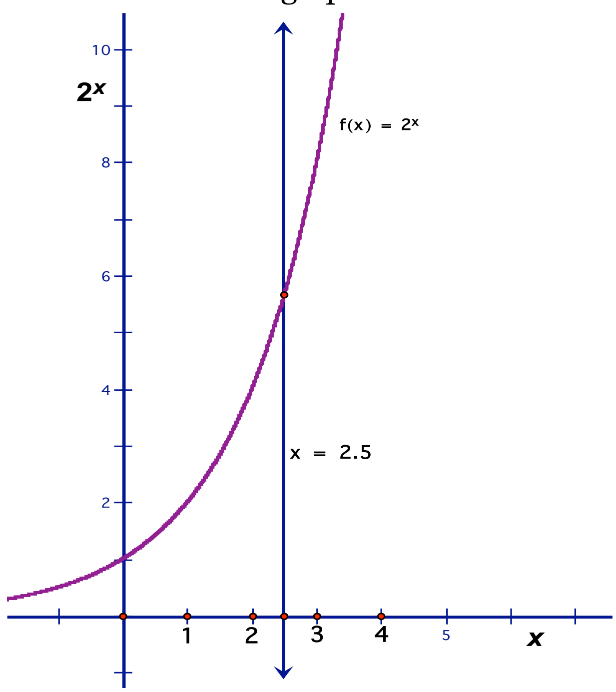
Consistency is maintained in this focus. The choices that are made in the extension of the definition and properties of exponents are made so that they are preserved across different sets of numbers, i.e. from whole numbers to rational numbers. That is, we can define 2^n, with *n* rational, in a way that allows the properties of exponents to hold. By deciding that the properties must hold, we can then use them to express $2^{2.5}$ as $\left(2^5\right)^{\frac{1}{2}}$ or $\left(2^{\frac{1}{2}}\right)^5$.

Mathematical Focus 3

A graphical representation of the function f with rule $f(x) = 2^x$ *allows for more accurate estimation of* 2^n *where* $n \in §^*$.

It should be recognized that at this point one is using the assumption that the domain of the function f with rule $f(x) = 2^x$ can be extended from the set of integers to the set of real numbers. In particular, the resulting graph of the function with the new domain will be represented by a smooth, continuous graph of the function. This graph allows one to obtain an estimate for f(2.5) with varying degrees of accuracy depending on the technology or method employed.

For example, one can estimate the value of f(2.5) from a calculator-generated graph of *f* and the trace option to obtain a value of f(2.5). Alternatively, one can look at the intersection of the function graph with the vertical line x = 2.5.



Post – Commentary

It is also useful to consider the fact that $f(x) = a^x$ would behave differently for particular values of "a" and "x." In the discussion below, "a" is assumed to be positive and "x" is a rational number. If "a" had been negative, the arguments would not apply. Furthermore, different mathematical discussions would be necessary if "x" had been irrational, transcendental, or complex.

Also, when considered equations of the form $a^x \cdot a^y = a^z$, it is important to consider the domain of the variables x, y, and z. Only with the appropriate domains can one make claims about the exponent rules such as $a^x \cdot a^y = a^{x+y}$.